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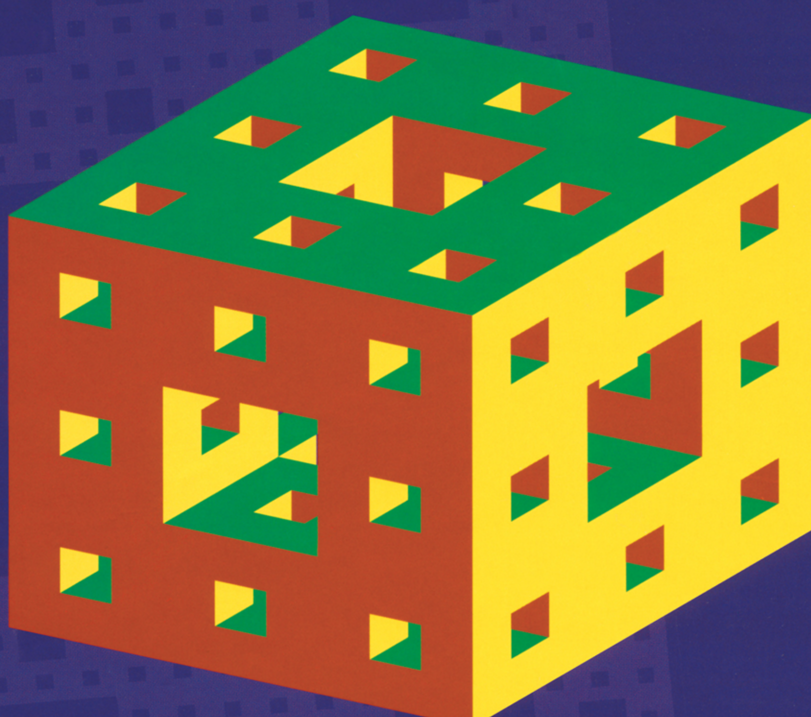

Edited by

K.P. Hart, J. Nagata & J.E. Vaughan

Encyclopedia of
General Topology

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Contents

Preface

General Topology has experienced rapid growth during the past fifty years, and nowadays its language and concepts pervade much of modern day mathematics. This book is intended for a broad community of scholars and students working in mathematics and other areas who want to become acquainted quickly with the terminology and ideas of General Topology that may be relevant to their work. Thus the book provides a source where the specialist and non-specialist alike can find short introductions to both the basic theory and the newest developments in General Topology.

Because the book is designed for the reader who wants to get a general view of the terminology with minimal time and effort there are very few proofs given; on occasion a sketch of an argument will be given, more to illustrate a notion than to justify a claim. We assume that the reader has a rudimentary knowledge of Set Theory, Algebra, Geometry and Analysis. A reader who wants to study the subject matter of one or more of the articles systematically (or who wants to see the proof of a particular result) will find sufficient references at the end of each article as well as in the books in our list of standard references.

Guide to the reader

Titles of articles are given in the Table of Contents under the following ten headings that roughly follow Section 54 of the 2000 Mathematics Subject Classification as used by *Mathematical Reviews* and *Zentralblatt MATH*.

- A – Generalities
- B – Basic constructions
- C – Maps and general types of spaces defined by maps
- D – Fairly general properties
- E – Spaces with richer structures
- F – Special properties
- G – Special spaces
- H – Connection with other structures
- J – Influences of other fields
- K – Connections with other fields

Topological terms used in the encyclopedia are listed in the Index. Terms defined within an article are indicated thus: **compactness**. Terms used in an article but defined elsewhere will be typeset thus: ***compactness*** (the first occurrence only); their definitions can be located by consulting the index.

There is a list of standard references, given below, that are cited uniformly by a letter system. Thus [E, 3.1] would refer to the first section on compactness in Engelking's *General Topology* and [HvM, Chapter 18] to the chapter on compact spaces in *Recent Progress in General Topology* (edited by Hušek and van Mill).

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A: Generalities

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a-1 Topological Spaces

The following is a brief survey of basic terminology used in general topology. Engelking's book [E] is one of the standard texts on general topology. In the book [E, pp. 18–20] the readers can quickly review origins and background of general topology.

1. Topological spaces

Let X be a set and $\mathcal{T}$ be a family of subsets on X. If $\mathcal{T}$ satisfies the following conditions:

(O1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
(O2) If $U_1 \in \mathcal{T}$ and $U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$,
(O3) If $\mathcal{T}' \subset \mathcal{T}$, then $\bigcup \mathcal{T}' \in \mathcal{T}$,

then $\mathcal{T}$ is called a **topology** on X and the pair $(X, \mathcal{T})$ is called a **topological space** (or **space** for short). Every element of $(X, \mathcal{T})$ is called a **point**. Every member of $\mathcal{T}$ is called an **open set** of X or open in X. If $\{x\} \in \mathcal{T}$, then the point x is called an **isolated point** of X. The complement of an open set is called a **closed set** of X or **closed** in X. If a set is open and closed in a topological space, then it is called **open-and-closed** or **closed-and-open** (or **clopen** for short). Let $\mathcal{C}$ be the family of closed sets of a topological space $(X, \mathcal{T})$. Then $\mathcal{C}$ satisfies the following conditions:

(C3) $\emptyset \in \mathcal{C}$ and $X \in \mathcal{C}$,
(C3) If $F_1 \in \mathcal{C}$ and $F_2 \in \mathcal{C}$, then $F_1 \cup F_2 \in \mathcal{C}$,
(C3) If $\mathcal{C}' \subset \mathcal{C}$, then $\bigcap \mathcal{C}' \in \mathcal{C}$.

The intersection of countably many open sets of a topological space need not be open. The intersection of countably many open sets is called a G_δ**-set**. The union of countably many closed sets of a topological space need not be closed. The union of countably many closed sets is called an F_σ**-set**. The complement of a G_δ-set is an F_σ-set. A topological space in which every closed set is a G_δ-set is called a **perfect space**. Countable unions of G_δ-sets are $G_{\delta\sigma}$**-sets** and their complements are $F_{\sigma\delta}$**-sets**. This procedure can be continued to form the hierarchy of ***Borel sets***.

Let $(X, \mathcal{T})$ be a topological space. A subfamily $\mathcal{B}$ of $\mathcal{T}$ is called a **base** or **basis** for X (or even an **open base**) if for every $x \in X$ and arbitrary $U \in \mathcal{T}$ containing the point x there exists $V \in \mathcal{B}$ such that $x \in V \subset U$, in other words every open set of X is the union of a subfamily of $\mathcal{B}$. A topological space X having a countable base is called **second-countable** or we say that X satisfies the **second axiom of countability**. A base $\mathcal{B}$ for X satisfies the following conditions:

(B1) If $U_1, U_2 \in \mathcal{B}$ and $x \in U_1 \cap U_2$, then there exists $U \in \mathcal{B}$ such that $x \in U \subset U_1 \cap U_2$,
(B2) $X = \bigcup \mathcal{B}$.

A subfamily $\mathcal{S}$ of $\mathcal{T}$ is called a **subbase** (or an **open subbase**) for X if the family of all finite intersections of members in $\mathcal{S}$ is a base for X.

Let X be a topological space and $x \in A \subset X$. We say that A is a **neighbourhood** of x if there exists an open set U in X such that $x \in U \subset A$. Every open set containing a point x is, of course, a neighbourhood of x. Such a neighbourhood is called an **open neighbourhood** of x. A family $\mathcal{B}(x)$ of neighbourhoods of $x \in X$ is called a **neighbourhood base** or a **local base** at the point x if for every open set U containing x there exists $V \in \mathcal{B}(x)$ such that $x \in V \subset U$. A family is a **local subbase** at a point if its finite intersections form a local base at that point.

If every point of X has a countable neighbourhood base, then X is called **first-countable** or we say that X satisfies the **first axiom of countability**. Every second-countable space is first-countable. Let $\{\mathcal{B}(x)\}_{x \in X}$ be a collection of neighbourhood bases of the points of X, which is called a **neighbourhood system** for X. It satisfies the following conditions:

(NB1) Every $\mathcal{B}(x)$ is non-empty and every member of $\mathcal{B}(x)$ contains x,
(NB2) If $U, V \in \mathcal{B}(x)$, then there exists $W \in \mathcal{B}(x)$ such that $W \subset U \cap V$,
(NB3) If $U \in \mathcal{B}(x)$, then there exists a set V such that $x \in V \subset U$ and for every $y \in V$ there exists $W \in \mathcal{B}(y)$ satisfying $W \subset V$.

Let $A \subset B \subset X$. We say that B is a **neighbourhood** of a set A if there exists an open set U in X such that $A \subset U \subset B$. A **neighbourhood base** at A sometimes called an **outerneighbourhood base** for A is a family $\mathcal{B}$ of neighbourhoods such that every neighbourhood of A contains a member of $\mathcal{B}$.

Let A be a subset of a topological space X. We denote by $\operatorname{Int} A$ ($\operatorname{int} A$ or A°) the union of all open sets of X contained in A. The set $\operatorname{Int} A$ is called the **interior** of A. It is the largest open set of X contained in A. It is easy to check that a point x belongs to $\operatorname{Int} A$ if and only if there exists a neighbourhood U of x such that $U \subset A$. The operator Int, called the **interior operator**, satisfies the following conditions:

(IO1) $\operatorname{Int} X = X$,
(IO2) $\operatorname{Int} A \subset A$,
(IO3) $\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B$,
(IO4) $\operatorname{Int}(\operatorname{Int} A) = \operatorname{Int} A$.

We denote by $\bar{A}$ ($\mathrm{cl}_X A$ or $\mathrm{Cl}_X A$) the intersection of all closed sets of X containing A. The set $\bar{A}$ is called the **closure** of A. It is the smallest closed set of X containing A. It is easy to check that a point x belongs to $\bar{A}$ (is an **adherent point** of A) if and only if every neighbourhood of x intersects A. A point not in $\bar{A}$ is an **exterior point** of A.

The set $\bar{A} \cap \overline{X - A}$ is called the **boundary** of A and denoted by $\operatorname{Fr} A$ ($\operatorname{Br} A$, ∂A). The operator $\bar{\cdot}$, called the **closure operator**, satisfies the following conditions:

(CO1) $\overline{\emptyset} = \emptyset$,
(CO2) $A \subset \bar{A}$,
(CO3) $\overline{A \cup B} = \bar{A} \cup \bar{B}$,
(CO4) $\overline{(\bar{A})} = \bar{A}$.

EXAMPLE 1. Let $X = \{a, b, c\}$ and

$$\mathcal{T} = \big\{\emptyset, X, \{a, c\}, \{b, c\}, \{c\}\big\}.$$

Since $\mathcal{T}$ satisfies (O1), (O2) and (O3), $(X, \mathcal{T})$ is a topological space. The family of closed subsets of X is $\{\emptyset, X, \{b\}, \{a\}, \{a, b\}\}$. Obviously $\operatorname{Int}\{a, b\} = \emptyset$ and $\operatorname{cl}_X\{c\} = X$.

EXAMPLE 2. Let $\mathbb{R}$ be the set of real numbers. Let $\mathcal{T}$ be the family of all subsets U of $\mathbb{R}$ satisfying the property that for each $x \in U$ there exists an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset U$. Since $\mathcal{T}$ satisfies (O1), (O2) and (O3), $(\mathbb{R}, \mathcal{T})$ is a topological space. The topology $\mathcal{T}$ is called the **natural topology** on $\mathbb{R}$.

A subset U of a topological space X is called a **regular open** set or an **open domain** if $U = \operatorname{Int} \overline{U}$ holds. A subset F of a topological space X is called a **regular closed** set or a **closed domain** if $F = \overline{\operatorname{Int} A}$ holds.

A point x of a topological space X is called an **accumulation point** or a **cluster point** of a set $A \subset X$ if $x \in \overline{A - \{x\}}$ holds, in other words every neighbourhood of x contains a point of A besides x. The set of all accumulation points of A is called the **derived set** of A, and denoted by A^d. A point x of a topological space X is called a **complete accumulation point** of a set $A \subset X$ if for every neighbourhood U of x, $|U \cap A| = |A|$ holds. If every neighbourhood of x contains uncountably many points of A, then the point is called a **condensation point** of A.

A subset A of a topological space X is called a **dense set** (a **nowhere dense set** respectively) in X if $\bar{A} = X$ ($\operatorname{Int} \bar{A} = \emptyset$ respectively) holds. If a topological space X has a countable dense subset, then X is called a **separable space**. Any countable union of nowhere dense subsets is called a **set of first category** or a **meager set**. Any set not of the first category is called a **set of second category**.

Let X be a topological space and $\mathcal{A}$ a family of subsets of X. If $X = \bigcup \mathcal{A}$ holds, then $\mathcal{A}$ is called a **cover** or **covering** of X, and a cover whose members are open (closed respectively) in X is called an **open cover** (a **closed cover** respectively) of X. If every point of X is contained in at most finitely many (countably many respectively) members of $\mathcal{A}$, then $\mathcal{A}$ is called **point-finite** (**point-countable** respectively). If for every $x \in X$ there exists a neighbourhood of x which intersects at most one member (finitely many members respectively) of $\mathcal{A}$, then $\mathcal{A}$ is called a **discrete family** (a **locally finite family** respectively). If for every subfamily $\mathcal{A}' \subset \mathcal{A}$,

$$\bigcup\big\{\bar{A}\colon A \in \mathcal{A}'\big\} = \overline{\bigcup \mathcal{A}'}$$

holds, then $\mathcal{A}$ is called **closure-preserving**. More strongly, if for every subfamily $\mathcal{A}' = \{A_\alpha\colon \alpha \in \Gamma\} \subset \mathcal{A}$ and any $B_\alpha \subset A_\alpha$ ($\alpha \in \Gamma$),

$$\bigcup\big\{\overline{B_\alpha}\colon \alpha \in \Gamma\big\} = \overline{\bigcup\{B_\alpha\colon \alpha \in \Gamma\}}$$

holds, then $\mathcal{A}$ is called **hereditarily closure-preserving**. If $\mathcal{A}$ can be represented as a countable union of discrete families, then $\mathcal{A}$ is called **σ-discrete**. In a similar way we define **σ-disjoint**, **σ-locally finite**, **σ-point-finite**, **σ-closure-preserving**, **σ-hereditarily closure-preserving**, etc. The following implication holds.

discrete
↓
locally finite
↙ ↘
point-finite hereditarily closure-preserving
↓ ↓
point-countable closure-preserving

There are several ways to give a set X a topology.

Let $\mathcal{B}$ be a family of subsets of X satisfying the conditions (B1) and (B2). Let

$$\mathcal{T} = \Big\{U\colon U = \bigcup \mathcal{B}' \text{ for some } \mathcal{B}' \subset \mathcal{B}\Big\}.$$

Then $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a base for the topological space $(X, \mathcal{T})$. The topology $\mathcal{T}$ is called the **topology generated by the base** $\mathcal{B}$. Let $\mathcal{S}$ be a family of subsets of X satisfying (B2). Then the family $\mathcal{B}$ of all finite intersections of members of $\mathcal{S}$ satisfies (B1) and (B2). The topology generated by $\mathcal{B}$ is called the **topology generated by the subbase** $\mathcal{S}$.

Let $\{\mathcal{B}(x)\colon x \in X\}$ be a collection of families of subsets of a set X satisfying the conditions (NB1), (NB2) and (NB3).

$$\mathcal{T} = \Big\{U\colon \text{if } x \in U,\ \text{then } x \in V \subset U \text{ for some } V \in \mathcal{B}(x)\Big\}.$$

Then $\mathcal{T}$ is a topology on X and each $\mathcal{B}(x)$ is a neighbourhood base of x in the topological space $(X, \mathcal{T})$. The topology $\mathcal{T}$ is called the **topology generated by the neighbourhood system** $\{\mathcal{B}(x)\colon x \in X\}$.

Let X be a set and Int is an operator assigning to every set $A \subset X$ a set $\operatorname{Int} A \subset X$ ruled by (IO1), (IO2), (IO3) and (IO4). Let

$$\mathcal{T} = \{U\colon \operatorname{Int} U = U\}.$$

Then $\mathcal{T}$ is a topology on X and $\operatorname{Int} A$ is the interior of A in the topological space $(X, \mathcal{T})$. The topology $\mathcal{T}$ is called the **topology generated by the interior operator** Int.

Let X be a set and $\bar{\cdot}$ is an operator assigning to every set $A \subset X$ a set $\bar{A} \subset X$ ruled by (CO1), (CO2), (CO3) and (CO4). Let

$$\mathcal{T} = \{X - A \colon \bar{A} = A\}.$$

Then $\mathcal{T}$ is a topology on X and $\bar{A}$ is the closure of A in the topological space $(X, \mathcal{T})$. The topology $\mathcal{T}$ is called the **topology generated by the closure operator** $\bar{\cdot}$.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X. If $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say that $\mathcal{T}_2$ is **finer** (**stronger** or **larger**) than $\mathcal{T}_1$, or $\mathcal{T}_1$ is **coarser** (**weaker** or **smaller**) than $\mathcal{T}_2$. The finest topology on X is the family of all subsets of X, and it is called the **discrete topology**. A topological space with this topology is called the **discrete space**. The coarsest topology on X consists of $\emptyset$ and X only, and it is called the **anti-discrete topology** or **indiscrete topology**. A topological space with this topology is called the **anti-discrete space** or **indiscrete space**. All topologies on a set X are partially ordered by inclusion $\subset$.

Let $(X, \mathcal{T})$ be a topological space and Y a subset of X. Let

$$\mathcal{T}_Y = \{Y \cap U \colon U \in \mathcal{T}\}.$$

It is easy to check that $\mathcal{T}_Y$ satisfies the conditions (O1), (O2) and (O3), so it is a topology on Y. The topological space $(Y, \mathcal{T}_Y)$ is called a **subspace** of X. The topology $\mathcal{T}_Y$ is called the **induced topology** or **subspace topology**. If Y is open (closed respectively) in X, then it is called an **open subspace** (a **closed subspace** respectively) of X. Obviously a set $A \subset Y$ is open (closed respectively) in the subspace Y if and only if there exists an open set U (a closed set F respectively) of X such that $A = Y \cap U$ ($A = Y \cap F$ respectively). If a subspace Y of a topological space X does not have any isolated points, then it is called **dense-in-itself**. A closed and dense in itself subspace is called a **perfect set**.

Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces such that $X_\alpha \cap X_\beta = \emptyset$, where A is an index set. For the union $X = \bigcup_{\alpha \in A} X_\alpha$, let

$$\mathcal{T} = \{U \colon U \subset X,\ U \cap X_\alpha \text{ is open in } X_\alpha \text{ for any } \alpha \in A\}.$$

It is easy to check that $\mathcal{T}$ satisfies the conditions (O1), (O2) and (O3), so it is a topology on X. The topological space $(X, \mathcal{T})$ is called the **topological sum** (also **discrete sum**) of the spaces $\{X_\alpha\}_{\alpha \in A}$ and denoted by $\bigoplus_{\alpha \in A} X_\alpha$.

Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. For the Cartesian product $X = \prod_{\alpha \in A} X_\alpha$, let

$$\mathcal{B} = \left\{\prod_{\alpha \in A} U_\alpha \colon U_\alpha \text{ is open in } X_\alpha,\ U_\alpha \neq X_\alpha \text{ only for finitely many } \alpha \in A\right\}.$$

It is easy to check that $\mathcal{B}$ satisfies the conditions (B1) and (B2), so it generates a topology $\mathcal{T}$ on X. The topological space $(X, \mathcal{T})$ is called the **product space** of the spaces $\{X_\alpha\}_{\alpha \in A}$ and the topology $\mathcal{T}$ is called the **Tychonoff product topology** on X. The base $\mathcal{B}$ is called the **canonical base** for X.

2. Networks, k-networks and weak bases

We recall the notions of networks, k-networks and weak bases of topological spaces. They behave like bases, but every member of such families need not be open. Therefore such families are easier to deal with than bases.

A family $\mathcal{N}$ of subsets of a topological space X is called a **network** for X if for every $x \in X$ and any neighbourhood U of x there exists $N \in \mathcal{N}$ such that $x \in N \subset U$. A base for X is a network. The family $\{\{x\} \colon x \in X\}$ is, for instance, a network for X. The notion of a network was introduced in [3] to investigate metrization conditions of ***compact*** spaces. It was proved in [3] that the ***network weight*** and the ***weight*** are equal for compact spaces. Thus every compact space with a countable network is second-countable, hence ***metrizable*** by Urysohn's theorem [E, 4.2.9]. Michael named a ***regular*** space having a countable network as a **cosmic space** [16]. It is known that a regular space X is the image of a separable ***metric space*** under a ***continuous map*** if and only if X is cosmic, see [16]. A regular space with a σ-locally finite network is called a ***σ-space***, which was introduced by Okuyama [18]. The class of σ-spaces is important in the theory of ***generalized metric spaces***.

A family $\mathcal{P}$ of subsets of a topological space X is called a **k-network** (**pseudobase** respectively) for X if whenever $C \subset U$ with C compact and U open in X, there exists $\mathcal{P}'$ of finitely many members of $\mathcal{P}$ ($P \in \mathcal{P}$ respectively) such that $C \subset \bigcup \mathcal{P}' \subset U$ ($C \subset P \subset U$ respectively). The notions of a k-network and a pseudobase are due to [20] and [16] respectively. Obviously a pseudobase is a k-network, and a k-network is a network. According to [16], a regular space having a countable pseudobase is called an **$\aleph_0$-space**. If $\mathcal{P}$ is a countable k-network for a topological space X, then the family of the unions of finitely many members of $\mathcal{P}$ is obviously a pseudobase for X. Hence we may say that an $\aleph_0$-space is a topological space having a countable k-network. Since every separable metric space is second-countable, it is an $\aleph_0$-space. Conversely every first-countable $\aleph_0$-space is separable and metrizable [16]. According to [20], a regular space with a σ-locally finite k-network is called an **$\aleph$-space**. Since every metric space has a σ-locally finite base, it is an $\aleph$-space. Conversely every first-countable $\aleph$-space is metrizable [19].

As represented by Nagata–Smirnov's metrization theorem, metrizability of topological spaces can be characterized by means of the existence of special bases. Similarly we can make use of a k-network to give inner characterizations of images of metric spaces by special continuous maps (e.g.,

quotient maps, ***closed maps***, etc). We recall two typical results in this direction. Michael proved in [16] that a regular space X is the image of a separable metric space by a quotient map if and only if X is a ***k-space*** and an $\aleph_0$-space. Foged proved in [12] that a regular space X is the image of a metric space by a closed map if and only if X is a ***Fréchet space*** with a σ-hereditarily closure-preserving k-network.

The notion of a k-network is useful in investigating function spaces. For topological spaces X and Y, let $C(X, Y)$ be the **function space** of all continuous maps from X to Y with the ***compact-open topology***. Michael proved in [16] that if X and Y are $\aleph_0$-spaces, then $C(X, Y)$ is also an $\aleph_0$-space. Foged proved in [11] that if X is an $\aleph_0$-space and Y is an $\aleph$-space, then $C(X, Y)$ is a ***paracompact*** $\aleph$-space.

The field of the theory of a k-network is extensive. It is closely related to other branches in general topology. Regular spaces with a point-countable k-network were studied in detail in [13]. Shibakov proved in [22] that a ***Hausdorff sequential topological group*** with a point-countable k-network is metrizable if its ***sequential order*** is less than ω_1. Further properties of sequential topological groups with a point-countable k-network were studied in [15]. The readers can find many papers on k-networks in the references of [24] and [25].

Let X be a topological space. For every $x \in X$ let $\mathcal{T}_x$ be a family of subsets of X containing x. If the collection $\{\mathcal{T}_x: x \in X\}$ satisfies

(1) for every $x \in X$ the intersections of finitely many members of $\mathcal{T}_x$ belong to $\mathcal{T}_x$ and
(2) $U \subset X$ is open in X if and only if $x \in U$ implies $x \in T \subset U$ for some $T \in \mathcal{T}_x$,

then it is called a **weak base** for X and each individual $\mathcal{T}_x$ a **weak neighbourhood base**. This notion was introduced by Arhangel′skiĭ [6, p. 129] to study ***symmetrizable*** spaces. A topological space X is said to satisfy the **weak first axiom of countability** or X is **weakly first-countable** if it has a weak base $\{\mathcal{T}_x: x \in X\}$ such that each $\mathcal{T}_x$ is countable. Every symmetrizable space is weakly first-countable and a topological space is first-countable if and only if it is a weakly first-countable Fréchet space [6]. A regular space with a σ-locally finite weak base is called a **g-metrizable space** [23]. Foged proved in [10] that a topological space is g-metrizable if and only if it is a weakly first-countable $\aleph$-space.

3. Special bases

Since Alexandroff–Urysohn's metrization theorem [E, 5.4.9], many special bases related to ***developable*** spaces have been studied.

A base $\mathcal{B}$ for a topological space X is a **uniform base** or a **point-regular base** if for every point $x \in X$ and every neighbourhood U of x the set

$$\{B \in \mathcal{B}: x \in B,\ B \cap (X - U) \neq \emptyset\}$$

is finite. It is easy to check that a base $\mathcal{B}$ for a topological space X is uniform if and only if for every $x \in X$, every family of countably infinite members of $\mathcal{B}$ containing x is a neighbourhood base at the point x. The notion of a uniform base is due to Alexandroff [1] and he proved in the paper that a topological space is ***metrizable*** if and only if it is ***collectionwise normal*** and has a uniform base. Topological spaces having a uniform base are closely related to developable spaces. Heath proved in [14] that a ***T_2-space*** has a uniform base if and only if it is ***metacompact*** and developable. Topological spaces having a uniform base were characterized as the images of ***metric spaces*** under ***continuous compact open maps*** [6].

Arhangel'skiĭ introduced in [4] a stronger version of a uniform base. A base $\mathcal{B}$ for a topological space X is called a **regular base** if for every point $x \in X$ and every neighbourhood U of x there exists a neighbourhood V of x such that $V \subset U$ and the set

$$\{B \in \mathcal{B}: B \cap V \neq \emptyset \text{ and } B \cap (X - U) \neq \emptyset\}$$

is finite. He proved in that paper that a topological space is metrizable if and only if it is a ***T_1-space*** and has a regular base.

A base $\mathcal{B}$ for a topological space X is called a **base of countable order** if for every point $x \in X$ and any decreasing family $\{B_n: n \in \omega\}$ of distinct members of $\mathcal{B}$ containing x the family is a neighbourhood base at the point x. Obviously a uniform base is a base of countable order. This notion was introduced by Arhangel'skiĭ in [5]. He proved in [5] that every ***regular*** developable space has a base of countable order, and that a ***paracompact*** T_2-space with a base of countable order is metrizable. Worrell and Wicke gave a characterization of developable spaces in terms of bases of countable order. Indeed they proved in [26] that a T_1-space is developable if and only if it is ***submetacompact*** and has a base of countable order. Hence, by Bing's metrization theorem [E, 5.4.1], a collectionwise normal space is metrizable if and only if it is submetacompact and has a base of countable order [26].

For a family $\mathcal{A}$ of subsets of a set X and $x \in X$ we put $\operatorname{ord}(x, \mathcal{A}) = |\{A \in \mathcal{A}: x \in A\}|$. A family $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ of open subsets of a topological space X is called a **θ-base** (**$\delta\theta$-base** respectively) if for every point $x \in X$ and any neighbourhood U of x there exist $n \in \omega$ and $B \in \mathcal{B}_n$ such that $x \in B \subset U$ and $1 \leqslant \operatorname{ord}(x, \mathcal{B}_n) < \omega$ ($1 \leqslant \operatorname{ord}(x, \mathcal{B}_n) \leqslant \omega$ respectively). Every σ-point finite base is a θ-base. A $\delta\theta$-base is a common generalization of a θ-base and a point-countable base. The notion of a θ-base is due to Worrell and Wicke [26] and they proved in this paper that a topological space X is developable if and only if X has a θ-base and every closed subset of X is a G_δ-set. Hence, by Bing's metrization theorem [E, 5.4.1], a collectionwise normal space is metrizable if and only if it has a θ-base and every closed subset is a G_δ-set [26]. Later Bennett and Lutzer proved in [8] that a regular space has a θ-base if and only if it is ***quasi-developable***. The notion of a $\delta\theta$-base was introduced by Aull [7]. Aull proved in [7] that a topological space has a θ-base if and only if it is a weak σ-space and has a $\delta\theta$-base, where a topological space X is called a **weak σ-space** if it has a σ-disjoint network such that each disjoint family is discrete in its union. Chaber proved in [9] that a submetacompact ***β-space*** with a $\delta\theta$-base is developable. In particular, by Bing's metrization theorem [E, 5.4.1], every ***compact*** T_2-space with a $\delta\theta$-base is metrizable .

Point-countable bases were introduced by Alexandroff and Urysohn [2]. They proved in [2] that a ***locally separable*** space with a point-countable base is metrizable. Ponomarev proved in [21] that a T_1-space is an image of a metric space by a continuous open ***s-map*** if and only if it has a point-countable base. Miščenko proved in [17] that every ***countably compact*** T_1-space with a point-countable base is metrizable.

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a-2 Modified Open and Closed Sets (Semi-Open Set etc.)

1. θ-closed and δ-closed sets

A subset A of a ***topological space*** (X, τ) is called **regular open** if $A = \mathrm{Int}(\mathrm{Cl}(A))$. The complement of a regular open set is called **regular closed**. A point $x \in X$ is called a **θ-cluster** (**δ-cluster**) point of A if $A \cap \mathrm{Cl}(U) \neq \emptyset$ ($A \cap \mathrm{Int}(\mathrm{Cl}(U)) \neq \emptyset$) for every ***open set*** U of X containing x. The set of all θ-cluster (δ-cluster) points of A is called the **θ-closure** (**δ-closure**) and is denoted by $\mathrm{Cl}_\theta(A)$ ($\mathrm{Cl}_\delta(A)$). A subset A is called **θ-closed** (**δ-closed**) [14] if $\mathrm{Cl}_\theta(A) = A$ ($\mathrm{Cl}_\delta(A) = A$). The complement of a θ-closed (δ-closed) set is called **θ-open** (**δ-open**). The family of regular open sets of (X, τ) is not a ***topology***. But it is a ***base*** for a topology τ_s called the **semiregularization** of τ. If $\tau_s = \tau$, then (X, τ) is called **semiregular**. For any subset $S \subset X$, $\mathrm{Cl}(S) \subset \mathrm{Cl}_\delta(S) \subset \mathrm{Cl}_\theta(S)$ and for any $U \in \tau$ $\mathrm{Cl}(U) = \mathrm{Cl}_\delta(U) = \mathrm{Cl}_\theta(U)$ [14].

The following are equivalent: (a) (X, τ) is ***Hausdorff***; (b) for any distinct points $x, y \in X$ $\mathrm{Cl}_\theta(\{x\}) \cap \mathrm{Cl}_\theta(\{y\}) = \emptyset$; (c) every ***compact*** set of X is θ-closed. A space (X, τ) is called **almost regular** if for any regular closed set F and a point $x \notin F$, there exist disjoint $U, V \in \tau$ such that $F \subset U$ and $x \in V$. The following are equivalent: (a) (X, τ) is ***regular***; (b) $\mathrm{Cl}_\theta(A) = \mathrm{Cl}(A)$ for every subset $A \subset X$; (c) (X, τ) is almost-regular and semiregular. The following are equivalent: (a) (X, τ) is almost-regular; (b) $\mathrm{Cl}_\theta(A) = \mathrm{Cl}_\delta(A)$ for every subset $A \subset X$; (c) (X, τ_s) is regular; (d) $\mathrm{Cl}_\theta(A) = A$ for every regular closed set A of X.

2. Semi-open sets

A subset A of (X, τ) is **semi-open** [9] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. The family $\mathrm{SO}(X, \tau)$ of semi-open sets of (X, τ) is not a topology. The complement of a semi-open set is called **semi-closed**. A semi-open and semi-closed set is called **semiregular** [4] or **regular semi-open**. The **semi-closure** $\mathrm{sCl}(A)$ of A is defined as

$$\mathrm{sCl}(A) = \bigcap \bigl\{F\colon A \subset F,\ X \setminus F \in \mathrm{SO}(X, \tau)\bigr\}.$$

If $U \in \mathrm{SO}(X, \tau)$, then $\mathrm{sCl}(U)$ is semiregular. A subset A of (X, τ) is called **semi-θ-open** if for each $x \in A$ there exists $U \in \mathrm{SO}(X, \tau)$ such that $x \in U \subset \mathrm{sCl}(U) \subset A$. By using semi-open sets, many separation axioms, covering properties and functions have been introduced and investigated. For example, we can mention semi-Hausdorff, semiregular, **s-compact** (every semi-open ***cover*** has a finite ***subcover***), semi-continuous, irresolute, etc. A function $f: X \to Y$ is called **semi-continuous** or **quasi-continuous** (respectively **irresolute**) if $f^{-1}(V)$ is semi-open in X for every open (respectively semi-open) set V of Y.

A space (X, τ) is called **S-closed** [13] (**s-closed** [4]) if for every cover $\{V_\alpha\colon \alpha \in \Delta\}$ of X by semi-open sets there exists a finite subset Δ_0 of Δ such that $X = \bigcup_{\alpha \in \Delta_0} \mathrm{Cl}(V_\alpha)$ ($X = \bigcup_{\alpha \in \Delta_0} \mathrm{sCl}(V_\alpha)$). Every s-closed space is S-closed and every S-closed space is **quasi H-closed** (for every ***open cover*** $\{V_\alpha\colon \alpha \in \Delta\}$ of X there exists a finite subset Δ_0 of Δ such that $X = \bigcup_{\alpha \in \Delta_0} \mathrm{Cl}(V_\alpha)$). For a Hausdorff space (X, τ), the following are equivalent: (a) (X, τ) is s-closed; (b) (X, τ) is S-closed; (c) (X, τ) is quasi H-closed ***extremally disconnected***. For a space (X, τ), the following are equivalent: (a) (X, τ) is S-closed; (b) every regular closed cover of X has a finite subcover; (c) every proper regular open set of (X, τ) is an S-closed subspace. Compactness and S-closedness are independent of each other. S-closedness does not behave like compactness. The following are typical properties: a ***closed set*** of an S-closed space is not always S-closed; the ***product space*** of two S-closed spaces need not be S-closed; S-closedness need not be preserved by ***continuous*** surjections; the inverse images of S-closed spaces under ***perfect maps*** are not necessarily S-closed.

For a space (X, τ), the following are equivalent: (a) (X, τ) is s-closed; (b) every cover of X by semiregular sets has a finite subcover; (c) every cover of X by semi-θ-open sets has a finite subcover. Moreover, s-closedness has the following properties: s-closedness is preserved by ***open*** continuous surjections and if the product space $\prod_{\alpha \in \Delta} X_\alpha$ is s-closed then each space X_α is s-closed but the converse is not true.

3. Preopen sets

A subset A of (X, τ) is called **preopen** [11], **locally dense** or **nearly open** if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$. The complement of a preopen set is called **preclosed**. The family $\mathrm{PO}(X, \tau)$ of preopen sets of (X, τ) is not a topology. For a subset A of (X, τ), the following are equivalent: (a) $A \in \mathrm{PO}(X, \tau)$; (b) A is the intersection of an open set and a ***dense set***; (c) A is a dense subset of some open subspace; (d) $\mathrm{sCl}(A) = \mathrm{Int}(\mathrm{Cl}(A))$. Here are basic properties of preopen sets: (1) every singleton is either preopen or ***nowhere dense***; (2) the arbitrary union of preopen sets is preopen; (3) the finite intersection of preopen sets need not be preopen; (4) preopenness and semi-opennes are independent of each other; (5) a subset A is regular open if and only if it is semi-closed and preopen.

Extremally disconnected spaces are characterized by preopen and semi-open sets. For a space (X, τ), the following are equivalent: (a) (X, τ) is extremally disconnected; (b) every regular closed subset of X is preopen; (c) every semi-open set of X is preopen; (d) the ***closure*** of every preopen set is open.

A space (X, τ) is called **hyperconnected**, **irreducible** or a **D-space** if every nonempty open set of X is dense. For a space (X, τ), the following are equivalent: (a) (X, τ) is hyperconnected; (b) $\mathrm{sCl}(A) = X$ for every nonempty preopen set A; (c) every nonempty preopen set of X is dense; (d) every nonempty semi-open set is dense; (e) X is not expressed as the union of two disjoint nonempty semi-open sets of X.

By using preopen sets, many separation axioms and functions are defined and investigated. However, we deal with only important covering properties: a space (X, τ) is a **strongly compact space** (or a **p-closed space** [7]) if for every cover $\{V_\alpha \colon \alpha \in \Delta\}$ of X by preopen sets there exists a finite subset Δ_0 of Δ such that $X = \bigcup_{\alpha\in\Delta_0} V_\alpha$ ($X = \bigcup_{\alpha\in\Delta_0} \mathrm{pCl}(V_\alpha)$). Every strongly compact space is p-closed and every p-closed space is quasi H-closed. If (X, τ) is p-closed and extremally disconnected, then it is s-closed.

4. α-open sets

A subset A of (X, τ) is called **α-open** [12] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. The family τ^α of α-open sets of (X, τ) is a topology which is finer than τ. The complement of an α-open set is called **α-closed**. For a subset A of (X, τ), the following are equivalent: (a) $A \in \tau^\alpha$; (b) A is semi-open and preopen; (c) $A \cap B \in \mathrm{SO}(X, \tau)$ for every $B \in \mathrm{SO}(X, \tau)$; (d) $A = O \setminus N$, where $O \in \tau$ and N is nowhere dense; (e) there exists $U \in \tau$ such that $U \subset A \subset \mathrm{sCl}(U) = \mathrm{Int}(\mathrm{Cl}(U))$. Here are some fundamental properties of α-open sets: (1) the intersection of semi-open (preopen) set and an α-open set is semi-open (preopen), (2) $\mathrm{SO}(X, \tau) = \mathrm{SO}(X, \tau^\alpha)$ and $\mathrm{PO}(X, \tau) = \mathrm{PO}(X, \tau^\alpha)$, (3) α-open sets A, B are disjoint if and only if $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ and $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))$ are disjoint. As a consequence of (3) we can obtain new characterizations of Hausdorffness, regularity, normality and connectedness by replacing open sets with α-open sets.

5. b-open and β-open sets

A subset A of (X, τ) is called **b-open** [3] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a b-open set is called **b-closed**. The family $\mathrm{BO}(X, \tau)$ of b-open sets of (X, τ) is not a topology. Every semi-open (preopen) set is b-open but the converses are not true. There are some basic properties: (1) the union of any family of b-open sets is b-open, (2) the intersection of a b-open set and an α-open set is b-open, (3) $\mathrm{BO}(X, \tau) = \mathrm{BO}(X, \tau^\alpha)$.

A subset A of (X, τ) is called **β-open** [1] or **semi-preopen** [2] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open (semi-preopen) set is called **β-closed** (**semi-preclosed**). The family $\mathrm{SPO}(X, \tau)$ of semi-preopen sets of (X, τ) is not a topology. For a subset A of (X, τ), the following are equivalent: (a) $A \in \mathrm{SPO}(X, \tau)$; (b) $\mathrm{Cl}(A)$ is regular closed; (c) there exists $U \in \mathrm{PO}(X, \tau)$ such that $U \subset A \subset \mathrm{Cl}(U)$; (d) $\mathrm{sCl}(A) \in \mathrm{SO}(X, \tau)$; (e) $A = S \cap D$ for $S \in \mathrm{SO}(X, \tau)$ and a dense set D. There are some basic properties of semi-preopen sets: (1) the arbitrary union of semi-preopen sets is semi-preopen, (2) $\mathrm{BO}(X, \tau) \subset \mathrm{SPO}(X, \tau)$, (3) $\mathrm{SPO}(X, \tau) = \mathrm{SPO}(X, \tau^\alpha)$, (4) the intersection of an α-open set and a semi-preopen set is semi-preopen.

A space (X, τ) is called **β-compact** if every cover of X by β-open sets has a finite subcover. However, it was shown in [8] that infinite β-compact spaces do not exist.

The following relations hold among modifications of open sets stated above.

$$\begin{array}{ccccccccc}
 & & \text{regular open} & \rightarrow & \text{semiregular} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\theta\text{-open} & \rightarrow & \delta\text{-open} & \rightarrow & \text{semi-open} & \rightarrow & b\text{-open} & \rightarrow & \beta\text{-open} \\
 & & \downarrow & & \uparrow & & \uparrow & & \\
 & & \text{open} & \rightarrow & \alpha\text{-open} & \rightarrow & \text{preopen} & &
\end{array}$$

6. Generalized closed sets

In 1970, Levine [10] introduced the concept of generalized closed sets as a generalization of closed sets in topological spaces. Recently, the study of modifications of generalized closed sets has found considerable interest among general topologists. One of the reasons is that these concepts are quite natural. The following is the original definition: A subset A of (X, τ) is called **generalized closed** (**g-closed**) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$. There are many modifications of g-closed sets.

By replacing $\mathrm{Cl}(A)$ with $\mathrm{Cl}_\delta(A)$ and $\mathrm{Cl}_\theta(A)$, **δ-generalized closed** [5] sets and **θ-generalized closed** [6] sets are obtained. These sets play an important role in study of the digital line. The **digital line** $(\mathbb{Z}, \kappa)$ is the set $\mathbb{Z}$ of all integers equipped with the topology κ generated by $\{\{2n-1, 2n, 2n+1\} \colon n \in \mathbb{Z}\}$. A space (X, τ) is called a **$T_{3/4}$-space** [5] if every δ-generalized closed set of X is δ-closed. A $T_{3/4}$-space places strictly between a ***T_1-space*** and a **$T_{1/2}$-space** (every g-closed set is closed). The Khalimsky line or so called digital line is a $T_{3/4}$-space but it is not T_1.

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a-3 Cardinal Functions, Part I

A function φ which assigns a cardinal $\varphi(X)$ to each ***topological space*** X is called a **cardinal function** or a **cardinal invariant** if it is a topological invariant, i.e., if we have $\varphi(X) = \varphi(Y)$ whenever X and Y are ***homeomorphic***.

Here we assume that the values of cardinal functions are always infinite cardinals. The simplicity of infinite cardinal arithmetic (i.e., $\kappa + \lambda = \kappa \cdot \lambda = \lambda$ whenever κ and λ are infinite cardinal numbers with $\kappa \leqslant \lambda$) helps us to simplify our statements.

We adopt the following set-theoretic notation: **cardinal numbers** are initial ordinals, i.e., κ is a cardinal if and only if it is the smallest ordinal of the cardinality $|\kappa|$. $\kappa, \lambda, \ldots$ always denote cardinal numbers. ω and ω_1 are used to denote the first infinite ordinal (cardinal) and the first uncountable ordinal (cardinal) respectively. So, for each cardinal κ, we have $\kappa + \omega = \kappa$ if κ is infinite, and $\kappa + \omega = \omega$ if κ is finite. To avoid typesetting problems, 2^κ is often denoted by $\exp \kappa$. Thus $\exp \omega = 2^\omega$ is the cardinality $\mathfrak{c}$ of the continuum, in other words, the cardinality of the set of real numbers.

Let (X, τ) be a topological space. Obviously, the **cardinality** $|X|$ of X is the simplest cardinal function. Recall that a subfamily $\mathcal{B}$ of τ is a ***base*** if every member of τ is the union of a subfamily of $\mathcal{B}$. The most important cardinal function is the **weight** $w(X)$ of X, which is defined by $w(X) = \min\{|\mathcal{B}|: \mathcal{B}$ is a base for $X\} + \omega$. Here ω is added to make the value infinite. A space X satisfies the **second axiom of countability**, or is **second-countable** if $w(X) \leqslant \omega$, in other words, if it has a countable base. In the early stage of general topology, first, ***Euclidean spaces*** and its ***subspaces*** were studied. In the following stage, separable ***metrizable spaces*** played an important role. Indeed, for example, Kuratowski mainly concentrated on separable metrizable spaces in his book [4]. Kinds of countability played an important role in the classical theory. For a ***regular T_1-space*** X, the following are equivalent:

(a) X is separable metrizable,
(b) $w(X) = \omega$ (i.e., X is second-countable),
(c) X can be embedded in the ***product*** I^ω of countably many copies of the unit interval $I = [0, 1]$.

More generally, for a ***completely regular*** T_1-space X and an infinite cardinal κ, the following (b′) and (c′) are equivalent: (b′) $w(X) = \kappa$, (c′) X can be embedded in the product I^κ of κ many copies of the unit interval $I = [0, 1]$, which is called **Tychonoff cube** of weight κ. Hence weight is a cardinal function which measures, in some sense, the size of a typical space in which the space can be embedded.

In Cardinal functions Part I, we give a brief introduction to simple and well-known cardinal functions, which are obtained as generalizations of classical countability axioms (for more information see [3], [KV, Chapter 1], [E, Problems, Cardinal functions I, II, III, IV]). The inequalities between the cardinal functions defined here are summarized in [KV, Chapter 1, §3, Figure 1].

Now we turn our attention to other cardinal invariants. For a metrizable space X, the separability of X can be characterized by some other cardinal functions. Let X be a topological space. The **Lindelöf degree** (**Lindelöf number**) $L(X)$ (sometimes denoted by $l(X)$) of X is defined by $L(X) = \min\{\kappa$: every ***open cover*** of X has a ***subcover*** of cardinality $\leqslant \kappa\} + \omega$. If $L(X) = \omega$, i.e., every open cover has a countable subcover, we say that X is ***Lindelöf***. Define the **density** $d(X)$ of X by $d(X) = \min\{|D|$: D is a ***dense*** subset of $X\}$. If $d(X) = \omega$, we say that X is a **separable space**. Define the **cellularity** (**Souslin number**) $c(X)$ of X by $c(X) = \min\{\kappa$: every family of pairwise disjoint nonempty ***open*** subsets of X has cardinality $\leqslant \kappa\} + \omega$. If $c(X) = \omega$, we say that X has the **countable chain condition** (**Souslin property**). Countable chain condition is abbreviated as **ccc**. The **spread** $s(X)$ and the **extent** $e(X)$ are defined as follows: $s(X) = \sup\{|D|$: D is a ***discrete*** subset of $X\} + \omega$, and $e(X) = \sup\{|D|$: D is a discrete ***closed*** subset of $X\} + \omega$.

For a metrizable space X, we have

$$w(X) = L(X) = d(X) = c(X) = s(X) = e(X).$$

For a ***completely metrizable*** space X, we have either $|X| = w(X)$ or $|X| = w(X)^\omega$ (all these results can be found in [KV, Chapter 1, §8]).

Those cardinal functions are not equal for general topological spaces. However there are some relations between them. For example, we have the following inequalities: $L(X) \leqslant w(X)$, $c(X) \leqslant d(X) \leqslant w(X)$, and $d(X) \leqslant |X|$.

We give here some examples showing the difference between such cardinal functions. The **Sorgenfrey line** S is defined to be the real line $\mathbb{R}$ as a set. The base for S is a family consisting of all half open intervals of the form $[p, q)$ with $p, q \in \mathbb{R}$ and $p < q$. S satisfies $L(S) = \omega < 2^\omega = w(S)$. Let D be an uncountable ***discrete space*** and X be the ***one-point compactification*** of D. Then $L(X) = \omega < |D| = c(X)$. Let

$$Y = I^{\exp\exp\omega} \quad (\text{or } \{0, 1\}^{\exp\exp\omega})$$

be the ***Tychonoff product*** of $\exp\exp\omega$ many copies of the unit interval $I = [0, 1]$ (or the two point discrete space $\{0, 1\}$), then $c(Y) = L(Y) = \omega$, $d(Y) = \exp\omega$ and $w(Y) = \exp\exp\omega$.

The cardinal functions defined above are called **global cardinal functions**. Now we introduce **local cardinal functions**. Let X be a topological space and $x \in X$. Recall that a family $\mathcal{U}$ of neighbourhoods of x is called a **neighbourhood base** of x in X if for every neighbourhood V of x, there is $U \in \mathcal{U}$ with $U \subset V$. The **character** $\chi(x, X)$ of x in X is defined by $\chi(x, X) = \min\{|\mathcal{U}|: \mathcal{U}$ is a neighbourhood base of x in $X\} + \omega$. Furthermore the **character** $\chi(X)$ of X is defined by $\chi(X) = \sup\{\chi(x, X): x \in X\}$. In other words, $\chi(X) = \min\{\kappa: \kappa \geqslant \omega$, any point $x \in X$ has a neighbourhood base of cardinality $\leqslant \kappa\}$. A space X satisfies the **first axiom of countability**, or is **first-countable** if $\chi(X) \leqslant \omega$, i.e., if every point of X has a countable neighbourhood base. The **tightness** $t(x, X)$ of x in X is defined by $t(x, X) = \min\{\kappa: \kappa \geqslant \omega$, for any set $A \subset X$ with $x \in \operatorname{cl} A$, there is a subset B of A with $|B| \leqslant \kappa$ and $x \in \operatorname{cl} B\}$, and define the **tightness** $t(X)$ of X by $t(X) = \sup\{t(x, X): x \in X\}$. In other words, $t(X) = \min\{\kappa: \kappa \geqslant \omega$, for any point $x \in X$ and any subset A of X with $x \in \operatorname{cl} A$, there is a subset B of A of cardinality $\leqslant \kappa$ satisfying $x \in \operatorname{cl} B\}$. Every metrizable space satisfies the first axiom of countability, and every first-countable space has countable tightness.

For ***compact*** spaces, some cardinal functions can be characterized by the existence of a family with some weaker condition. A family $\mathcal{N}$ of subsets of a space (X, τ) is called a **network** if every member of τ is a union of a subfamily of $\mathcal{N}$, i.e., if for any $V \in \tau$ and any $x \in V$, there is $A \in \mathcal{N}$ with $x \in A \subset V$. Hence, a family $\mathcal{N}$ of subsets of X is a base for X if and only if it is a network for X consisting of open sets. Define the **network weight** $nw(X)$ of X by $nw(X) = \min\{|\mathcal{N}|: \mathcal{N}$ is a network for $X\} + \omega$. It is interesting to see that $w(X) = nw(X)$ for any compact space X. Since the family $\{\{x\}: x \in X\}$ is a network for any space X, we have $w(X) = nw(X) \leqslant |X|$ for any compact space X. Note that the equality $w(X) = nw(X)$ also holds for any metrizable space X. Define the **pseudocharacter** $\psi(x, X)$ of x in X by $\psi(x, X) = \min\{|\mathcal{U}|: \mathcal{U}$ is a family of neighbourhoods of x with $\{x\} = \bigcap \mathcal{U}\} + \omega$. The **pseudocharacter** of X is defined by $\psi(X) = \sup\{\psi(x, X): x \in X\}$. Then, by the definition of compactness, it is easy to see that $\chi(X) = \psi(X)$ for any compact T_2-space X.

Some cardinal functions are bounded by some powers containing other cardinal functions. For example, we have $|X| \leqslant \exp w(X)$ for any T_1-space X. This is because each point can be decided by the family of all open sets containing the point.

It can be shown that $w(X) \leqslant \exp d(X)$ for any regular space X. Indeed, let $\mathcal{B}$ be a base and D a dense set of a regular space X. Then $\{\operatorname{int} \operatorname{cl} B: B \in \mathcal{B}\}$ is a base consisting of regular open sets, where a **regular open** set is an open set U with $U = \operatorname{int} \operatorname{cl} U$. Since each regular open set can be decided by its intersection with D, we have the inequality.

Some global cardinal functions are bounded by some combinations of global functions and local functions. For example, we have $|X| \leqslant d(X)^{\chi(X)}$ for any T_2-***space*** X. In 1969, Arkhangel'skiĭ proved a highly nontrivial result that the cardinality of any compact first-countable T_2-space is $\leqslant 2^{\omega}$, answering Alexandroff and Urysohn's problem that had been unanswered for about thirty years. More generally, it can be shown that

$$|X| \leqslant \exp\bigl(L(X) \cdot \psi(X) \cdot t(X)\bigr)$$

for any T_2-space X. His original proof is difficult to understand. We give here the idea of a simplified proof due to R. Pol. Let X be a Lindelöf first-countable T_2-space. We show that $|X| \leqslant 2^{\omega}$. Let $\mathcal{U}_x$ be a countable neighbourhood base for each $x \in X$, and

$$\mathcal{A} = \bigl\{A: A \text{ is a subset of } X \text{ with } |A| \leqslant 2^{\omega}\bigr\}.$$

Note that $\operatorname{cl} A \in \mathcal{A}$ for any $A \in \mathcal{A}$, because each point $x \in \operatorname{cl} A$ is decided by a ***sequence*** in A ***converging*** to x and there are $\leqslant 2^{\omega}$ many sequences in A. Consider an operation Φ which assigns to each closed set $A \in \mathcal{A}$, an element $\Phi(\mathcal{A}) \in \mathcal{A}$ with the following property: $A \subset \Phi(A)$; and for any countable subfamily $\mathcal{V}$ of $\bigcup_{x \in A} \mathcal{U}_x$ of X with $A \subset \bigcup \mathcal{V}$ and $X - \bigcup \mathcal{V} \neq \emptyset$, we have $\Phi(A) - \bigcup \mathcal{V} \neq \emptyset$ (the Lindelöf property is used to show the existence of such $\mathcal{V}$ whenever $X - A \neq \emptyset$). An operation Φ exists because there are only $\leqslant 2^{\omega}$ many choices of such $\mathcal{V}$ and so we can pick a point $x_{\mathcal{V}}$ from $X - \bigcup \mathcal{V}$ for such $\mathcal{V}$. Let $\Phi(A)$ be the union of A and the set of such $x_{\mathcal{V}}$'s. Now take any point $x_0 \in X$ and define $A_0 = \{x_0\}$. By using transfinite induction, for each $\alpha < \omega_1$, define $A_\alpha = \operatorname{cl}(\Phi(\operatorname{cl}(\bigcup_{\beta<\alpha} A_\beta)))$. Then we can show that $A_\alpha \in \mathcal{A}$ and $X = \bigcup_{\alpha<\omega_1} A_\alpha$, which implies $|X| \leqslant 2^{\omega}$.

The argument above is called "a **closing-off argument**" or "a **closure argument**". Here, the term "closure" does not mean the topological ***closure*** but closure under a family of operations. A general way of simplifying closing-off arguments is via "***elementary submodels*** of the universe". For example, to prove Arkhangel'skiĭ's inequality by using an elementary submodel, just take one, M say, of cardinality 2^{ω} such that $X \in M$ and $A \in M$ for any countable $A \subset M$. Here we need not mention the operation Φ above. To complete the proof, what we should do is show that $X \subset M$ (see [2]).

Now let us consider the behavior of cardinal functions under topological operations. It is interesting for us to observe that each cardinal function behaves quite differently.

(1) continuous images: Let $\varphi(X)$ be a cardinal function and $f: X \to Y$ be a ***continuous map*** onto Y. In this case, we say that φ is **preserved by the map** f if $\varphi(Y) \leqslant \varphi(X)$. It is easy to see that the Lindelöf number L, density d and cellularity c are preserved under any continuous map. If X is compact, then weight w is preserved by any continuous map $f: X \to Y$ onto Y. Indeed, since X and Y are compact, we have $w(X) = nw(X)$ and $w(Y) = nw(Y)$. It is easy to check that network weight is preserved by any continuous map.

However, generally speaking, weight w is not preserved under continuous maps. For example, let $X = \mathbb{R}^2$ and let Y be the space obtained from X by collapsing the x-axis to one point denoted by ∞. Let $f: X \to Y$ be the ***quotient map***. Then X is second-countable and f is a ***closed map***, but Y is not second-countable, because Y is not first-countable at ∞. At early stage of general topology, spaces satisfying some axiom of countability were studied. However uncountability naturally appears in such a situation. An easy way to describe this phenomenon is to consider sequential fans. The **sequential fan** $S(\kappa)$ with κ spines is defined by the space obtained from the ***topological sum*** $T(\kappa)$ of κ many copies of the ***convergent sequence*** by identifying all the ***limit point***s to a point denoted by ∞. Let $f : T(\kappa) \to S(\kappa)$ be the quotient map. If $\kappa = \omega$, then f is a closed map from a second-countable (hence, a separable metrizable) space $T(\omega)$ onto a space $S(\omega)$. But $S(\omega)$ is not first-countable at ∞. The character $\chi(\infty, S(\omega))$ is equal to the cardinal denoted by $\mathfrak{d}$, and it can be shown that we cannot prove $\mathfrak{d} = 2^\omega$ or $\mathfrak{d} < 2^\omega$ by using our usual mathematics (i.e., ZFC set theory) (see [KV, Chapter 3]).

(2) subspaces: A cardinal function φ is a **monotone cardinal function** if $\varphi(Y) \leqslant \varphi(X)$ for any subspace Y of X. Weight w and character χ are monotone. But Lindelöf number L, density d and cellularity c are not monotone. For example, let D be an uncountable discrete space and X a one-point compactification of D. Then $L(D) > L(X)$. Let Y be the ***Niemytzki Plane*** and Z the x-axis of the plane. Then Z is a closed discrete set of Y and

$$d(Z) = c(Z) = 2^\omega > \omega = d(Y) = c(Y).$$

For each cardinal function φ which is not monotone, define a new cardinal function $h\varphi$ by $h\varphi = \sup\{\varphi(Y): Y \subset X\}$. Note that $hc(X) = he(X) = s(X)$ for any space X. $hL(X)$ and $hd(X)$ will be discussed in Cardinal functions Part II.

(3) products: Let φ be a cardinal function and κ, λ cardinals. The property $\varphi \leqslant \kappa$ is said to be a **λ-multiplicative property** if we have $\varphi(\prod_{\alpha<\lambda} X_\alpha) \leqslant \kappa$ whenever $\varphi(X_\alpha) \leqslant \kappa$ for each $\alpha < \lambda$. Likewise $\varphi \leqslant \kappa$ is said to be a **multiplicative property** if we have $\varphi(\prod_{\alpha<\lambda} X_\alpha) \leqslant \kappa$ whenever λ is any cardinal and $\varphi(X_\alpha) \leqslant \kappa$ for each $\alpha < \lambda$. It is easy to see that $w \leqslant \kappa$ is κ multiplicative.

While compactness is multiplicative, the Lindelöf property is not multiplicative. The Sorgenfrey line S is Lindelöf, but the square $S \times S$ contains a closed discrete subset $\{(p, -p): p \in S\}$ of cardinality 2^ω, which implies that $S \times S$ is not Lindelöf.

Next we consider density. For example, separability is $\mathfrak{c}$ ($= 2^\omega$) multiplicative but not $\mathfrak{c}^+$ multiplicative. More generally, $d \leqslant \kappa$ is 2^κ multiplicative (see [E, 2.3.15]), but not $(2^\kappa)^+$ multiplicative because of the inequality $w(X) \leqslant \exp d(X)$.

Surprisingly, the multiplicativity of cellularity depends on your set theory. It is known that a product has the countable chain condition if and only if any finite subproduct has the countable chain condition. So the point is whether cellularity is finitely multiplicative or not. Under ***Martin's Axiom*** (more precisely $\mathsf{MA}(\aleph_1)$), any Tyhonoff product of spaces with the countable chain condition has the countable chain condition (see [Ku, Chapter 2, 2.24]). On the other hand, assume the negation of the **Souslin hypothesis**, i.e., there is a **Souslin tree**, which is an ***ω_1-tree*** with no uncountable ***antichain***s and no uncountable ***branch***es. For example, if we assume $\diamondsuit$, then there is such a ***tree***. The set theoretical axiom $\diamondsuit$ holds under the assumption $V = L$ (i.e., every set is constructible). The negation of the Souslin hypothesis is equivalent to the existence of a **Souslin line** L, that is a non-separable ***linearly ordered topological space*** with the countable chain condition. The square $L \times L$ of a Souslin line does not satisfies the countable chain condition (see [Ku, Chapter 2, 4.3]). For more results on chain conditions, see [6].

(4) ***compactifications***: For any compactification Y of X, we have $|Y| \leqslant \exp\exp d(X)$ and $w(Y) \leqslant \exp d(X)$. For example, let $D(\kappa)$ be the infinite discrete space of cardinality κ, and $\beta D(\kappa)$ be the ***Čech–Stone compactification*** of $D(\kappa)$. Then $|\beta D(\kappa)| = \exp\exp\kappa$ and $w(\beta D(\kappa)) = \exp\kappa$ (see [E, 3.5, 3.6]).

(5) **function spaces** (see [5, Chapter 4], [1]): We mention here only some typical results. All spaces here are assumed to be completely regular T_1-spaces. Let $C_k(X, Y)$ (respectively $C_p(X, Y)$) be the space of continuous functions from X to Y with the ***compact-open topology*** (respectively the ***topology of pointwise convergence***). $C_k(X, R)$ (respectively $C_p(X, R)$) is sometimes denoted by $C_k(X)$ (respectively $C_p(X)$), while $C_u(X)$ is the space of continuous real functions on X with the ***topology of uniform convergence***. Then $|X| = \chi(C_p(X)) = w(C_p(X))$ for any space X with $|X| \geqslant \omega$, and

$$nw\big(C_p(X, Y)\big) \leqslant nw\big(C_k(X, Y)\big) \leqslant w(X) \cdot w(Y).$$

There are beautiful relations between tightness and Lindelöf number as follows: For any space X and any $n \in N$, we have $t(X^n) \leqslant L(C_p(X))$ for any $n \in N$. Furthermore $t(C_p(X)) \leqslant \kappa$ if and only if $L(X^n) \leqslant \kappa$ for any $n \in N$. A kind of duality between ***hereditary density*** and ***hereditary Lindelöf degree*** holds:

$$\sup\{hd(X^n): n \in N\} = \sup\{hL\big((C_p(X))^n\big): n \in N\},$$

and

$$\sup\{hL(X^n): n \in N\} = \sup\{hd\big((C_p(X))^n\big): n \in N\}.$$

We have already mentioned the relations between cardinal functions for metrizable spaces and for compact spaces. Now we consider the relations between cardinal functions for other classes of spaces.

(1) ***LOTS***: For any linearly ordered topological space (LOTS) X, we have $\chi(X) = \psi(X) \leqslant c(X) \leqslant d(X) \leqslant w(X) = nw(X) \leqslant |X|$ (see [E, 3.12.4]).

(2) ***Topological group***s (see [KV, Chapter 24, §3]): For a topological group G, $\chi(G) = \omega$ holds if and only if G is metrizable. For a topological group G, we have $w(G) = d(G) \cdot \chi(G)$. For every compact infinite topological group G with $w(G) = \kappa$, we have $|G| = \exp\kappa$, $d(G) = \log\kappa$, and $c(G) = \omega$.

(3) Dyadic spaces (see [E, 3.12.12], [KV, Chapter 24, §1]): A compact space X is called a **dyadic space** if it is a continuous image of the ***Cantor cube*** D^κ for some infinite κ, where $D = \{0, 1\}$ with the ***discrete topology***. It is

known that compact topological groups are dyadic. For any dyadic space X, we have $c(X) = \omega$ and surprisingly, we have $w(X) = \chi(X) = t(X)$.

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a-4 Cardinal Functions, Part II

Here in Cardinal functions II, several topics on ***cardinal function***s are introduced. Some of them are rather set theoretical, where set theoretical notions, for example, CH (the ***Continuum Hypothesis***), MA (***Martin's Axiom***), large cardinals, the forcing method, $V = L$ etc, play important roles. However we do not assume that readers are familiar with such notions. The set theoretical background can be found in [Ku] and [7].

1. π-weight, π-character

(See [KV, Chapter 1] and [1].) The cardinal function π-weight is interesting not only from the topological point of view, but also from the Boolean algebraic point of view. A family $\mathcal{U}$ of non-empty ***open sets*** of a space X is called a **π-base** if for any non-empty open set V of X, there is $U \in \mathcal{U}$ with $V \subset U$. Define the **π-weight** $\pi w(X)$ of X by

$$\pi w(X) = \min\{|\mathcal{U}|: \mathcal{U} \text{ is a } \pi\text{-base for } X\} + \omega.$$

In terms of the theory of Boolean algebras, π-weight is "the density" (in the sense of Boolean algebras) of the Boolean algebra $RO(X)$ of all ***regular open*** sets of X. In such a way, results on topologies can be translated to results on Boolean algebras and vice versa (see [11, 12]).

A family $\mathcal{U}$ of non-empty open sets of X is called a **local π-base** of x in X if for every ***neighbourhood*** V of x, there is $U \in \mathcal{U}$ with $U \subset V$. Note that elements of a local π-base of x need not contain x.

The **π-character** $\pi\chi(x, X)$ of x in X is defined by

$$\pi\chi(x, X) = \min\{|\mathcal{U}|: \mathcal{U} \text{ is a local } \pi\text{-base of } x \text{ in } X\} + \omega.$$

Furthermore the **π-character** $\pi\chi(X)$ of X is defined by $\pi\chi(X) = \sup\{\pi\chi(x, X): x \in X\}$.

We have $d(X) \leqslant \pi w(X) \leqslant w(X)$ for any ***topological space*** X.

Unlike ***weight*** and ***character***, π-weight and π-character are not ***monotone***. For example, while $\pi w(\beta\omega) = \pi\chi(\beta\omega) = \omega$, we have $\pi w(\beta\omega - \omega) \geqslant \pi\chi(\beta\omega - \omega) \geqslant \mathrm{cf}(\mathfrak{c}) > \omega$, where $\beta\omega$ is the ***Čech–Stone compactification*** of the countable ***discrete space*** ω and $\mathrm{cf}(\mathfrak{c})$ is the cofinality of the cardinal $\mathfrak{c}$ (see [KV, Chapter 11, 4.4.3]).

For any ***topological group*** G, we have $\pi w(G) = w(G)$ and $\pi\chi(G) = \chi(G)$ see [KV, Chapter 24, §3]).

2. Tightness

The **tightness** $t(X)$ of a space X is defined as follows: For any $x \in X$, define $t(x, X) = \min\{\kappa: \kappa \geqslant \omega$, for any set $A \subset X$ with $x \in \mathrm{cl}\, A$, there is a subset B of A with $|B| \leqslant \kappa$ and $x \in \mathrm{cl}\, B\}$, and define $t(X) = \sup\{t(x, X): x \in X\}$. Let α be an ordinal. A ***sequence*** of points of a space X is called a **free sequence** of length α in X if for each $\beta < \alpha$ $(\mathrm{cl}\{x_\gamma: \gamma < \beta\}) \cap (\mathrm{cl}\{x_\gamma: \gamma \geqslant \beta\}) = \emptyset$. Define $F(X) = \sup\{\kappa: \kappa$ is a cardinal and there is a free sequence of length κ in $X\} + \omega$.

Arkhangel'skiĭ proved that $t(X) = F(X) \leqslant s(X)$ for any ***compact*** space X, and Šhapirovskiĭ proved that $t(X) = h\pi\chi(X)$ for any compact space X (see [KV, Chapter 1, §7] and [1]).

Now we consider the tightness of ***products*** of spaces. Tightness behaves well for compact spaces. Indeed, if $\{X_i: i < n\}$ is a finite family of compact spaces with $t(X_i) \leqslant \kappa$ for each $i < n$, then $t(\prod_{i<n} X_i) \leqslant \kappa$.

But in general, tightness behaves quite badly. For example, let $S(\kappa)$ be the ***sequential fan*** with κ spines. It is easy to check that $S(\kappa)$ has countable tightness. Indeed it has a stronger property called Fréchet. A space X is a **Fréchet space** if for any $x \in X$ and $A \subset X$ with $x \in \mathrm{cl}\, A$, there is a sequence $\{x_n: n \in N\}$ ***converging*** to x. However $t(S(\omega) \times S(2^\omega))$ is uncountable. It is known that

$$t\big(S(\omega) \times S(\omega_1)\big) = \omega_1$$

if and only if the set theoretical assumption $\mathfrak{b} = \omega_1$ holds. Thus $t(S(\omega) \times S(\omega_1))$ depends on your set theory. In general, it is difficult to decide the tightness of finite products of sequential fans (cf. [4]).

3. Hereditary Lindelöf degree vs. hereditary density

For a cardinal function φ, we define the corresponding **hereditary cardinal function** $h\varphi$ by

$$h\varphi = \sup\{\varphi(Y): Y \subset X\}.$$

For example, we have **hereditary Lindelöf degree** hL and **hereditary density** hd. It is natural to ask whether $hL = hd$ or not. This is not true in general. Indeed, there are spaces X and Y satisfying $\omega < hL(X) < hd(X)$ and $\omega < hd(Y) < hL(Y)$ (see [14, 0.5]).

The problem of when hL or hd is countable is called the S and L problem and it has turned out to be quite difficult (see [KV, Chapter 7], [13]).

An **S-space** is a ***regular*** T_1 ***hereditarily separable*** space which is not ***Lindelöf*** (or, not ***hereditarily Lindelöf***). An **L-space** is a regular T_1 hereditarily Lindelöf space which is not ***separable*** (or, not hereditarily separable). Note that, for example, the existence of a regular T_1 hereditarily separable space which is not hereditarily Lindelöf is equivalent to the existence of a regular T_1 hereditarily separable space which is not Lindelöf.

A space X is called **right separated** (respectively **left separated**) if there is a well ordering $<$ of X such that $\{y \in X\colon y < x\}$ is open (respectively ***closed***) for any $x \in X$.

A ***linearly ordered topological space*** X is called a **Souslin line** if it has the ***countable chain condition*** (***ccc***), i.e., $c(X) \leqslant \omega$, but it is not separable. It is known that $c(X) = hc(X) = hL(X)$ for any linearly ordered topological space. Hence every Souslin line is an L-space. It is known that the **Souslin hypothesis** (= the hypothesis that there are no Souslin trees) is equivalent to the hypothesis that there are no linearly ordered L-spaces. Under the existence of the Souslin line, Rudin constructed an S-space. Hajnal and Juhász constructed an L-space under CH (the Continuum Hypothesis) (see [KV, Chapter 7]).

Todorčević proved that under PFA (the ***Proper Forcing Axiom***), there are no S-spaces. In other words, every regular T_1 hereditarily separable space is Lindelöf. It is still unknown whether "there are no L-spaces" is consistent (i.e., cannot be disproved by our usual axioms of ZFC) or not. It had been believed for a long time that the S-space problem and the L-space problem are the same problem, i.e., if there is an S-space in some model of set theory, then there is an L-space in the same model and vice versa. But this is not true. Indeed, Todorčević showed that there is a model of set theory with Martin's Axiom where there is an L-space but there are no S-spaces (see [14]).

Martin's Axiom and "the S and L problem" can be regarded as a kind of partition problem (see [14], [2]). A basic form of a partition problem is the following Ramsey's theorem: If there are six people, then either three of them mutually know each other or three of them mutually do not know each other.

For any set X, define $[X]^2 = \{\{a, b\}\colon a, b \in X,\ a \neq b\}$. Let $6 = \{0, 1, 2, 3, 4, 5\}$. The following is a mathematical expression of Ramsey's theorem above: For any partition $K_0 \cup K_1 \subset [6]^2$, there is a subset A of 6 with cardinality 3 such that either $[A]^2 \subset K_0$ or $[A]^2 \subset K_1$. It is interesting to see that such a way of thinking is essential to our set theory and topology (for the theory of partition relations, see [5], or Appendix 4 in [8]).

4. Reflection theorems

(See [6] and [3].) Generally speaking, a **reflection theorem** is a theorem of the form "if a set (or a space) X has a property P, then there is a subset (or a ***subspace***) Y of a small size (in some sense) satisfying P". Reflection principle plays an important key role in set theory.

In topology, Hajnal–Juhász proved the following beautiful reflection theorem: If X does not have a countable ***base***, then there is a subspace of X of cardinality $\leqslant \omega_1$ which does not have a countable base. To state a more general result due to them, we need a definition. Let φ be a cardinal function and $\mathcal{C}$ a class of spaces. Then φ is said to **reflect a cardinal** κ if whenever $\varphi(X) \geqslant \kappa$, we have $Y \subset X$ with $|Y| \leqslant \kappa$ and $\varphi(Y) \geqslant \kappa$. For a class $\mathcal{C}$ of spaces, φ reflects a cardinal κ for $\mathcal{C}$ if whenever $X \in \mathcal{C}$ and $\varphi(X) \geqslant \kappa$, we have $Y \subset X$ with $|Y| \leqslant \kappa$ and $\varphi(Y) \geqslant \kappa$. By using this definition, we can restate Hajnal–Juhász's result as follows: weight w reflects ω_1. Furthermore they showed that weight w reflects any infinite cardinal.

Cardinal functions ***cellularity*** c, ***extent*** e, ***spread*** s, hereditary density hd and hereditary Lindelöf degree hL reflects all infinite cardinals. ***Density*** d reflects every regular cardinal but need not reflect a singular cardinal for the class of T_1-spaces. For the class of ***compact Hausdorff spaces***, tightness t and character χ reflects any infinite cardinal.

The main technique to show reflection theorems is a closing-off argument (in other words, a method by using elementary submodels) mentioned in Cardinal functions, Part I.

5. sup = max **problem**

Recall that the cellularity $c(X)$ is defined by $c(X) = \min\{\kappa\colon \kappa \geqslant \omega$, every family of pairwise disjoint nonempty open sets of X has cardinality $\leqslant \kappa\}$. In other words, $c(X) = \sup\{\kappa\colon$ there is a pairwise disjoint family of cardinality κ consisting of open sets of $X\} + \omega$. A natural question arises: Can the "supremum" in this definition be replaced by the "maximum"? In other words, if $c(X) = \kappa$, then does a pairwise disjoint family of open sets of cardinality κ really exist? Such kind of problems are called "sup = max **problems**". Note that such a question is not trivial only when κ is a limit cardinal (i.e., a singular cardinal or a regular limit cardinal). Erdös–Tarski proved that if $c(X) = \kappa$ for a singular cardinal κ, then X has a pairwise disjoint family of cardinality κ consisting of nonempty open sets. For other results on sup = max problems, see [KV, Chapter 1, §12], [9, Chapter 4].

6. The number of open sets, compact sets, etc

Let $o(X)$ be the number of open sets of X, $C(X)$ the family of all ***continuous*** real valued functions, and $RO(X)$ the family of all regular open sets of X. Here a subset U of X is called a regular open set if $U = \operatorname{int}\operatorname{cl} U$. It is known that $|C(X)|^\omega = |C(X)|$, and $|RO(X)|^\omega = |RO(X)|$ for any infinite Hausdorff space (see [KV, Chapter 1, §10]). It is natural to ask whether $o(X)^\omega = o(X)$ or not. For a ***metrizable space*** X, it can be shown that $o(X) = 2^{w(X)}$, hence this equality holds. In 1986, Shelah proved that if X is an infinite compact T_2***-space***, then $o(X)^\omega = o(X)$ (see [10]).

Let $K(X)$ be the family of all compact subsets of a space X. For a T_2-space X, we have $|K(X)| \leqslant 2^{hL(X)}$ and $|K(X)| \leqslant 2^{e(X)\cdot\psi(X)}$ (see [KV, Chapter 1, §9]).

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a-5 Convergence

A **sequence** in X is a map from the set $\mathbb{N}$ of natural numbers to X. A sequence $s:\mathbb{N}\to X$ will often be identified with its image $\{s(n): n\in\mathbb{N}\}$, and we will use a shorter notation (s_n) to refer to sequences. A sequence (t_n) is a **subsequence** of a sequence (s_n) if there exists an increasing map $k:\mathbb{N}\to\mathbb{N}$ such that $t_n = s_{k(n)}$ for all n. A sequence (x_n) of points of a ***topological space*** is a **convergent sequence** with limit x, if for every ***neighbourhood*** W of x there exists a natural number n_W such that $x_n\in W$ for each $n\geqslant n_W$. Convergence of sequences in a topological space X satisfies the following three conditions:

(S1) if a sequence (x_n) converges to x then every subsequence of (x_n) also converges to x,
(S2) if $x_n = x$ for each n then (x_n) converges to x,
(S3) if a sequence (x_n) does not converge to x then there exists a subsequence (y_n) of (x_n) such that no subsequence of (y_n) converges to x.

Convergence of sequences can be introduced axiomatically. In [7] M. Fréchet defined an **$\mathcal{L}$-space** as a set X and a relation "(x_n) converges to x" (in symbols, $x=\lim_n x_n$) between sequences (x_n) of elements of X and elements x of X satisfying conditions (S1), (S2) and the following condition

(S0) a sequence (x_n) cannot converge to two different points.

An **$\mathcal{L}^*$-space** is an $\mathcal{L}$-space that fulfills (S3), the additional axiom of P. Alexandroff and P. Urysohn [2]. An **$\mathcal{S}^*$-space** is an $\mathcal{L}^*$-space satisfying the following diagonal condition:

(S4) $x=\lim_n x_n$ and $x_n=\lim_k x_{n,k}$ for every n implies that there exist an increasing sequence (n_m) of natural numbers and a sequence (k_m) of natural numbers with $x=\lim_m x_{n_m,k_m}$.

A subset S of an $\mathcal{L}$-space is closed whenever the limit of every convergent sequence of points of S belongs to S; a set is open if it is the complement of a closed set. The collection of open subsets of an $\mathcal{L}$-space is a (not necessarily ***Hausdorff***) ***topology***. Conversely, the convergent sequences in a Hausdorff topological space X constitute an **$\mathcal{L}^*$-space generated by** X. A subset S of a topological space X is **sequentially closed** (respectively, **sequentially open**) if it is closed (respectively, open) with respect to the $\mathcal{L}^*$-space generated by X. A topological space is called **sequential** if every sequentially open set is open. A topological space is a **Fréchet space** or a **Fréchet–Urysohn space** if $x\in\operatorname{cl}A$ (where $\operatorname{cl}A$ denotes the ***closure*** of a subset A) implies the existence of a sequence of points from A that converges to x. The convergent sequences of a Hausdorff Fréchet topology constitute an $\mathcal{S}^*$-space; conversely, each $\mathcal{S}^*$-space determines a Fréchet (not necessarily a Hausdorff) topology.

The concept of net, introduced by E.H. Moore and H.L. Smith in [15], is an extension of that of sequence. A partially ordered set $(D,\leqslant)$ is said to be **directed** if for every $\alpha,\beta\in D$ there exits $\delta\in D$ such that $\alpha\leqslant\delta$ and $\beta\leqslant\delta$. If $(D,\leqslant)$ is a directed set, then a map $t:D\to X$ is called a **net** on X. Every sequence can be identified with a net defined on $(\mathbb{N},\leqslant)$, the set of natural numbers directed by the natural order. A net is constant if the corresponding map is constant. A net $s:E\to X$ is a **subnet** of $t:D\to X$ if there exists a map $f:E\to D$ such that $s=t\circ f$ and for every $\delta\in D$ there is $\varepsilon\in E$ with $f(\varepsilon)\geqslant\delta$. If X is a topological space, then a net $t:D\to X$ **converges** to $x\in X$ (in symbols, $x\in\lim t=\lim_\delta t(\delta)$) provided that for every neighbourhood W of x there is $\delta_W\in D$ such that $t(\alpha)\in W$ for each $\alpha\geqslant\delta_W$. A sequence (x_n) in a topological space converges to x if and only if the net $t:\mathbb{N}\to X$ defined by $t(n)=x_n$ converges to x.

Let $(D,\leqslant_D)$ and $(E_\delta,\leqslant_\delta)$ for each $\delta\in D$ be directed sets. For $(\delta,\varphi),(\delta',\varphi')\in D\times\prod_{\delta\in D}E_\delta$ define $(d,\varphi)\preceq(d',\varphi')$ if and only if $d\leqslant_D d'$ and $\varphi(\delta)\leqslant_\delta\varphi'(\delta)$ for each $\delta\in D$. Then $(D\times\prod_{\delta\in D}E_\delta,\preceq)$ is a directed set. If $f_\delta:E_\delta\to X$ is a net for each $\delta\in D$, then the map

$$\triangle\{f_\delta\colon \delta\in D\}: D\times\prod_{\delta\in D}E_\delta\to X$$

defined by $\triangle\{f_\delta\colon\delta\in D\}(d,\varphi)=f_d(\varphi(d))$ is a net called the **diagonal net** of the family $\{f_\delta\colon\delta\in D\}$.

Convergence of nets in a topological space is often called **Moore–Smith convergence** and has the following properties (N1)–(N4) similar to properties (S1)–(S4), respectively:

(N1) if a net converges to x, then every subnet converges to x,
(N2) every constant net converges to its constant value,
(N3) if $x\notin\lim t$ then there exists a subnet s of t such that $x\notin\lim r$ for each subnet r of s, and
(N4) if $x\in\lim t$ and $t(\delta)\in\lim f_\delta$ for each $\delta\in D$, then $x\in\lim\triangle\{f_\delta\colon\delta\in D\}$.

A non-empty family $\mathcal{F}$ of subsets of a set X is called a **filter** on X if: (i) $F\in\mathcal{F}$ and $F\subset G\subset X$ implies $G\in\mathcal{F}$, (ii) $F_0\cap F_1\in\mathcal{F}$ provided that $F_0,F_1\in\mathcal{F}$, and (iii) $\emptyset\notin\mathcal{F}$. The family $\mathcal{N}(x)$ of neighbourhoods of x is a filter for each point x of a topological space: its **neighbourhood filter**. A family $\mathcal{C}$ of non-empty sets is a **centered family** provided that $C_1,C_2\in\mathcal{C}$ implies the existence of $C_3\in\mathcal{C}$ with $C_3\subset C_1\cap C_2$. If $\mathcal{C}$ is a centered family of subsets of X, then $\mathcal{F}=\{F\subseteq X\colon \exists C\in\mathcal{C},\ C\subseteq F\}$ is a filter on X generated by $\mathcal{C}$ (and we say that $\mathcal{C}$ generates $\mathcal{F}$). A centered subfamily $\mathcal{B}$ of a filter $\mathcal{F}$ is a **filter base** provided that $\mathcal{B}$ generates $\mathcal{F}$.

For every sequence (x_n) of elements of a set X, the family $\{\{x_k: k \geqslant n\}: n \in \mathbb{N}\}$ is centered, and thus generates a filter on X called the **filter generated by the sequence** (x_n). Maximal filters (with respect to the set inclusion) are called **ultrafilters**. For a subset $A \subset X$, the filter $\dot{A} = \{B \subset X: A \subset B\}$ is called the **principal filter** of A. A **free filter** is a filter with empty intersection.

If X is a set, then we use $\mathbb{F}(X)$ to denote the set of all filters on X. If $\Phi: Y \to \mathbb{F}(X)$ is a map and $\mathcal{F} \in \mathbb{F}(Y)$, then

$$\Phi(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} \bigcap_{y \in F} \Phi(y) \in \mathbb{F}(X)$$

is the **sum** or **contour** of Φ over $\mathcal{F}$; in other words, $A \in \Phi(\mathcal{F})$ whenever there exists $F \in \mathcal{F}$ such that $A \in \Phi(y)$ for every $y \in F$.

A subset ξ of $\mathbb{F}(X) \times X$ is called a **filter convergence** (or a **convergence structure**) on a set X provided that $(\mathcal{F}, x) \in \xi$ and $\mathcal{F} \subset \mathcal{G}$ imply $(\mathcal{G}, x) \in \xi$ and $(\dot{x}, x) \in \xi$ for every $x \in X$, where $\dot{x}$ stands for the principal filter of $\{x\}$. If $(\mathcal{F}, x) \in \xi$, then we say that $\mathcal{F}$ **converges** to x (x is a limit of $\mathcal{F}$) with respect to ξ. We often write $x \in \lim_\xi \mathcal{F}$ instead of $(\mathcal{F}, x) \in \xi$, and when ξ has been fixed, simply $x \in \lim \mathcal{F}$. In other words, lim represents a filter convergence on X if it satisfies

(F1) if $\mathcal{F}, \mathcal{G} \in \mathbb{F}(X)$, $\mathcal{F} \subset \mathcal{G}$ and $x \in \lim \mathcal{F}$, then $x \in \lim \mathcal{G}$, and

(F2) $x \in \lim \dot{x}$ for every $x \in X$.

(The original definition of convergence structure in [13] by H.-J. Kowalsky included also the condition $\lim(\mathcal{F}_0 \cap \mathcal{F}_1) \supset \lim \mathcal{F}_0 \cap \lim \mathcal{F}_1$ for each filters $\mathcal{F}_0$ and $\mathcal{F}_1$.) By a **convergence space** we understand a set and a filter convergence on that set. A filter convergence is a **Hausdorff convergence** if $\lim \mathcal{F}$ is at most a singleton for every filter $\mathcal{F}$. We say $\mathcal{F}$ is a **convergent filter** if $\lim \mathcal{F} \neq \emptyset$. The set $\lim \mathcal{F}$ is often called the **limit set** of $\mathcal{F}$.

Every $\mathcal{L}$-space determines a filter convergence as follows: $x \in \lim \mathcal{F}$ if and only if there exists a sequence (x_n) that converges to x such that $\mathcal{F}$ contains the filter generated by (x_n). Filter convergences determined by $\mathcal{L}$-spaces are Hausdorff (due to the condition (S0)).

Let X be a topological space. A filter $\mathcal{F}$ on X **converges** to a point $x \in X$ if every neighbourhood of x belongs to $\mathcal{F}$. This special filter convergence satisfies (F1), (F2) and also

(F3) $\bigcap\{\lim \mathcal{F}: \mathcal{F} \in \mathbb{F}\} \subset \lim \bigcap\{\mathcal{F}: \mathcal{F} \in \mathbb{F}\}$ for every $\mathbb{F} \subset \mathbb{F}(X)$, and

(F4) $\lim \mathcal{F} \subset \lim \Phi(\mathcal{F})$ for every $\mathcal{F} \in \mathbb{F}(X)$ and each map $\Phi: X \to \mathbb{F}(X)$ such that $x \in \lim \Phi(x)$ for every $x \in X$.

For a topological space X, convergence of nets and filter convergence are equivalent. Indeed, if $(D, \leqslant)$ is a directed set and $s: D \to X$ is a net, then the family $\{B_\delta: \delta \in D\}$, where $B_\delta = \{s(\alpha): \alpha \in D,\ \alpha \geqslant \delta\}$, is centered, and thus generates a filter $\mathcal{F}$ on X. The net s converges to $x \in X$ if and only if the filter $\mathcal{F}$ converges to x. Conversely, if $\mathcal{F}$ is a filter on X, then the set

$$D = \{(x, F): x \in F\} \subset X \times \mathcal{F}$$

partially ordered by $(x_0, F_0) \leqslant (x_1, F_1)$ whenever $F_0 \supset F_1$, is a directed set. The map $s: D \to X$ defined by $s(x, F) = x$ is a net that converges to some element $x' \in X$ if and only if the filter $\mathcal{F}$ converges to x'. If the filter corresponding to a net is an ultrafilter then the net itself is often called an **ultranet** or a **universal net**.

If $\mathcal{T}$ is a topology on X, then $C\mathcal{T}$ denotes the filter convergence for which $x \in \lim_{C\mathcal{T}} \mathcal{F}$ if and only if $O \in \mathcal{F}$ for every $O \in \mathcal{T}$ such that $x \in O$. Conversely, every filter convergence ξ defines a topology $\mathcal{O}_\xi$ by declaring $O \in \mathcal{O}_\xi$ whenever $\lim_\xi \mathcal{F} \cap O \neq \emptyset$ implies that $O \in \mathcal{F}$. A filter convergence ξ is said to be **topological** (or simply, a **topology**) if $\xi = C\mathcal{O}_\xi$. A convergence is topological if and only if it fulfills (F1) through (F4). A filter convergence is: (i) a **pseudotopology** if $x \in \lim \mathcal{F}$ provided that $x \in \lim \mathcal{U}$ for every ultrafilter $\mathcal{U} \supset \mathcal{F}$, (ii) a **paratopology** if $x \notin \lim \mathcal{F}$ implies the existence of a filter $\mathcal{H}$ with a countable filter base such that $F \cap H \neq \emptyset$ whenever $F \in \mathcal{F}$ and $H \in \mathcal{H}$, and $x \notin \lim \mathcal{G}$ for every $\mathcal{G} \supset \mathcal{H}$, (iii) a **pretopology**, or a Moore **closure space**, if (F3) holds, and (iv) **diagonal** if it fulfills (F4). Hence a filter convergence is a topology if and only if it is a diagonal pretopology. The notion of pretopology is due to F. Hausdorff [11], while that of pseudotopology to G. Choquet [3].

Properties of convergent filters correspond to some properties of convergent nets, namely (N1) amounts to (F1), while (N2) is equivalent to (F2). The diagonal condition (N4) is equivalent to the conjunction of (F3) and (F4). Finally, (N3) is equivalent to being a pseudotopology and is implied by (N4).

Given a filter convergence ξ on a set X, a subset A of X is ξ**-compact** if for every filter $\mathcal{H}$ such that $A \in \mathcal{H}$ there exists a filter $\mathcal{G} \supset \mathcal{F}$ such that $\lim_\xi \mathcal{G} \cap A \neq \emptyset$. If X is ξ-compact then we call ξ a **compact convergence** and if every ξ-convergent filter contains a ξ-compact set then ξ is a **locally compact convergence**. A convergence is **sequential** (respectively, of **countable character**) if $x \in \lim \mathcal{F}$ implies the existence of a filter $\mathcal{E}$ generated by a sequence (respectively, having a countable filter base) such that $\mathcal{E} \subset \mathcal{F}$ and $x \in \lim \mathcal{E}$.

If $f: X \to Y$ and $\mathcal{F}$ is a filter on X, then $f(\mathcal{F})$ denotes the filter on Y generated by $\{f(F): F \in \mathcal{F}\}$. If X and Y are convergence spaces, then a map $f: X \to Y$ is **continuous** if $f(\lim \mathcal{F}) \subset \lim f(\mathcal{F})$ for every filter $\mathcal{F}$ on X. A filter convergence ξ is **finer** than a filter convergence τ, and τ is **coarser** than ξ (in symbols, $\xi \geqslant \tau$), if the identity map from ξ to τ is continuous. The set of filter convergences on a fixed set, ordered by $\geqslant$, is a **complete lattice** (every subset has the least upper bound and the greatest lower bound). If X and Y are sets and $f: X \to Y$ is a map, then for every filter convergence on X, there exists the finest filter convergence on Y for which f is continuous (**final convergence**); as well, for each filter convergence on Y, there exists the coarsest filter convergence on X for which f is continuous (**initial convergence**). The notions of product, final convergence or quotient, subconvergence and so on, are natural consequences of the definitions above.

From the point of view of ***Category Theory***, filter convergences with continuous maps as ***morphisms*** form a category [1]. A ***subcategory*** **F** of convergences is **(concretely) reflective** if the least upper bounds of sets of convergences from **F**, and the initial convergences of convergences from **F** remain in **F**, and is **(concretely) coreflective** if the greatest lower bounds of sets of convergences from **F**, and the final convergences of convergences from **F** remain in **F**. If **F** is a reflective (coreflective) subcategory of convergences, then for every convergence ζ there exists the finest among the convergences ξ in **F** such that $\xi \leqslant \zeta$ (respectively, the coarsest among the convergences ξ in **F** such that $\zeta \leqslant \xi$). Topologies, pretopologies, paratopologies and pseudotopologies with continuous maps as morphisms form reflective subcategories of convergences. In particular, for any convergence ζ we can define $T\zeta$, $P\zeta$, $P_\omega\zeta$ and $S\zeta$ to be respectively the finest topology, pretopology, paratopology and pseudotopology coarser than ζ. The ***covariant functor***s T, P, P_ω and S are ***reflector***s. Sequential convergences, convergences of countable character and locally compact convergences with continuous maps as morphisms form coreflective subcategories of convergences. In particular, for any convergence ζ we can define $\mathrm{Seq}\,\zeta$, $\mathrm{First}\,\zeta$ and $K\zeta$ to be the coarsest convergence among sequential convergences, convergences of countable character and locally compact convergences, respectively, that is finer than ζ. The functors Seq, First and K are ***coreflectors***.

Many classical types of topologies can be characterized in terms of various reflectors and coreflectors in the category of convergences. For example [5], a topological space (X, τ) is: **bisequential** if and only if $\tau \geqslant S\,\mathrm{First}\,\tau$, ***strongly Fréchet*** if and only if $\tau \geqslant P_\omega\,\mathrm{First}\,\tau$, ***Fréchet–Urysohn*** if and only if $\tau \geqslant P\,\mathrm{First}\,\tau$, ***sequential*** if and only if $\tau \geqslant T\,\mathrm{First}\,\tau$, ***locally compact*** if and only if $\tau \geqslant SK\tau$ $(= K\tau)$, **countably** k'**-space** if and only if $\tau \geqslant P_\omega K\tau$, k'**-space** if and only if $\tau \geqslant PK\tau$, and ***k-space*** if and only if $\tau \geqslant TK\tau$.

For each convergence ξ on X and each convergence τ on Z there exists the coarsest convergence $[\xi, \tau]$ among all convergences on the set $C(\xi, \tau)$ of continuous maps from ξ to τ for which the canonical evaluation map $e : X \times C(\xi, \tau) \to Z$, defined by $e(x, f) = f(x)$ for $(x, f) \in X \times C(\xi, \tau)$, is continuous. (The convergence $[\xi, \tau]$ on $C(\xi, \tau)$ is called the **continuous convergence** induced by ξ and τ.) In other words, the category of convergences is ***Cartesian closed***. The category of pseudotopologies is also Cartesian closed, but those of topologies, pretopologies and paratopologies are not.

In the special case of sequences, continuous convergence was introduced by H. Hahn in [9]: A sequence (f_n) of maps converges to a map f if $x = \lim_n x_n$ for every x and each sequence (x_n) implies that $f(x) = \lim f_n(x_n)$. Another particular case of continuous convergence is that of hyperconvergence or upper Kuratowski convergence on the set of ***closed*** subsets (**hyperspace**) of a topological space [14, 10]. A subset A of a topological space X is closed if and only if the map $f_A : X \to \{0, 1\}$ defined by $f_A(x) = 0$ if $x \in A$ and $f_A(x) = 1$ if $x \notin A$ is continuous with respect to the **Sierpiński topology** $\$ = \{\emptyset, \{1\}, \{0, 1\}\}$ on $\{0, 1\}$. Therefore, if ξ is a topology on X, then $C(\xi, \$)$ can be identified with the collection of all ξ-closed sets. If ξ is a convergence, then continuous convergence $[\xi, \$]$ on $C(\xi, \$)$ is called the **upper Kuratowski convergence** with respect to ξ. The **lower Kuratowski convergence** $V^-(\xi)$ on $C(\xi, \$)$ induced by ξ is defined as follows: a filter $\mathcal{F}$ on $C(\xi, \$)$ converges to A provided that for every $x \in A$ there exists a filter $\mathcal{G}$ converging to x such that for every $G \in \mathcal{G}$ there is $F \in \mathcal{F}$ with $A \cap G \neq \emptyset$ whenever $A \in F$. If τ is a topology, then $B \in \lim_{[\tau,\$]} \mathcal{F}$ if and only if

$$\bigcap_{F \in \mathcal{F}} \mathrm{cl}_\tau \Bigl(\bigcup_{A \in F} A \Bigr) \subset B.$$

If τ is a ***regular*** topology, then $[\tau, \$]$ is a topology if and only if τ is ***locally compact***. If τ is a topology, then the lower Kuratowski convergence $V^-(\tau)$ is a topology called the **lower Vietoris topology**. One has $B \in \lim_{V^-(\tau)} \mathcal{F}$ whenever for every τ-open set O that intersects B, there exists $F \in \mathcal{F}$ such that $O \cap A \neq \emptyset$ for every $A \in F$. The **Kuratowski convergence** on the set of closed subsets of a topology τ is the supremum of the upper Kuratowski convergence $[\tau, \$]$ and the lower Kuratowski convergence $V^-(\tau)$.

If $(Y, \leqslant)$ is a complete lattice, then the **lower convergence** $y \in \lim_- \mathcal{F}$ and the **upper convergence** $y \in \lim_+ \mathcal{F}$ on Y are defined by

$$y \leqslant \sup\{\inf F \colon F \in \mathcal{F}\} \quad \text{and} \quad y \geqslant \inf\{\sup F \colon F \in \mathcal{F}\}$$

respectively, where $\sup A$ and $\inf A$ stand for the least upper bound and the greatest lower bound of a set $A \subseteq Y$. The upper Kuratowski convergence on the set of all closed subsets of a topological space X coincides with the upper convergence on the lattice of closed subsets of X; the lower convergence on the lattice of open subsets of X is called the **Scott convergence** and thus is ***homeomorphic*** with the upper Kuratowski convergence.

A convergence ξ on X is called a **regular convergence** if the following is satisfied: if $\mathcal{F}$ converges to x with respect to ξ, then the filter generated by $\{\mathrm{adh}_\xi F \colon F \in \mathcal{F}\}$, where

$$\mathrm{adh}_\xi F = \bigcup \bigl\{\lim_\xi \mathcal{H} \colon F \in \mathcal{H} \in \mathbb{F}(X)\bigr\},$$

also converges to x with respect to ξ. Unlike in the case of topologies, a compact Hausdorff convergence need not be even regular (for example, the Kuratowski convergence with respect of a Hausdorff topology). On the other hand, every compact Hausdorff regular pseudotopology is a topology [8].

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a-6 Several Topologies on One Set

If X is a set then the smallest ***topology*** on X is the ***indiscrete topology*** $\{\varphi, X\}$, and the largest topology on X is the family $\mathcal{P}(X)$ of all subsets of X (sometimes called the ***power set*** of X). This largest topology on X is called the ***discrete topology*** on X. If τ_1 and τ_2 are topologies on X then τ_1 is **smaller** than τ_2, and τ_2 is **larger** than τ_1, if and only if $\tau_1 \subset \tau_2$ as subsets of $\mathcal{P}(X)$. In other words, τ_1 is smaller than τ_2 if and only if each τ_1-***open*** subset of X is τ_2-open. In this case, it is often said that τ_1 is **coarser** than τ_2 and that τ_2 is **finer** than τ_1. [A word of warning is in order. Unfortunately, there is some confusion in the literature, and the situation defined immediately above is described by both of the statements "τ_1 is stronger than τ_2" and "τ_1 is weaker than τ_2".] If τ_1 and τ_2 are arbitrary topologies on the same set X, it may happen that τ_1 is neither smaller nor larger than τ_2, in which case τ_1 and τ_2 are not comparable. Probably the most common method used to show that two topologies τ_1 and τ_2 on the same set X are equal is a double comparison, namely $\tau_1 \subset \tau_2$ and $\tau_2 \subset \tau_1$. Such arguments are ubiquitous across General Topology.

If X is a set and τ_1 and τ_2 are topologies on X, then the triple (X, τ_1, τ_2) is defined to be a **bitopological space**. It seems that this term was first used by Kelly [6] in his classical paper on the topic. Bitopological spaces can be given a categorical treatment. The ***category*** Bitop has objects which are bitopological spaces and ***morphisms*** which are pairwise continuous functions. A function $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined to be **pairwise continuous** (or **bicontinuous**) if each of the functions between ***topological spaces*** $f:(X, \tau_1) \to (Y, \sigma_1)$ and $f:(X, \tau_2) \to (Y, \sigma_2)$ is ***continuous***. Brümmer [1] and Salbany [13] are two of the major contributors to such a treatment. If the topological space (X, τ) is identified with the bitopological space (X, τ, τ), then it is clear that the category Top is a subcategory of Bitop.

Bitopological spaces arise naturally whenever one considers a non-symmetrical topological structure. This is because the original structure and its conjugate each generate (usually different) topologies on the underlying set. Examples are ***quasi-metrics***, quasi-proximities, quasi-topological groups and ***quasi-uniformities***. A binary relation ξ_1 on $\mathcal{P}(X)$ is a **quasi-proximity** on X if the following axioms are satisfied:

(1) $A\,\xi_1\,(B \cup C)$ if and only if $A\,\xi_1\,B$ or $A\,\xi_1\,C$,
$(A \cup B)\,\xi_1\,C$ if and only if $A\,\xi_1\,C$ or $B\,\xi_1\,C$;
(2) $A\,\xi_1\,B$ implies A and B are non-empty;
(3) $A\,\not\xi_1\,B$ implies there exists an $E \subset X$ such that $A\,\not\xi_1\,E$ and $(X - E)\,\not\xi_1\,B$;
(4) $A \cap B \neq \varphi$ implies $A\,\xi_1\,B$.

If in addition ξ_1 satisfies

(5) $A\,\xi_1\,B$ implies $B\,\xi_1\,A$

then ξ_1 is a **proximity** on X. If ξ_1 is a quasi-proximity on X, its conjugate ξ_2 is defined in the usual way

$$A\,\xi_2\,B \quad \text{if and only if} \quad B\,\xi_1\,A.$$

Let $\tau_i = \tau(\xi_i)$, $i = 1, 2$, denote the corresponding topologies on X induced by the quasi-proximities ξ_i by means of Kuratowski ***closure*** operators $\tau_i \operatorname{cl} A = \{x \in X\colon \{x\}\xi_i A\}$. Then (X, τ_1, τ_2) is a quasi-proximizable bitopological space.

A group $(G, \cdot)$ endowed with a topology τ is a **quasi-topological group** if the group operation $(x, y) \to x \cdot y$ is continuous in both variables jointly. If (G, τ) is a quasi-topological group, then so is (G, τ'), where $\tau' = \{U \mid U^{-1} \in \tau\}$ is called the **conjugate topology** of τ. Furthermore, the map $x \to x^{-1}$ is a ***homeomorphism*** of (G, τ) onto (G, τ'), and the relationship between τ and τ' is symmetrical.

There is a well-developed theory of separation properties for bitopological spaces. Some authors distinguish between weak and strong versions of most of these properties. For example, the bitopological Hausdorff property may be defined as follows: (X, τ_1, τ_2) is **strong (weak) pairwise Hausdorff** if for each pair of distinct points x and y in X there are disjoint open sets $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$, $y \in V$ [(U contains one point and V contains the other point)]. Weston [17] first defined this notion in its strong form and used the term "**consistent space**". Kelly [6] first used the term pairwise Hausdorff. Fletcher Hoyle and Patty [4] defined (X, τ_1, τ_2) to be **strong pairwise** T_0 if for each pair of distinct points x, y in X there is either a τ_1 open set U such that $x \in U$ and $y \notin U$ or a τ_2 open set V such that $y \in V$ and $x \notin V$. The weak version of this property is defined as follows: (X, τ_1, τ_2) is **weak pairwise** T_0 if for each pair of distinct points of X there is a set which is either τ_1 open or τ_2 open containing one of the points but not the other. Similarly, (X, τ_1, τ_2) is defined to be **strong (weak) pairwise** T_1 if for each pair of distinct points x and y in X there are open sets $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$ (either [$x \in U$, $y \notin U$, $y \in V$ and $x \notin V$] or [$y \in U$, $x \notin U$, $x \in V$ and $y \notin V$]). In the bitopological space (X, τ_1, τ_2), Kelly [6] defined τ_1 to be **regular with respect to** τ_2 if for each point x in X and each τ_1 ***closed set*** P such that $x \notin P$ there is a τ_1 open set U and a τ_2 open set V disjoint from U such that $x \in U$ and $P \subset V$. Then (X, τ_1, τ_2) is **strong (weak) pairwise regular** if τ_1 is regular with respect to τ_2 and (or) τ_2 is regular with respect to τ_1. In (X, τ_1, τ_2), τ_1 is defined to be **completely regular** with respect to τ_2 if for each τ_1 closed set C and each point $x \notin C$ there is a real valued function f on X into [0, 1] such that $f(x) = 0$, $f(C) = 1$, and f is τ_1 ***upper semi-continuous*** and τ_2 ***lower semi-continuous***. Furthermore, (X, τ_1, τ_2) is

strong (weak) pairwise completely regular if τ_1 is completely regular with respect to τ_2 and (or) τ_2 is completely regular with respect to τ_1. Bitopological normality was defined by Kelly [6] as follows: (X, τ_1, τ_2) is **pairwise normal** if for each τ_1 closed set A and τ_2 closed set B disjoint from A there is a τ_1 open set V containing B and a τ_2 open set U disjoint from V containing A. Consider the bitopological space $(\mathbb{R}, \mathcal{U}, \mathcal{L})$ where $\mathbb{R}$ is the set of real numbers and $\mathcal{U}$ and $\mathcal{L}$ are the upper and lower topologies on $\mathbb{R}$, namely $\mathcal{U} = \{\varphi, \mathbb{R}, (a, \infty): a \in \mathbb{R}\}$ and $\mathcal{L} = \{\varphi, \mathbb{R}, (-\infty, a): a \in \mathbb{R}\}$. Then $(\mathbb{R}, \mathcal{U}, \mathcal{L})$ is pairwise normal, and satisfies the weak version of each of the other separation properties, but does not satisfy the strong form. By adding the appropriate form of the pairwise T_1 property to the higher separation properties, one obtains two hierarchies of bitopological separation properties – a weak one and a strong one. Section 2 of Kopperman [8] is a thorough discussion of bitopological separation properties.

Bitopological covering properties have proved to be much more intractable than the separation properties. Fletcher, Hoyle and Patty [4] provided an early definition of bitopological ***compactness***. A ***cover*** $\mathcal{U}$ of the bitopological space (X, τ_1, τ_2) is defined to be **pairwise open** if $\mathcal{U} \subset \tau_1 \cup \tau_2$ and $\mathcal{U}$ contains at least one non-empty member of τ_1 and at least one non-empty member of τ_2. If each pairwise open cover of (X, τ_1, τ_2) has a finite ***subcover*** then the space (X, τ_1, τ_2) is defined to be **pairwise compact**. Note that $(\mathbb{R}, \mathcal{U}, \mathcal{L})$ is pairwise compact. Cooke and Reilly [2] considered alternative definitions and characterizations of bitopological compactness. Salbany [13] has provided the most comprehensive early discussion of this topic, based on the stronger definition that (X, τ_1, τ_2) is **pairwise compact** if the topological space $(X, \tau_1 \vee \tau_2)$ is compact. Swart [16] also used this form of compactness for bitopological spaces. Salbany [13] has introduced the bitopological analogue of the ***Stone–Čech compactification***. A more recent development of these ideas is given in Sections 3 and 6 of Kopperman [8].

If $\mathcal{U}$ is a quasi-uniformity on X then the family $\mathcal{U}^{-1}$ defined by $\mathcal{U}^{-1} = \{V^{-1}: V \in \mathcal{U}\}$ is also a quasi-uniformity on X, called the **conjugate quasi-uniformity** of $\mathcal{U}$. The bitopological space (X, τ_1, τ_2) is **quasi-uniformizable** if there is a quasi-uniformity $\mathcal{U}$ on X such that $\tau_1 = \tau(\mathcal{U})$ and $\tau_2 = \tau(\mathcal{U}^{-1})$. The topology $\tau(\mathcal{U})$, generated on X by $\mathcal{U}$, is the family of all subsets P of X such that for each $x \in P$ there is a $V \in \mathcal{U}$ such that $V[x] \subset P$. Fletcher [3] and Lane [9] proved that a bitopological space is quasi-uniformizable if and only if it is pairwise completely regular. The corresponding quasi-metrization problem – which bitopological spaces are generated by a pair of conjugate quasi-metrics? – has proved more difficult. This problem was first considered by Kelly [6], and then by Lane [9], Patty [11] and Salbany [14]. The solution of this problem is due to Fox [5], and is presented as Theorem 1.1 of Kopperman [7], who has provided a full elaboration of the result and a discussion of its topological, as well as its bitopological, significance.

It is not surprising that bitopological arguments are useful in the consideration of the topological properties of non-symmetrical structures. Typical results of this kind are provided by Raghavan and Reilly [12]. For example, every quasi-metric space whose conjugate topology is ***sequentially compact*** is ***metrizable***, and every quasi-uniform space whose conjugate topology is R_0 and compact is uniformizable. Bitopological views of certain aspects of topology are provided by Salbany [15] and by Marin and Romaguera [10].

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a-7 Comparison of Topologies (Minimal and Maximal Topologies)

1. Introduction

Inherent in the study of ***topology*** is the notion of comparison of two different topologies on the same underlying set. This 'comparison' is carried out by the natural ordering of set inclusion. Thus, the family of all topologies definable for an infinite set X is a ***complete atomic*** and ***complemented lattice*** (under set inclusion) which we denote by $LT(X)$. If $\mathcal{T}$ and $\mathcal{S}$ are two topologies on X with $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S}$ is said to be **weaker** or **coarser** than $\mathcal{T}$ and $\mathcal{T}$ is said to be **stronger** or **finer** than $\mathcal{S}$. Given a ***topological invariant*** P, a member $\mathcal{T}$ of $LT(X)$ is said to be **minimal (maximal)** P if and only if $\mathcal{T}$ possesses property P but no weaker (stronger) member of $LT(X)$ possesses property P. P is said to be **expansive** (**contractive**) if and only if for each P-member of $LT(X)$, every stronger (weaker) member of $LT(X)$ is also P.

2. Maximal topologies

The concept of minimal topologies was first introduced in 1939 by Parhomenko [12] when he showed that ***compact Hausdorff spaces*** are minimal Hausdorff. In 1943, Hewitt proved that compact Hausdorff spaces are maximal compact also [7]. The converses were shown to be false but nonetheless this pleasing disposition of compact Hausdorff spaces motivated Larson's study of **complementary topological properties**: an expansive property P and a contractive property Q are called complementary when the minimal P-members of $LT(X)$ coincide with the maximal Q-members.

In 1948, Ramanathan [15] proved that a ***topological space*** is maximal compact if and only if its compact subsets are precisely the ***closed sets***. Further, it can be shown that in a compact space X, every compact subset is closed if and only if every ***continuous*** bijection from a compact space Y to X is a ***homeomorphism***. This can be generalized to the following: A topological space is maximal P if and only if every continuous bijection from a P-space Y to X is a homeomorphism. Other results concerning maximal properties occurred sporadically since then. In 1963 Smythe and Wilkins [17] constructed a maximal compact topology which was strictly weaker than a minimal Hausdorff topology. In 1965, Levine proved that $X \times X$ is maximal compact if and only if X is Hausdorff thus showing that maximal compact is not necessarily preserved by formation of ***products***. He also showed that the ***one-point compactification*** of the rationals is non-Hausdorff maximal compact. In 1966, Thron proved that a ***first-countable*** Hausdorff ***countably compact*** space is maximal countably compact and minimal first countable Hausdorff. The following year, Aull proved a stronger result, namely that a countably compact E_1-space is maximal countably compact and minimal E_1 (a topological space is E_1 if each point is the countable intersection of closed ***neighbourhoods***). In [6], Cameron discusses maximality with respect to the properties compactness, countable compactness, ***sequential compactness***, **Bolzano–Weierstrass compactness** (every infinite set has an ***accumulation point***) and ***Lindelöf***. For example, a topology is maximal countably compact, maximal sequentially compact, maximal Lindelöf if and only if the closed subsets of the topology are respectively precisely the countably compact subsets, the sequentially compact subsets and the Lindelöf subsets. Maximal topologies with respect to **order-induced topological properties** [1] (defined below in Section 3) and ***connectedness*** [13] have also been determined. For example, a topology is maximal connected if and only if it is connected, ***submaximal***, and for every **regular open** set V (i.e., such that $V = \operatorname{int} \overline{V}$) and for all $x \in \operatorname{Fr} V$ there is an ***open set*** C such that $\operatorname{Fr} C = \{x\}$ and there is a regular open neighbourhood N of x such that $N \cap V \cap C = \emptyset$ [8]. Note that $\overline{V}$, $\operatorname{int} V$ and $\operatorname{Fr} V$ denote the ***closure***, ***interior*** and ***boundary*** of V, respectively, where $\operatorname{Fr} V$ is given by $\overline{V} \setminus \operatorname{int} V$.

3. Minimal topologies

The study of minimal topological spaces has been much more intense than that of maximal spaces. (See, for example, [3, 4, 11, 14, 16] and [18].) Properties whose minimality has been investigated include ***normal***, ***Urysohn***, ***paracompact***, ***Tychonoff***, ***regular***, ***locally compact*** and **low separation axioms**, that is, those lying in logical strength between T_0 and Hausdorff. A brief survey of some minimality results may be found in [11]. These include the following: a topological space $(X, \mathcal{T})$ is

- minimal Hausdorff if and only if it is Hausdorff and every open ***filter base*** which has a unique ***cluster point*** is ***convergent*** to this point,
- minimal T_1 if and only if $\mathcal{T}$ is the **cofinite topology** $\mathcal{C}$ (i.e., the topology consisting of all sets whose complements are finite, together with $\emptyset$) on X,
- minimal regular if and only if it is regular and every regular filter-base which has a unique cluster point is convergent, where a **regular filter-base** is an open filter-

base α which is equivalent to a closed filter-base β (i.e., α and β generate the same ***filter***),
- minimal ***completely regular*** if and only if it is compact and Hausdorff,
- minimal normal if and only if it is compact and Hausdorff,
- minimal Urysohn if and only if it is Urysohn and every filter with a unique cluster point converges to this point,
- minimal (locally compact and Hausdorff) if and only if it is compact and Hausdorff,
- minimal paracompact if and only if it is compact and Hausdorff,
- minimal ***metric*** only if it is compact and Hausdorff,
- minimal ***completely normal*** if and only if it is compact and Hausdorff,
- minimal ***completely Hausdorff*** if and only if it is compact and Hausdorff,
- minimal T_0 if and only if it is T_0, **nested** and generated by the family $\{X \setminus \overline{\{x\}}\colon x \in X\} \cup \{\emptyset, X\}$,
- minimal T_D if and only if it is T_D and nested,
- minimal T_A if and only if it is T_A and **partially nested**,
- minimal T_F if and only if either there exists $x \in X$ such that $\mathcal{T} = \mathcal{C} \cap \mathcal{I}(x)$ or there exists a non-empty proper non-singleton subset Y of X such that $\mathcal{T} = \mathcal{D}(Y)$.

We note some definitions for the above: $(X, \mathcal{T})$ is

(i) nested if and only if either $G \subseteq H$ or $H \subseteq G$ for all $G, H \in \mathcal{T}$;
(ii) partially nested if and only if there exists a cofinite subset A of X such that
  - if F_1 and F_2 are each $\mathcal{T}$-closed subsets of A, then either $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$,
  - $y \in X \setminus A$ implies $\overline{\{y\}} = \{y\} \cup A$,
  - if $A = \overline{\{t\}}$ for some t, then $|X \setminus A| = 1$;

(iii) T_D if and only if for all $x \in X$, $\overline{\{x\}} \setminus \{x\}$ is $\mathcal{T}$-closed;
(iv) T_A if and only if for all $x \in X$, either $\{x\}$ is $\mathcal{T}$-closed, or $\mathcal{T}$-open, or $\overline{\{x\}} \setminus \{x\}$ is a point-closure (i.e., the closure of a singleton);
(v) T_F if and only if for all $x \in X$, either $\{x\}$ is $\mathcal{T}$-closed or $\{x\}$ is the intersection of all $\mathcal{T}$-open subsets of X containing x;
(vi) $\mathcal{I}(Y) = \{G \subseteq X\colon Y \subseteq G\} \cup \{\emptyset\}$ where $Y \subseteq X$. When $Y = \{y\}$, we simply write $\mathcal{I}(y)$;
(vii) $\mathcal{D}(Y) = \{G \subseteq X\colon G \subseteq Y$ and $Y \setminus G$ is finite, or $Y \subseteq G$ and $X \setminus G$ is finite$\} \cup \{\emptyset\}$;
(viii) $\mathcal{C}$ is the cofinite (or minimum T_1) member of $LT(X)$.

Motivation for such an investigation is provided by realising that it is in seeking to identify those members of $LT(X)$ which minimally satisfy an invariant that we are, in a very real sense, examining the topological essence of the invariant. A closely allied question to that of minimality is whether or not a given invariant P is a **Katětov invariant**: P is said to be Katětov if and only if every P-member of $LT(X)$ is stronger than a minimal P-member of $LT(X)$. Other related questions include whether or not minimal P is ***hereditary*** or ***productive***. For example, it is known that a minimal Hausdorff ***subspace*** of a Hausdorff space is closed, while a clopen subspace of a minimal Hausdorff space is also minimal Hausdorff. Again, can an arbitrary P-space be ***embedded*** in some minimal P-space? The approaches taken in these investigations vary widely, leading to a great diversity of characterisations. So, for example, while it is known that the compact Hausdorff topologies are a precise identification for minimal normal, minimal paracompact, minimal Tychonoff and minimal locally compact, minimal Hausdorff topologies are characterised by those Hausdorff topologies in which every ***open filter*** with a unique cluster point converges [5]. In particular, the description of many low separation axioms affords an alternative order-theoretic viewpoint as many of these can be recast in terms of the associated **specialisation order** on the underlying set. This order is the natural partial order induced on X by any T_0-member of $LT(X)$:

$$x \leqslant y \iff x \in \overline{\{y\}}, \quad x, y \in X.$$

Thus, invariants expressed solely in terms of point-closures may be interpreted naturally in terms of the induced partial order so offering a new insight into the problem of minimality. A partial order is immediately available which may be readily exploited and also lends a welcome visual aspect to the discussion. This was the approach taken by Andima and Thron [1] in their study of order-induced topological properties; that is, a topological property P which has an associated order property K such that a topology has property P if and only if its specialisation order has property K. Thus, given $\mathcal{T} \in LT(X)$ and its associated specialisation order $\leqslant$, for
- $P = T_1$, $K =$ 'is a **diverse partial order**', i.e., for all distinct $x, y \in X$, $x \nleqslant y$ and $y \nleqslant x$;
- $P = T_F$, $K =$ 'is a partial order such that all chains consist of at most two elements'.

Thus we have an alternative order-theoretic minimality description for such properties. For example, given $\mathcal{T} \in LT(X)$ and its associated specialisation order $\leqslant$,
- $\mathcal{T}$ is minimal T_0 if and only if $\leqslant$ is a chain and $\mathcal{T}$ is the topology whose closed sets are generated by the family $\{\emptyset, X, \{y \in X\colon y \leqslant x\}\colon x \in X\}$.
- $\mathcal{T}$ is minimal T_F if and only if $\leqslant$ is a partial order such that (i) all chains consist of at most two elements, (ii) if x is maximal, then $y \leqslant x$ for each non-maximal element $y \in X$, and (iii) $\mathcal{T}$ is the topology whose closed sets are generated by the family $\{\emptyset, X, \{y \in X\colon y \leqslant x\}\colon x \in X\}$.

4. Low separation axioms

We conclude with a discussion of some techniques that have been used to good and illuminating effect for this type of topological invariant. Examples include T_0, T_1, T_D, T_A and sober. Given $\mathcal{T} \in LT(X)$, $\mathcal{T}$ is said to be
- **sober** if and only if every non-empty **$\mathcal{T}$-closed irreducible** subset of X is the closure of a unique singleton. A subset is $\mathcal{T}$-closed irreducible if and only if it cannot

be expressed as a union of two non-empty proper $\mathcal{T}$-closed subsets of X.

We observe that T_D is not an order-induced topological property. Not surprisingly, these techniques require and exploit a knowledge of the members of the lattice $LT(X)$.

Separation axioms within $[T_0, T_1]$

Notable in the lattice structure are the (incomparable) topologies comprising the **anti-atoms**, that is, the maximal non-discrete topologies. These are called the **ultratopologies** because they derive from ***ultrafilter***s: specifically, given an ultrafilter $\mathcal{U}$ on X, and $x \in X$ such that $\{x\} \notin \mathcal{U}$, with $\mathcal{E}(x) = \mathcal{P}(X \setminus \{x\}) \cup \{X\}$, the ultratopologies are precisely of the form $\mathcal{U} \cup \mathcal{E}(x)$. Thus, the nature of the ultratopology is determined by that of the corresponding ultrafilter. Note that $\mathcal{P}(Y)$ denotes the **power set** of Y, i.e., the family of all subsets of Y. If $\mathcal{U}$ is a **principal ultrafilter**, that is, generated by a singleton $\{y\}$ for some $y \neq x$, then the resulting ultratopology has the form $\mathcal{I}(y) \cup \mathcal{E}(x)$ where $\mathcal{I}(y) = \{G \subseteq X\colon y \in G\} \cup \{\emptyset\}\}$ and is accordingly referred to as principal. It is worthwhile noting that each non-principal topology is T_1 while no principal ultratopology is. The significance of this band of topologies is apparent as we describe a general scheme due to McCartan and McCluskey [10] for establishing minimality with respect to a given property P. They assume the existence of a minimal P-topology $\mathcal{T}$ and then attempt to identify the critical features of such a structure by intersecting with a suitably chosen principal ultratopology. Principal ultratopologies, being 'almost discrete', are very rich in topological structure and so, by such an intersection, there is a very good chance of the resulting strictly weaker topology retaining the property P and thus providing the desired contradiction. This intersection has a subtle effect on the $\mathcal{T}$-closures of singletons (or points); a representative point $\mathcal{T}$-closure is increased by the addition of a unique element, and this has an obvious 'knock-on' effect. Formally, if $A \subseteq X$ and $\mathcal{T}^* = \mathcal{T} \cap (\mathcal{I}(y) \cup \mathcal{E}(x))$, then the $\mathcal{T}^*$-closure of A is described by

$$\bar{A}^* = \begin{cases} \bar{A}, & \text{if } y \notin \bar{A}, \\ \bar{A} \cup \overline{\{x\}}, & \text{if } y \in \bar{A}. \end{cases}$$

Thus if A is a singleton, then its $\mathcal{T}^*$-closure is either unchanged (if y is not in its $\mathcal{T}$-closure) or is increased by the addition of $\{x\}$. The knock-on effect simply refers to the consequent inclusion of all points in the $\mathcal{T}$-closure of $\{x\}$ (since $\bar{A}^*$ is also $\mathcal{T}$-closed). This construction must of course be carefully set up; its success relies heavily upon the internal structure of P. Moreover, this construction has a strong order-theoretic appeal. This is particularly well-illustrated and exploited when used as a tool to determine minimality with respect to separation axioms whose characterisations are in terms of the behaviour of point-closures (and so are essentially order-induced properties. See the earlier description of the specialisation order). The topological significance of the above described intersection has a direct order-theoretic translation when the given space is regarded as a ***partially ordered set***. Essentially, the intersection 'forces x beneath y' and consequently all of $\overline{\{x\}}$ beneath every element lying above y in the original order, that is, the specialisation order induced by $\mathcal{T}$.

The technique adopted by Larson [9] in solving minimal T_0 and T_D is an interesting comparison. Given a T_0, respectively T_D, topology $\mathcal{T}$, his approach was to show that for an arbitrary $\mathcal{T}$-open set B, the topology which results from intersecting $\mathcal{T}$ with $\mathcal{M}(B) = \{G \subseteq X\colon G \subseteq B \text{ or } B \subseteq G\}$ is T_0, respectively T_D. It can be shown that $\mathcal{M}(B)$ is in fact an intersection of principal ultratopologies; specifically, $\mathcal{M}(B) = \bigcap\{\mathcal{I}(y) \cup \mathcal{E}(x)\colon y \in B,\ x \notin B\}$. Such an intersection $\mathcal{T} \cap \mathcal{M}(B)$ has rather more dramatic consequences for the original point-closures in that, for any $z \in B$, its closure is increased by $X \setminus B$. The closures of elements outside B remain unchanged. The order-theoretic interpretation is that every point in $X \setminus B$ is forced beneath every point in B while no extra points are added beneath any point of $X \setminus B$.

As mentioned earlier, intersection with a single principal ultratopology has only a marginal, subtle effect on the size and structure of the original topology and since the original topology is in many cases assumed to be minimal, such a delicate manipulation is appropriate. Intersection with a topology of the form $\mathcal{M}(B)$ may in general be too severe a reduction to be confident of retaining the invariant under consideration. It is interesting to note that there are several known axioms for which precisely one of the above two methods is successful [10]. For example, the properties T_δ and T_ζ (for all $x \in X$, either $\{x\}$ is closed or $\overline{\{x\}} \setminus \{x\}$ is a union of **incommensurable** point-closures, i.e., no member of the union is contained in nor contains any other member of that union) both display nestedness in their minimal structure. However, the technique which proves this for T_ζ, namely intersection with a single principal ultratopology, will not work for T_δ. However Larson's technique as described above will establish nestedness for this latter case.

Without T_1 as a logical upper bound, the above techniques have only partial success in the quest for minimality with respect to a given property. However, intersection with a topology of the form $\mathcal{C} \cup \mathcal{E}(x)$, where $\mathcal{C}$, the cofinite topology, consists of all sets whose complements are finite, together with $\emptyset$ provides a useful alternative.

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B: Basic constructions

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b-1 Subspaces (Hereditary (P)-Spaces)

Let X be a given ***topological space*** with a ***topology*** $\mathcal{O}$ (the collection of open subsets) and X' a subset of X. Put

$$\mathcal{O}' = \{X' \cap O\colon O \in \mathcal{O}\}.$$

Then $\mathcal{O}'$ satisfies the conditions to be a topology on X'; that is, X' is a topological space with the topology $\mathcal{O}'$. This topological space X' is called a **subspace** of X and this topology $\mathcal{O}'$ is called the **relative topology** (the **induced topology**, the **subspace topology**) of $\mathcal{O}$ with respect to X'. Sometimes, a ***closed*** subset (***open*** subset) is called a **closed subspace** (**open subspace**) in view of subspaces. The following facts are direct consequences of definitions of subspaces.

(1) A subset E' of X' is a closed (open) subset of the space X' if and only if $E' = X' \cap E$ for some closed (open) subset E of the space X.
(2) Let A be a subset of X' and $\bar{A}'$ denote the ***closure*** of A in X'. Then $\bar{A}' = X' \cap \bar{A}$ holds, where $\bar{A}$ denotes the ***closure*** of A in X.
(3) For a point x' of X' a subset U' of X' is a neighbourhood of x' in X' if and only if $U' = X' \cap U$ for some neighbourhood U of x' in X.
(4) For a ***filter*** $\mathcal{F}$ in X' is ***convergent*** in X' if and only if as a filter in X, $\mathcal{F}$ converges to a point of X'.
(5) For a ***net*** $\mathcal{N}$ in X', $\mathcal{N}$ is ***convergent*** in X' if and only if as a net in X, $\mathcal{N}$ converges to a point of X'. In particular, for a ***sequence*** $\mathcal{S}$ in X', $\mathcal{S}$ is ***convergent*** in X' if and only if $\mathcal{S}$ converges to a point of X'.
(6) If $\mathcal{B}$ is a ***base*** for X then $\mathcal{B}' = \{B \cap X'\colon B \in \mathcal{B}\}$ is a base for X'.

There is a notion related to that of a base that is occasionally useful when dealing with subspaces: an **outer base** for a subspace X' of X is a family $\mathcal{B}$ of open sets in X such that for every point y of X' and every open set O in X with $x \in O$ there is a $B \in \mathcal{B}$ with $x \in B \subseteq O$. If $\mathcal{B}$ is a base for X then $\{B \in \mathcal{B}\colon B \cap X' \neq \emptyset\}$ is an outer base for X'.

Let X be the Euclidean plane $\mathbb{R}^2$ and let X' be the x-axis of $\mathbb{R}^2$. Then X' is a closed subspace of X. More generally, for any two natural numbers m and n with $m \leqslant n$, the m-dimensional Euclidean space $\mathbb{R}^m$ is a closed subspace of the n-dimensional Euclidean space $\mathbb{R}^n$.

Similarly, subspaces are defined for ***uniform spaces*** and ***proximity spaces***; that is, the restriction of the uniformity or the proximity relation to a subset forms a uniformity or a proximity relation on the subspace, respectively.

Let X be a topological space and Y a subspace of X. If $\overline{Y} = X$ holds, then Y is called a **dense subspace** of X.

For the real line $\mathbb{R}^1$ the subspace consisting of all rational numbers is a dense subspace of $\mathbb{R}^1$ and the subspace consisting of all irrational numbers is also a dense subspace of $\mathbb{R}^1$. On the other hand, it is easily seen that $\mathbb{R}^1$ is not a dense subspace of $\mathbb{R}^n$ for any $n \geqslant 2$.

For a ***topological property*** P we say that P is a **hereditary property** (**closed-hereditary property** or a **open-hereditary property**) if any subspace (closed subspace or open subspace) of any topological space possessing property P also has the property P.

Any hereditary property is always a closed hereditary property and an open hereditary property, as well. Converses of above directions are very often not true.

The following properties are hereditary properties which are easily seen from their definitions.

(1) T_0,
(2) T_1,
(3) T_2,
(4) ***regularity***,
(5) ***complete regularity***,
(6) ***metrizability***.

The following properties are closed hereditary ones but not hereditary ones.

(1) ***normality***,
(2) ***compactness***,
(3) ***paracompactness***,
(4) ***Lindelöf*** property,
(5) ***local compactness***,
(6) ***completeness***.

To show that above properties (1), (2), (3) and (4) are not hereditary the following space is useful.

Tychonoff plank
Let X be the product space $[0, \omega_1] \times [0, \omega_0]$ and $Y = X - \{(\omega_1, \omega_0)\}$. Then X is a compact T_2 space and Y is an open subspace of X. Y is sometimes called the **Tychonoff plank** [E, 3.12.20]. Since X is a compact T_2 space, X satisfies properties (1), (2), (3) and (4). On the other hand, Y is not normal, because the two edges $\{\omega_1\} \times [0, \omega_0)$ and $[0, \omega_1) \times \{\omega_0\}$ are not separated by disjoint open subsets of Y. Since Y is not normal, Y does not satisfy any of the properties (2), (3) and (4).

To show that the local compactness is not hereditary, put $X = \mathbb{R}^1$ (real line) and $Y = \mathbb{Q}$ (the set of all rational numbers) as a subspace of X. Then X is a locally compact space, but Y is not locally compact subspace of X.

To show that the completeness is not hereditary, put $X = [0, 1]$ (the **unit interval** as a subspace of $\mathbb{R}^1$) and $Y = [0, 1)$ as a subspace of X. Then X is a complete metric space with respect to the usual metric, but Y is not a complete subspace, because any sequence in Y, convergent to 1 is a ***Cauchy sequence*** in Y which does not converge in Y. In the case of completeness it is to be noted that the Cauchyness depends on the metric.

The following properties are open hereditary but not hereditary ones.

(1) ***local compactness***,
(2) ***local connectedness***,
(3) ***separability***.

To show that above properties (1) and (2) are not hereditary, put $X = \mathbb{R}^1$ and $Y = \mathbb{Q}$ as a subspace of X. Then X satisfies properties (1) and (2), but Y does not satisfy either of the properties (1) or (2).

To show that the property (3) is not hereditary, the following space is used as a representative example.

Niemytzki plane
Let $X = \{(x, y) \in \mathbb{R}^2: y \geqslant 0\}$ be the subset of $\mathbb{R}^2$. As a base for open subsets at a point (x, y) for which $y > 0$, we choose the interiors of circles around (x, y), and for a point $(x, 0)$ on x-axis, we choose as a base sets consisting of the point $(x, 0)$ and the interior of a circle tangent to the x-axis at the point $(x, 0)$. The space X is called the **Niemytzki Plane** [E, 1.2.4] or the **Niemytzki space** [N, Example II.5, Example III.3]. Then X is separable, because the set of points, both of whose coordinates are rational numbers is a countable dense subset of X. On the other hand, Y is not a separable subspace, because Y is an uncountable ***discrete*** space.

There are many topological properties which are not closed hereditary and not open hereditary, as well. For example, ***connectedness*** is such a property. To see this take $\mathbb{R}^1$ and its subspaces $[0, 1] \cup [2, 3]$ and $(0, 1) \cup (2, 3)$. Then $\mathbb{R}^1$ is a connected space, and the first subspace is closed and the second one is an open subspace, but neither of them are connected.

For a hereditary property P, a space with the property P is called a **hereditary (P)-space** [N, p. 102]. A most familiar such property is metrizability; that is, any subspace of any metrizable space is metrizable.

For a property (P) which is not necessarily hereditary in general, we can classify spaces according to whether the property (P) hold hereditarily or not. For example, although the separability is not a hereditary property, the real line $\mathbb{R}^1$ is a hereditarily separable space. More generally, a new space is defined as the hereditarily (P)-space; for instance, a ***hereditarily normal*** space means the space in which any subspace is normal.

Generally speaking, there are two types of topological properties. The first one is the property which characterizes spaces themselves; for instance, compactness, paracompactness and normality etc. are conditions concerning for whole spaces, and the second one is the property which is characterized using subsets; for instance, separability and ***nowhere density*** etc. are characterized by the conditions for subsets.

For every topological space X and every subspace Y of X the formula $i_Y(x) = x$ defines a map i_Y of Y into X. The map i_Y is called the **embedding** of Y in X.

Let X and X_1 be two topological spaces. If for a subspace Y of X there exists a ***homeomorphism*** $f : X_1 \to Y$, then we call the space X_1 **embeddable** in X and the map $i_Y f : X_1 \to X$ a **homeomorphic embedding** of X_1 in X. Sometimes, X is called an **extension** of X_1 [E, p. 67].

Let P be a topological property and X a topological space. If for every point x of X there exists a neighbourhood V_x of x which has the property P, then we say that X has **property P locally**. A subset A of X is called a **locally closed subset**, if A is closed locally. This means that every point x of A has a neighbourhood such that $V_x \cap A$ is a closed subset of V_x. In other words, A is closed locally if and only if $A = O \cap F$ for some open subset O of X and some closed subset F of X. As an example of a locally closed subset, let $A = [0, 1)$ be the right open interval as the subset of $\mathbb{R}^1$. Then A is a locally closed subset of $\mathbb{R}^1$; in other word, A is the intersection of a closed interval $[0, 1]$ and an open interval $(-1, 1)$ [E, 2.7.1].

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b-2 Relative Properties

In many areas of general topology, we encounter pairs consisting of a space and a ***subspace***, or a pair consisting of a space and a larger space (i.e., a space containing it as a subspace). For example, a ***Tychonoff space*** X is a subspace of its ***Stone–Čech compactification***. Every ***separable metric space*** is a subspace of the ***Hilbert cube***.

Throughout this article, X will denote a ***topological space*** and Y a subspace of X.

1. Elementary relative separation properties

In order to understand the concepts of relativity, we start with the notion of relatively Hausdorff. A subspace Y is **Hausdorff in** (respectively **strongly Hausdorff in**) X, if for each distinct two points $y_1, y_2 \in Y$ (respectively $y_1 \in Y$ and $y_2 \in X$), there are disjoint ***open*** sets U_1, U_2 of X with $y_i \in U_i$ $(i = 1, 2)$. These concepts are different. In fact, if we let X be a space $\{1, 2, 3\}$ with an open base $\{\{1, 3\}, \{2\}\}$ and $Y = \{1, 2\}$ its subspace, then Y is Hausdorff in X but Y is not strongly Hausdorff in X.

A subspace Y is **regular in** (respectively **superregular in**) X if, for each $y \in Y$ and each closed in X subset F of X with $y \notin F$, there are disjoint open subsets U and V of X such that $y \in U$ and $F \cap Y \subset V$ (respectively $F \subset V$). Clearly we have:

(a) If X is a ***Hausdorff*** space, then Y is Hausdorff in X;
(b) If X is a ***regular*** space, then Y is superregular in X, and if Y is superregular in X, then Y is regular in X.

The inverse implications in the above statements are not necessarily true. For example, in the gap between regularity and superregularity, let X be a space consisting of all real numbers with the ***base*** $\{(a, b)\colon a, b \in X,\ a < b\} \cup \{X \setminus \{1/n\colon n \in N\}\}$ and $Y = \{0\}$ a subspace of X, where N denotes the set of all natural numbers. Then Y is regular in X but Y is not superregular in X [E, 1.5.6].

As another version of relative regularity, a subspace Y is **strongly regular in** X, if for each $x \in X$ and each closed in X subset F of X with $x \notin F$, there are disjoint open subsets U and V of X such that $x \in U$ and $F \cap Y \subset V$. In this way, we can consider many versions of relative $\mathcal{P}$ for a topological property $\mathcal{P}$.

As a consequence, there are two types of problems with respect to relativity, as below:

(1) Suppose X is a given space and $\mathcal{P}$ a topological property. Then find a condition $\mathcal{Q}$ for subspace Y of X in order that Y be relatively $\mathcal{P}$ in X.
(2) For a topological property $\mathcal{P}$, characterize Y in order that Y be relatively $\mathcal{P}$ in every space which contains Y as a subspace.

As for (2), a representative result is the following: A Hausdorff space Y is regular in every larger Hausdorff space if and only if Y is ***compact*** [1, Theorem 48].

2. Relative normality

A subspace Y is **normal in** (respectively **quasi-normal in**) X if for each pair E and F of disjoint closed subsets of X, there are disjoint open subsets U and V of X (respectively disjoint open in Y subsets U and V of Y) such that $E \cap Y \subset U$ and $F \cap Y \subset V$. A subspace Y is **strongly normal in** X if, for each pair E and F of disjoint closed subsets of Y, there are disjoint open subsets U and V of X such that $E \subset U$ and $F \subset V$. It is clear that if Y is strongly normal in X, then Y is normal, and if Y is normal, then Y is quasi-normal in X. In the ***Niemytzki Plane*** L, the subspace $L_1 = \{(a, 0) \in L\colon a \in \mathbb{R}\}$ is normal and is quasi-normal in L, but L_1 is not normal in L. If we consider any Tychonoff space Y which is not normal (for example, Niemytzki plane or the square of the ***Sorgenfrey line***), then Y is normal in its Stone–Čech compactification $\beta(Y)$ [E, 3.5.10], and Y is not strongly normal in $\beta(Y)$. This means that the normality of the larger space need not imply the relatively strong normality. On the other hand, if Y is ***dense*** in X and is normal in itself (respectively quasi-normal in X), then Y is strongly normal in X (respectively normal in X) ([3, 1.2.5] and [3, 1.2.6]).

Relating to relative regularity, we have: Y is regular in X and the space Y is ***Lindelöf***, then Y is strongly normal in X.

Concerning the extension property of real-valued continuous functions, a subspace Y is **weakly C-embedded** in X [3] if every real-valued continuous function f on Y has a Y-continuous extension f^* over X, where f^* is said to be **Y-continuous** if for each point $y \in Y$ and each ***neighbourhood*** V of $f^*(y) = f(y)$, there exists a neighbourhood U of y in X with $f^*(U) \subset V$. A subspace Y is **C-embedded** in X if f^* can be chosen to be continuous on X in the above definition. It is known that, if Y is a dense subset of a Tychonoff space, then Y is weakly C-embedded in X [1, Theorem 13].

A subspace Y is called to be **realnormal in** (respectively **strongly realnormal in**) X [3] if, for every disjoint closed in X (respectively in Y) subsets A and B of X (respectively of Y), there exists a Y-continuous map $f: X \to R$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Y is **weakly realnormal in** X if, for every disjoint closed subsets A and B of X, there exists a continuous map $f: Y \to R$ such that $f(A \cap Y) = \{0\}$ and $f(B \cap Y) = \{1\}$. The following statements are clear from the definitions:

(a) If Y is normal, then Y is weakly realnormal in X,
(b) If Y is realnormal in X, then Y is normal in X.

In general, the weak realnormality of Y in X does not imply the realnormality of Y in X [1]. However, for a dense subspace Y of a Tychonoff space X, the weak realnormality of Y in X implies the realnormality of Y in X. Furthermore, Y is strongly normal in X if and only if Y is a normal space and Y is weakly C-embedded in X [3].

Relating to the strong normality, Y is strongly realnormal in X if and only if Y is strongly normal in X [1].

As for the case in which Y is normal in every larger regular space, I.Yu. Gordienko proved: A ***Moore space*** Y is normal in every larger regular space if and only if Y is separable and ***metrizable*** [1, Theorem 48].

3. Relative compactness

In this part, we shall introduce the relative compactness type properties and how they are related with other relative separation properties. Y is **compact in** (respectively **Lindelöf in**) X if, every ***open cover*** of X has a finite (respectively countable) subcollection which is a ***cover*** of Y. It is seen that Y is compact in X if and only if, for every collection $\mathcal{F}$ of subsets of Y with the ***finite intersection property***, $\bigcap\{\overline{F}^X: F \in \mathcal{F}\}$ is not empty.

If Y is compact (respectively Lindelöf), then Y is compact (respectively Lindelöf) in X. But the converse need not be true, because every Tychonoff space Y is always compact in its Stone–Čech compactification $\beta(Y)$. D.V. Ranchin showed that if X is a regular ***T_1-space***, then the compactness of Y in X is equivalent to the compactness of $\overline{Y}^X$, where $\overline{Y}^X$ means the closure of Y in X [1, p. 95]. Furthermore, the following hold [1, p. 95]:

(a) If X is Hausdorff (respectively regular) and Y is compact (respectively Lindelöf) in X, then Y is normal in X.
(b) If Y is strongly regular in X and Y is Lindelöf in X, then Y is normal in X ([1, Theorem 34] and [1, Theorem 35]).

By the above fact (a), Y is regular if Y is compact in a Hausdorff space X, but Y need not be ***complete regularity*** under the same conditions (see [2]). Therefore a natural question is: Under what conditions, does the relative compactness imply the Tychonoff property? In regard this question, A.V. Arkhangel'skiĭ and I.V. Yaschenko proved that: If Y is compact in a Urysohn space, then Y is Tychonoff [1, Theorem 44], where a space X is a **Urysohn space**, if for every two points $x, y \in X$ $(x \neq y)$ there are neighbourhoods U and V of x and y with y$\overline{U} \cap \overline{V} = \emptyset$.

As for (b), the assumption that X is strongly regular in X cannot be weakened to Y to be regular in X even if X is Hausdorff.

It is natural to define relative ***countable compactness*** and to study its properties. A subspace Y of a space X is **countably compact in** X, if every countable open cover of X has a finite subcollection whose union contains Y. A.V. Arkhangel'skiĭ [2] gave some interesting results concerning relative countable compactness.

4. Relative paracompactness

A family $\mathcal{A}$ of subsets of a space X is said to be **locally finite at** Y **in** X, if every point $y \in Y$ has a neighbourhood U in X which intersects with at most finitely many members of $\mathcal{A}$. There are three basic versions of relative paracompactness [3]: Y is **1-paracompact in** (respectively **3-paracompact in**) X, if every open cover $\mathcal{U}$ of X has an open in X (respectively in Y) cover of X (respectively of Y) which is a refinement (respectively a partial refinement) of $\mathcal{U}$ and locally finite at Y in X. A subspace Y is **2-paracompact in** X, if every open cover $\mathcal{U}$ of X has an open in X cover of Y which is a partial refinement of $\mathcal{U}$ and locally finite at Y in X.

In some papers (for example, [1]), if Y is 2-paracompact in X, then Y is said to be **paracompact in** X. It is easily seen that:

(a) If Y is 1-paracompact in X, then Y is 2-paracompactness in X, and if Y is 2-paracompact in X, then Y is 3-paracompact in X.
(b) If Y is Lindelöf (respectively paracompact), then Y is 2-paracompact (respectively 3-paracompact) in X.

In the above statements, none of the inverse implications need be true [7]. Moreover, there is an example of a relative compact subspace which is not 1-paracompact in a larger space (see [3, 3.3.2]). By adding some conditions in the assumptions the following are true: If Y is dense in X and 3-paracompact in X, then Y is 2-paracompact in X [3, 3.1.4], and if Y is 2-paracompact (respectively 3-paracompact) in X and strongly regular in X, then Y is normal (respectively quasi-normal) in X [3, 3.1.5] (respectively [3, 3.1.8]).

Some characterizations of paracompactness can be extended to relative cases; for example, Y is 1-paracompact in X if and only if every open cover of X has an open ***refinement*** of X which is σ-locally finite at Y in X [7].

Suppose that a topological property $\mathcal{P}$ and a space Y are given. Then a natural question arises: What is the condition on Y in order that Y be relative $\mathcal{P}$ in every larger space containing Y? Concerning this question, the following results are known [7]: For a regular space Y, Y is 1 (respectively 2)-paracompact in every larger regular space if and only if Y is compact (respectively Lindelöf). For a space Y, Y is 3-paracompact in every larger space if and only if Y is para compact.

A subspace Y of X is **countably i-paracompact in** X $(i = 1, 2, 3)$ if every countable open cover satisfies the condition in the definition of i-paracompactness $(i = 1, 2, 3)$, respectively.

One of the most useful characterizations of ***countable paracompactness*** is F. Ishikawa's one [E, 5.2.1]. For the relative countable paracompactness, Y. Yasui showed that [13]: Y is countably 1 (respectively 2 , 3)-paracompact in X if and only if every countable increasing open cover $\{U_n: n \in N\}$ of X has a family $\{V_n: n \in N\}$ of open sets in X, which is a cover of Y and $\overline{V_n}^X \subset U_n$ (respectively $V_n \subset U_n$ and

$Y \cap \overline{V_n}^X \subset U_n$, $\overline{Y \cap V_n}^Y \subset U_n$) for each $n \in N$. A subspace Y is **s-normal in** X, if for every disjoint closed subsets E and F of X, there are disjoint open subsets U and V of X such that $E \subset U$ and $F \cap Y \subset V$. Relating to Dowker's characterization of countable paracompactness using the product with $[0, 1]$, the following hold: For an s-normal subspace Y in X and the ***convergent sequence*** $I_0 = \{0\} \cup \{1/n: n \in N\}$, Y is countably 2-(respectively 3-) paracompact in X if and only if the product $Y \times I_0$ is normal (respectively quasi-normal) in X [13].

As an example of a result concerning relative countable 1-paracompactness in every larger space, M.V. Matveev proved the following result: A Tychonoff space Y is countably 1-paracompact in every Tychonoff space which contains Y as a closed subspace if and only if Y is Lindelöf [2, Theorem 13.5].

5. Relative dimensions

For a subspace Y of X, the **relative covering dimension** of Y in X does not exceed n (denoted by $\dim(Y, X) \leqslant n$), if every finite open cover $\mathcal{U}$ of X has a finite open in Y cover $\mathcal{V}$ of Y, partial refinement such that $\operatorname{ord}(y, \mathcal{V}) \leqslant n + 1$ for each $y \in Y$. The **strong covering dimension of** Y **in** X does not exceed n (denoted by $\operatorname{Dim}(Y, X) \leqslant n$), if every finite open cover $\mathcal{U}$ of X has a finite open in X cover $\mathcal{V}$ of Y, partial refinement such that $\operatorname{ord}(x, \mathcal{V}) \leqslant n + 1$ for each $x \in X$.

There exist a normal space X and a subspace Y of X such that $\dim(Y, X) \neq \dim(Y)$ and $\dim(Y, X) \neq \dim(X)$ (see [11]). Since the inequality $\dim(Y_1, X_1) \leqslant \dim(Y_2, X_2)$ holds for $Y_1 \subset Y_2 \subset X_1 \subset X_2$, we have $\dim(Y, X) \leqslant \dim(Y)$ and $\dim(Y, X) \leqslant \dim(X)$. Although equality does not hold in general, the following are sufficient conditions for $\dim(Y, X) = \dim(Y)$: (a) Y is closed, or (b) Y is a normal subspace in itself and an ***F_σ-set*** in X, or (c) Y is ***C^*-embedded*** in X [6]. It is also known that if Y is a normal subspace in itself and ***z-embedded*** in X [MN, p. 65], then

$$\frac{1}{2}\operatorname{Dim}(Y, X) \leqslant \dim(Y, X) \leqslant \dim(Y)$$

[6]. Of course, dim and Dim do not coincide; that is, there exist a Tychonoff space X and a subspace Y of X with $\dim(Y, X) \neq \operatorname{Dim}(Y, X)$ [11, Example 5]. On the other hand, if Y is either strongly normal in X and $\dim(Y, X)$ is defined, or Y is a Lindelöf subspace in itself, then the equality $\dim(Y, X) = \operatorname{Dim}(Y, X)$ holds [11]. Furthermore, the inequality

$$\operatorname{Dim}(Y \cup Z, X) \leqslant \operatorname{Dim}(Y, X) + \operatorname{Dim}(Z, X) + 1$$

holds for any subspaces Y and Z of X [11].

Let X be a regular space, Y a subspace and $n \geqslant 0$. The **relative small inductive dimension** of Y in X [9], denoted by $\operatorname{ind}(Y, X)$, is defined as follows (by induction): $\operatorname{ind}(Y, X) \leqslant n$ if for any $y \in Y$ and any open neighbourhood U of y in X there exists an open neighbourhood V of y in X such that $y \in V \subset U$ and $\operatorname{ind}(\operatorname{Fr}_X(V) \cap Y, X) \leqslant n - 1$, where $\operatorname{ind}(\emptyset, X) = -1$.

Let X be a normal space, Y a normal subspace and n a non-negative integer. The **relative large inductive dimension** of Y in X [6], denoted by $\operatorname{Ind}(Y, X)$, is defined as follows: $\operatorname{Ind}(Y, X) \leqslant n$ if for any closed set $A \subset Y$ and any open set $U \subset X$ with $U \supset A$ there exists an open set V in X such that $A \subset V \subset U$ and $\operatorname{Ind}(\operatorname{Fr}_X(V) \cap Y, X) \leqslant n - 1$, where $\operatorname{Ind}(\emptyset, X) = -1$.

If either X is hereditarily normal, or Y is a ***perfectly normal*** subspace of a normal space X, then $\operatorname{ind}(Y, X) = \operatorname{ind}(Y)$ [9]. If Y is a separable metrizable subspace of a Tychonoff space (respectively a normal space) X, then $\operatorname{ind}(Y, X) = \operatorname{ind}(Y)$ [10] (respectively $\operatorname{Ind}(Y, X) = \operatorname{Ind}(Y)$ [6]). If X is completely regular, then $\operatorname{ind}(Y, X) \leqslant \operatorname{ind}(Y) + 1$ [10]. Furthermore, the following basic facts hold for $\operatorname{ind}(Y, X)$ [9]:

(a) $\operatorname{ind}(Y) \leqslant \operatorname{ind}(Y, X) \leqslant \operatorname{ind}(X)$, and
(b) $\operatorname{ind}(Z, Y) \leqslant \operatorname{ind}(Z, X) \leqslant \operatorname{ind}(Y, X)$ for $Z \subset Y \subset X$, and
(c) $\operatorname{ind}(Y_1 \cup Y_2, X) \leqslant \operatorname{ind}(Y_1, X) + \operatorname{ind}(Y_2, X) + 1$.

For Ind-versions in the above fact (a), there exist a compact space X and its dense metrizable subspace Y such that $\operatorname{Ind}(Y, X) > \operatorname{Ind}(X)$ [6]. For Ind-version of the above (c), if X is hereditarily normal, then we have $\operatorname{Ind}(Y_1 \cup Y_2, X) \leqslant \operatorname{Ind}(Y_1, X) + \operatorname{Ind}(Y_2, X) + 1$ [6].

6. Others

A.V. Arkhangel'skiĭ and I.Yu. Gordienko introduced the relatively local finiteness [4]. A subset A of a space X is **finitely located in** X, if any closed in X subset B of A is finite. For $x \in X$ and a subset Y of X, Y is said to be **relatively locally finite at** x **in** X if there exists a neighbourhood U of x in X such that $U \cap Y$ is finitely located in Y. Y is **relatively locally finite in** X, if Y is relatively locally finite at x in X for any $x \in Y$. Then we have: If every countable subspace Y of a relatively locally finite Hausdorff space X is relatively locally finite in X, then X is discrete [4, Theorem 1]. If a Lindelöf Hausdorff space X is relatively locally finite in X, then X is discrete [4, Theorem 2].

It is also known that every point x of a relatively locally finite space X has a neighbourhood U which is countably compact in X [4]. On the other hand, there is a countably compact and Hausdorff space X which is not finite (see [12]). The relatively local finiteness is not finitely productive [12]. To prove this, the following is the essential fact [12]: If a Hausdorff space X is relatively locally finite, then each point $x \in X$ has an open neighbourhood U such that every infinite subset $A \subset U$ has infinitely many ***accumulation point***s in $X \setminus U$.

A.V. Arkhangel'skiĭ, I.Yu. Gordienko, T. Nogura, C.E. Aull and many other mathematicians obtained results about relative metrizability, relative ***symmetrizability***, relative ***sequentiality*** and so on (see [5]). For the relative ***subparacompactness***, see [8].

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b-3 Product Spaces

1. The product topology on $X \times Y$

Let X and Y be ***topological spaces***. The **product topology** on $X \times Y$ is the ***topology*** having the collection $\mathcal{B} = \{U \times V\colon U$ is open in X, V is open in $Y\}$ as a ***base***. A product set $X \times Y$ with the product topology is called the **product** (or **product space**) of X and Y. Let $\mathbb{R}$ be the real line with the usual topology. Then the product topology on $\mathbb{R}^2$ $(= \mathbb{R} \times \mathbb{R})$ coincides with the usual topology of the plane $\mathbb{R}^2$.

Let $p_1 : X \times Y \to X$ be defined by the equation $p_1(\langle x, y\rangle) = x$ and let $p_2 : X \times Y \to Y$ be defined by the equation $p_2(\langle x, y\rangle) = y$. The maps p_1 and p_2 are called the **projection**s of $X \times Y$ onto its first and second factors, respectively. A map $f : X \to Y$ is called an **open map** (respectively a **closed map**) if for each ***open*** (respectively ***closed***) set U in X, $f(U)$ is an open (respectively a closed) set in Y. Each p_i is ***continuous*** and an open map. But p_i is not a closed map because the set $A = \{\langle x, y\rangle \in \mathbb{R}^2\colon xy = 1\}$ is a closed set of $\mathbb{R}^2$ but $p_i(A) = \mathbb{R} \setminus \{0\}$ is not a closed set of $\mathbb{R}$ for $i = 1, 2$.

It is an important problem whether $X \times Y$ has a topological property $\mathcal{P}$, or not, if both X and Y have $\mathcal{P}$, that is, whether $\mathcal{P}$ is a **productive property**.

A space X is a **T_1-space** if $\{x\}$ is a closed set of X for each $x \in X$. Since X is ***homeomorphic*** to $X \times \{y\}$ for a point $y \in Y$, if Y is a T_1-space, X is considered a closed subspace of $X \times Y$. Therefore, if X and Y are T_1-spaces and $\mathcal{P}$ is preserved by closed ***subspace*** and $X \times Y$ has $\mathcal{P}$, then X and Y have $\mathcal{P}$. A space X is a **regular space** if X is a T_1 and T_3-space. A space X is a **completely regular** space if X is a T_1 and $T_{3\frac{1}{2}}$-space. A space X is a **T_4-space** if any two disjoint closed sets are separated by open sets in X. A space X is a **normal space** if X is a T_1- and T_4-space. If X and Y are T_0 (respectively T_1, ***Hausdorff***, regular, completely regular), then so is $X \times Y$. The converse also holds.

Normality is not preserved by product spaces. In fact, the following example exists. The ***Sorgenfrey line*** S is the set $\mathbb{R}$ of all real numbers, having $\{[a, b)\colon a < b\}$ as a base. Then S is normal and $S^2 = S \times S$ is not normal. In fact, $A = \{\langle x, -x\rangle \in S^2\colon x \in \mathbb{Q}\}$ and $B = \{\langle x, -x\rangle \in S^2\colon x \in \mathbb{P}\}$ are disjoint closed sets but are not separated by open sets in S^2. Here $\mathbb{Q}$ is the set of all rationals and $\mathbb{P}$ is the set of all irrationals. The Sorgenfrey line S is a regular ***Lindelöf*** space. Since every regular Lindelöf space is a ***paracompact*** Hausdorff space and every paracompact Hausdorff space is normal, neither Lindelöfness nor paracompactness is preserved by product spaces in general. However, if X is ***compact*** and Y is paracompact (respectively Lindelöf, compact), then $X \times Y$ is paracompact (respectively Lindelöf, compact). Since normality, Lindelöfness, compactness and many other topological properties are preserved by closed subspaces, if $X \times Y$ has one of these properties and X and Y are T_1-spaces, then X and Y both have the same property.

For a finite family $\{X_i\colon i = 1, 2, \ldots, n\}$ of spaces, the product topology is defined analogously to the case of two spaces, i.e., having $\mathcal{B} = \{U_1 \times \cdots \times U_n\colon U_i$ is open in $X_i,\ i = 1, 2, \ldots, n\}$ as a base. The product topology of $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ and the usual topology on $\mathbb{R}^n$ are the same. Projections are defined analogously to the case of the product of two spaces.

In the following, all spaces are assumed to be Hausdorff spaces and all maps continuous and onto.

In 1951, C.H. Dowker proved that X is ***countably paracompact*** and normal if and only if $X \times \mathbb{I}$ is normal, where $\mathbb{I}$ is the closed unit interval with the usual topology. He posed the question whether every normal space is countably paracompact or not. A normal space X is called a **Dowker space** if it is not countably paracompact, in other word, if $X \times \mathbb{I}$ is not normal. In 1971, M.E. Rudin constructed a Dowker space [KV, Chapter 17]. Thus Dowker's problem was answered negatively.

In 1976, concerning the normality of product spaces, K. Morita posed the following three conjectures [MN, Chapter 3]:

MORITA'S CONJECTURE I. If $X \times Y$ is normal for any normal space Y, then X is a ***discrete space***.

This was answered affirmatively by M. Atsuji and M.E. Rudin.

In general, the product $X \times Y$ of a normal space X and a ***metric space*** Y need not be normal. This was shown by Michael [E, 5, 1.3] by constructing a hereditarily ***paracompact*** (i.e., every subspace is paracompact) space M, which is called the ***Michael line***, such that $M \times \mathbb{P}$ is not normal, where $\mathbb{P}$ is the space of all irrationals with the usual topology. On the other hand, Morita, by introducing the notion of ***P-spaces***, proved that a space Y is a normal P-space if and only if $X \times Y$ is normal for every ***metrizable space*** X. He also conjectured the following:

MORITA'S CONJECTURE II. A space X is metrizable if and only if $X \times Y$ is normal for every normal P-space Y.

K. Chiba, T.C. Przymusiński and M.E. Rudin [6] proved that under the assumption $V = L$, Morita's conjecture II holds. A related result of Morita states that a metrizable space X is **σ-locally compact** (the union of countably many ***locally compact*** closed subspaces) if and only if $X \times Y$ is normal for every normal and countably paracompact space Y [MN, Chapter 3, 3.6]. Morita conjectured that the assumption of metrizability of X can not be omitted:

MORITA'S CONJECTURE III. A space X is metrizable and σ-locally compact if and only if $X \times Y$ is normal for every countably paracompact space Y.

In 1998, Z. Balogh [1] proved that Morita's conjecture III holds. Further, in 2001, he [2] proved that Morita's conjecture II holds.

A space X is κ**-paracompact** (κ is an infinite cardinal number) if every ***open cover*** of cardinality $\leqslant \kappa$ has a ***locally finite*** open refinement. Morita proved that a normal space X is κ-paracompact if and only if $X \times \mathbb{I}^\kappa$ is normal. Since $\mathbb{I}^\kappa$ is compact, Morita generalized the notion of κ-paracompact to Z-paracompact, i.e., for a compact space Z, a space X is called Z**-paracompact** if $X \times Z$ is normal. It is known that normal κ-paracompactness is preserved by closed maps, i.e., if X is normal κ-paracompact and $f: X \to Y$ is a closed map, then Y is normal κ-paracompact. In 1967, Morita posed the conjecture: For any compact space Z, Z-paracompactness is preserved by closed maps. M.E. Rudin proved that this conjecture holds [MN, Chapter 4, 1.14]. She and M. Starbird proved that in case that Z is a metric space, the similar result holds, too. Moreover they proved the following: Let Y be a metric space and Z a compact space. Suppose that $X \times Y$ and $X \times Z$ are normal. Then $X \times Y \times Z$ is normal [MN, Chapter 4, 1.9]. See [MN, Chapter 4] for proofs.

The relationships between normality and countable paracompactness are interesting in the theory of product spaces. Morita proved: Suppose that X is normal and Y is nondiscrete metric. If $X \times Y$ is countably paracompact, then $X \times Y$ is normal. Rudin and Starbird proved the converse. So we have: (1) Suppose that X is normal and countably paracompact, and Y metric. Then $X \times Y$ is normal if and only if $X \times Y$ is countably paracompact [MN, Chapter 4, 1.4]. K. Nagami [22] proved that in the above Morita's theorem, $X \times Y$ can be replaced by an open set $G \subset X \times Y$, and T.C. Przymusiński [25] proved the converse. Thus we have: (2) Suppose that X is ***hereditarily normal*** (i.e., every subspace is normal) and hereditarily ***countably paracompact*** (i.e., every subspace is countably paracompact), Y metric, and G an open subspace of $X \times Y$. Then G is normal if and only if G is countably paracompact. T. Hoshina showed that the metric factor in (1) can be replaced by a Lašnev space [MN, Chapter 4, 2.8]. After that, A. Bešlagić and K. Chiba [3] showed that the metric factor in (2) can be replaced by a Lašnev space. Later, N. Zhong [31], H.J.K. Junnila and Y. Yajima [12] improved these results.

2. The product topologies on $\prod_{\lambda\in\Lambda} X_\lambda$

The Tychonoff topology and the box topology

Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of topological spaces and put $X = \prod_{\lambda\in\Lambda} X_\lambda = \{x = (x_\lambda)_{\lambda\in\Lambda} : x_\lambda \in X_\lambda \text{ for each } \lambda \in \Lambda\}$. This set X is called the **Cartesian product**. The **Tychonoff product topology** on X is the topology having the collection of sets of the form

$$\prod_{\lambda\in M} U_\lambda \times \prod_{\mu\in\Lambda\setminus M} X_\mu,$$

where U_λ is an open set in X_λ for each $\lambda \in M$ and M is a finite subset of Λ, as a base. The **Tychonoff product** (or **topological product**) of $\{X_\lambda : \lambda \in \Lambda\}$ is X with the Tychonoff topology and is denoted by $\prod_{\lambda\in\Lambda} X_\lambda$. This space also is called a **product space**; if all spaces are the same space Z then we speak of a **topological power** or simply a **power** of Z. The **box topology** on X is the topology having the collection of sets of the form

$$\prod_{\lambda\in\Lambda} U_\lambda,$$

where U_λ is an open set in X_λ for each $\lambda \in \Lambda$, as a base. If U_λ is an open subset of X_λ for each λ, then $\prod_{\lambda\in\Lambda} U_\lambda$ is called a **box** in $\prod_{\lambda\in\Lambda} X_\lambda$. The **box product** of $\{X_\lambda : \lambda \in \Lambda\}$ is X with the box topology and is denoted by $\square_{\lambda\in\Lambda} X_\lambda$. Suppose X is a topological space and ν is a cardinal number, then X^ν denotes the Tychonoff product and $\square^\nu(X)$ denotes the box product where ν is the index set and each factor X_λ is X. For each $\mu \in \Lambda$, let $p_\mu : \prod_{\lambda\in\Lambda} X_\lambda \to X_\mu$ be defined by the relation $p_\mu((x_\lambda)_{\lambda\in\Lambda}) = x_\mu$. The map p_μ is called the **projection** on X_μ. Each p_μ is continuous and an open map in both Tychonoff and box topologies.

If each X_λ is a T_1-space, then X_λ is a closed subspace of both $\prod_{\lambda\in\Lambda} X_\lambda$ and $\square_{\lambda\in\Lambda} X_\lambda$.

Let H be the set of all infinite sequences $(x_n)_{n=1}^\infty$ of real numbers such that $0 \leqslant x_n \leqslant 1/n$ for each n. For each $x = (x_n)_{n=1}^\infty$, $y = (y_n)_{n=1}^\infty \in H$, define

$$\rho(x, y) = \left(\sum_{n=1}^\infty (x_n - y_n)^2\right)^{1/2}.$$

Then ρ is a ***metric*** on H. The space H with this ***metric topology*** is called the **Hilbert cube**. The space H coincides with the Tychonoff product $\prod_{n=1}^\infty [0, 1/n]$. Therefore, H is homeomorphic to $\mathbb{I}^\omega$. On the other hand, let $\mathbb{R}^\infty$ be the set of all infinite sequences $(x_n)_{n=1}^\infty$ of real numbers with

$$\sum_{n=1}^\infty x_n^2 < \infty$$

with the metric defined similarly to the metric ρ above. Then $\mathbb{R}^\infty$ is called the **Hilbert space** and is homeomorphic to $\mathbb{R}^\omega$. The space $\mathbb{P}$ of irrationals with the subspace topology of $\mathbb{R}$ is homeomorphic to $\mathbb{N}^\omega$, where $\mathbb{N}$ is a countable discrete space. If each X_n is metrizable, then $\prod_{n\in\omega} X_n$ is metrizable. But, in general the Tychonoff product of an uncountable number of metrizable spaces is not metrizable. In fact 2^{ω_1} is not metrizable where 2 is the discrete space of two points. It is known that $\mathbb{N}^{\omega_1}$ is not normal.

Throughout this section we assume that all spaces contain at least two points. Let us consider normality, paracompactness and Lindelöfness of countable Tychonoff product $X = \prod_{n\in\omega} X_n$, where each X_n is a Hausdorff space. A necessary condition for X to be normal (respectively paracompact, Lindelöf) is that all **finite subproducts** of X, i.e., spaces $\prod_{n<m} X_n$ must be normal (respectively paracompact, Lindelöf). However, Przymusiński showed that this condition is not a sufficient condition by constructing the space X such that X^n is Lindelöf for all $n < \omega$ but X^ω is not normal (see [KV, Chapter 18, 2.6]). Concerning the relationship between normality and countable paracompactness of countable Tychonoff products (each space is a Hausdorff space), the following theorem is known: Suppose that all finite subproducts of a product space $X = \prod_{n<\omega} X_n$ are normal (respectively paracompact, Lindelöf). Then the followings are equivalent: (i) X is normal. (ii) X is countably paracompact. (iii) X is normal (respectively paracompact, Lindelöf) (see [KV, Chapter 18, 6.1]).

Comparison of the Tychonoff and the box topologies

I. Let $\{X_\lambda\colon \lambda \in \Lambda\}$ be a family of topological spaces and $F_\lambda \subset X_\lambda$ for each $\lambda \in \Lambda$. Then the following hold:

(1) $\prod_{\lambda\in\Lambda} F_\lambda$ is a closed subset of $\prod_{\lambda\in\Lambda} X_\lambda$ if and only if F_λ is a closed subset of X_λ for each $\lambda \in \Lambda$.
(2) $\prod_{\lambda\in\Lambda} F_\lambda$ is a closed (respectively open) subset of $\Box_{\lambda\in\Lambda} X_\lambda$ if and only if F_λ is a closed (respectively open) subset of X_λ for each $\lambda \in \Lambda$.
(3) For Tychonoff product, similar result for open sets does not hold. In fact, $\prod_{n\in\omega}(0,1)_n$ is not an open set in $\mathbb{R}^\omega$ where $(0,1)_n$ is the copy of $(0,1) = \{t\colon 0 < t < 1\}$.

II. Suppose that X is a topological space and $\{Y_\lambda\colon \lambda \in \Lambda\}$ is a family of topological spaces. Let $f : X \to \prod_{\lambda\in\Lambda} Y_\lambda$. Then f is continuous if and only if $p_\lambda \circ f$ is continuous for each $\lambda \in \Lambda$. This theorem does not hold if we replace $\prod_{\lambda\in\Lambda} Y_\lambda$ with $\Box_{\lambda\in\Lambda} Y_\lambda$. In fact, consider the function $f : \mathbb{R} \to \Box^\omega\mathbb{R}$ defined by the equation $f(t) = (t, t, \ldots, t, \ldots)$. Then $p_n \circ f$ is continuous for each n. But f is not continuous.

III. Separation axioms.

(1) $\prod_{\lambda\in\Lambda} X_\lambda$ is T_0 (respectively T_1, Hausdorff, regular, completely regular) if and only if X_λ is T_0 (respectively T_1, Hausdorff, regular, completely regular) for each $\lambda \in \Lambda$.
(2) $\Box_{\lambda\in\Lambda} X_\lambda$ is T_0 (respectively T_1, Hausdorff, regular, completely regular) if and only if X_λ is T_0 (respectively T_1, Hausdorff, regular, completely regular) for each $\lambda \in \Lambda$ (see [KV, Chapter 4, 1.2] and [13]).

IV. Products of compact spaces.

(1) Compactness preserved by Tychonoff product.
(2) Compactness can not be preserved in the box product. In fact, if Λ is infinite set and each X_λ is non-discrete regular space, then $\Box_{\lambda\in\Lambda} X_\lambda$ is not compact.
(3) $\Box^{\omega_1}(\omega + 1)$ is neither normal nor collectionwise Hausdorff [21]. Here $\omega + 1 = \{n\colon n \leqslant \omega\}$ with the ***order topology***, where ω is the first infinite ordinal. This space is a compact metric space. A space X is **collectionwise Hausdorff** if every closed discrete subset of X can be separated by disjoint open sets in X.
(4) It is not known whether $\Box^{\omega_1}(\omega + 1)$ is countably paracompact or not.

V. Products of countable number of metrizable spaces. If each X_n is metrizable, then $\prod_{n\in\omega} X_n$ is metrizable. But, if each X_n is an infinite non-discrete regular space, then $\Box_{n\in\omega} X_n$ is never metrizable. Thus

(1) $\Box^\omega\mathbb{R}$ is not metrizable.

Since every metrizable space is paracompact, "if every X_i is metrizable, then $\Box_{i\in\omega} X_i$ is paracompact" was conjectured. This conjecture was answered negatively by E.K. van Douwen. In 1975, he [10] proved that there is a family $\{X_n\colon n \in \omega\}$ of separable ***complete*** metric spaces such that $\Box_{n\in\omega} X_n$ is not normal, in fact:

(2) The box product of the family $\{\mathbb{P}\} \cup \{T_n\colon n \in \omega\}$, where T_n is a convergent sequence for each $n \in \omega$, is not normal.

M.E. Rudin [26] proved that assuming the ***Continuum Hypothesis*** (CH for short), the box product of countably many ***locally compact***, ***σ-compact***, metric spaces is paracompact. Therefore:

(3) (CH) $\Box^\omega\mathbb{R}$ is paracompact and $\Box^\omega(\omega + 1)$ is paracompact.
(4) If $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{d} = \mathfrak{c}$, then $\Box^\omega\mathbb{Q}$ is paracompact [20]. Here $\mathbb{Q}$ is the rationals with the usual topology and $\mathfrak{c}$ is the cardinality of $\mathbb{R}$. (See [20] for the definitions of $\mathfrak{b}$ and $\mathfrak{d}$.)

VI. Separability, ***first-countability*** and ***connectedness***.

(1) If each X_λ is separable and $|\Lambda| \leqslant \mathfrak{c}$, then $\prod_{\lambda\in\Lambda} X_\lambda$ is separable. Conversely, if $\prod_{\lambda\in\Lambda} X_\lambda$ is separable, then each X_λ is separable.
(2) If each X_n is first-countable, then $\prod_{n\in\omega} X_n$ is first-countable. The converse holds. However, 2^{ω_1} is not first-countable.
(3) $\prod_{\lambda\in\Lambda} X_\lambda$ is connected if and only if X_λ is connected for each $\lambda \in \Lambda$. Therefore, $\mathbb{R}^\omega$ is separable, first-countable and connected.
(4) Let $\{X_\lambda\colon \lambda \in \Lambda\}$ be a family of non-discrete regular spaces and Λ an infinite set. Then $\Box_{\lambda\in\Lambda} X_\lambda$ cannot be separable.
(5) Let $\{X_\lambda\colon \lambda \in \Lambda\}$ be a family of non-discrete regular spaces and Λ an infinite set. Then $\Box_{\lambda\in\Lambda} X_\lambda$ is neither first-countable nor connected. Therefore, $\Box^\omega\mathbb{R}$ is not separable, not first-countable and not connected.

A supplement of the study of box products

M.E. Rudin constructed interesting examples by using box products of ordinals. A space X is said to have the **property** $\mathcal{D}(\kappa)$ if each increasing open cover $\mathcal{U}$ of X with $|\mathcal{U}| = \kappa$ has a **shrinking**, i.e., there is an open cover $\{G(U)\colon U \in \mathcal{U}\}$ of X such that $\operatorname{cl} G(U) \subset U$ for each $U \in \mathcal{U}$. A normal space without the property $\mathcal{D}(\kappa)$ is called a **κ-Dowker space**. An

ω-Dowker space is a Dowker space. Rudin constructed a κ-Dowker space for each infinite cardinal κ. These spaces fullfiled the important role in the theory of normality of product spaces.

Moreover, she examined the paracompactness and normality of countable box products of ordinals. She obtained many results. For example, she [27] proved CH implies the box product of countably many σ-compact ordinals is paracompact. Normality and paracompactness of box products have been discussed in many papers. Extending results of Rudin [26], K. Kunun [19] proved that

(1) (CH) if each X_n is compact, then $\Box_n X_n$ is paracompact if and only if it is $L(X) \leqslant \omega_1$. Here $L(X)$ denotes the ***Lindelöf number*** of X,
(2) if each X_n is compact and $\Box_n X_n$ is paracompact, then $L(X) \leqslant \mathfrak{c}$,
(3) (under the assumption of the ***Martin's Axiom*** (MA for short)) if each X_n is compact and first-countable, then $\Box_n X_n$ is paracompact.

After that, van Douwen [11] proved:

(1) The following conditions are equivalent:
   (a) CH,
   (b) $\Box^\omega(2^\mathfrak{c})$ is paracompact,
   (c) $\Box^\omega(2^\mathfrak{c})$ is normal.
(2) $\Box^\omega(2^{\omega_2})$ is not normal.

An important tool in the study of separation and covering properties of box products is the so-called **nabla-product**: say $x \equiv y$ for $x, y \in X = \prod_{i=1}^\infty X_i$ iff $\{i: x_i \neq y_i\}$ is finite. The quotient of X by this equivalence relation is denoted $\nabla_{i=1}^\infty X_i$. If X carries the box topology then the ***quotient map*** is open, the space $\nabla_{i=1}^\infty X_i$ is a ***P-space***; it shares many properties with the box product.

3. Σ-products and σ-products

All spaces are assumed to be Hausdorff spaces. In 1959, H.H. Corson [7] introduced the definitions of Σ-products and σ-products and studied these spaces.

(a) Σ-products and σ-products with the Tychonoff topology. Let $\{X_\lambda: \lambda \in \Lambda\}$ be a family of topological spaces. Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be the Cartesian product with the Tychonoff topology. Take a point $p = (p_\lambda)_{\lambda\in\Lambda} \in X$. For each $x = (x_\lambda)_{\lambda\in\Lambda} \in X$, let $\mathrm{Supp}(x) = \{\lambda \in \Lambda: x_\lambda \neq p_\lambda\}$. Then the subspace

$$\Sigma(p) = \{x \in X: \text{ Supp}(x) \text{ is countable}\}$$

of X is called a **Σ-product** of $\{X_\lambda: \lambda \in \Lambda\}$ about p (p is called the **base point**). If "countable" is replaced by "finite", $\Sigma(p)$ is called a **σ-product**. We often write Σ instead of $\Sigma(p)$. For each countable (finite) set $F \subset \Lambda$, $\prod_{\lambda\in F} X_\lambda$ is called a **countable** (finite) **subproduct** of Σ. Σ-products and σ-products are ***dense*** subspaces of $\prod_{\lambda\in\Lambda} X_\lambda$. And a σ-product is a dense subspace of a Σ-product. For spaces $\{X_\lambda: \lambda \in \Lambda\}$, $\Sigma(p)$ may be different from $\Sigma(q)$ if $p \neq q$. For instance, suppose each X_λ is a copy of the closed unit interval $\mathbb{I}$ plus an isolated point ∞ and let Λ be uncountable, $p = (p_\lambda)_{\lambda\in\Lambda}$ with $p_\lambda = \infty$, $q = (q_\lambda)_{\lambda\in\Lambda}$ with $q_\lambda = 0$ for each $\lambda \in \Lambda$. Then $\Sigma(p)$ is not homeomorphic to $\Sigma(q)$.

If each X_λ is connected, then Σ-products and σ-products of $\{X_\lambda: \lambda \in \Lambda\}$ are connected. Every metric space can be embedded as a subspace of a Σ-product of copies of $\mathbb{I}$ [7].

A Σ-product (σ-product) is called proper (or non-trivial) if it is a proper subspace of X. A Σ-product is proper if and only if uncountably many spaces X_λ contain at least two points. Proper Σ-products contain the space ω_1 as a closed subspace. Here $\omega_1 = \{\alpha: \alpha < \omega_1\}$ with the order topology is ***countably compact*** and not compact. It is well known that if a countably compact space X is paracompact, then X is compact. Therefore proper Σ-products are never paracompact [KV, Chapter 18, 7.2]. By the same reason, proper Σ-products are never ***metacompact***, ***subparacompact***, ***submetacompact*** etc. Therefore normality-like properties of Σ-products have been discussed by many mathematicians. Σ-products of compact spaces may be non-normal. In fact, the Σ-product of uncountably many copies of $\omega_1 + 1$ is not normal [KV, Chapter18, 7.3]. Here $\omega_1 + 1$ is the space $\{\alpha: \alpha \leqslant \omega_1\}$ with the order topology. This space is compact.

A collection $\mathcal{F}$ of subsets of a space X is a **discrete family** if for every point $x \in X$, there is a ***neighbourhood*** U of x such that $|\{F \in \mathcal{F}: U \cap F \neq \emptyset\}| \leqslant 1$. A space X is **collectionwise normal** if for every discrete collection $\mathcal{F}$, there is a pairwise disjoint collection $\mathcal{U} = \{U(F): F \in \mathcal{F}\}$ of open sets such that $F \subset U(F)$ for each $F \in \mathcal{F}$. If a space X is collectionwise normal, then X is normal. Corson [7] proved that Σ-products of complete metric spaces are collectionwise normal. Therefore the Σ-product of copies of $\mathbb{R}$ is collectionwise normal. And he raised the question as to whether a Σ-product of copies of rational numbers is normal or not. In 1973, A.P. Kombarov and V.I. Malyhin solved this by showing that every Σ-product of separable metric spaces is (collectionwise) normal [KV, Chapter 18, p. 821]. In 1977, S.P. Gul'ko and M.E. Rudin independently proved that every Σ-product of metrizable spaces is normal [KV, Chapter 18, 7.4].

It is well known that every metric space is a paracompact ***p-space*** (due to Arhangel'skiĭ) with countable ***tightness***. A space X is a paracompact p-space if and only if it is a paracompact ***M-space*** in the sense of Morita if and only if X is the ***perfect*** preimage of a metric space, i.e., there exist a metric space T and a perfect map $f: X \to T$. In 1978, Kombarov generalized Gul'ko–Rudin's result by showing that every Σ-product of paracompact p-spaces $\{X_\lambda\}$ is (collectionwise) normal if and only if all spaces X_λ have countable tightness [KV. Chapter 18, 7.5].

A space X is said have the **shrinking property** if every open cover of X has a shrinking. If X is shrinking, then X is normal. In 1983, M.E. Rudin [28] proved that every Σ-product of metrizable spaces is shrinking. In 1985, A.L. Donne [9] generalized this result to the class of paracompact p-spaces with countable tightness.

It is not known whether a Σ-product of ***Lašnev spaces*** is normal or not. A Σ-product of copies of ω_1 is normal [15]. A Σ-product is not necessarily normal even if every countable subproduct is normal. In fact, in 1992, T. Daniel and G. Gruenhage [8] constructed the following example: A non-normal Σ-product in which every countable subproduct is paracompact, ***perfectly normal*** and first-countable.

However, if every finite subproduct is hereditarily ***separable*** (i.e., every subspace is separable), then the Σ-product is (collectionwise) normal if every countable subproduct is normal [18]. Every proper Σ-product is not hereditarily normal. In fact, Corson [7] proved: Let Σ be a Σ-product of uncountably many spaces, each space having at least two points and $x \in \Sigma$. Then $\Sigma \setminus \{x\}$ is not normal.

The real line $\mathbb{R}$ can be embedded as a closed subspace of the σ-product of countably many copies of $\mathbb{I}$ (see [5]). Corson [7] proved that

(1) if each X_λ is a separable metric space, then their σ-product is Lindelöf, and
(2) if each X_λ is σ-compact, then so is their σ-product. Therefore, the σ-product of copies of $\mathbb{R}$ is Lindelöf.

Concerning the normality of σ-products, Kombarov [15] proved the following theorem: If every finite subproduct of $\{X_\lambda\}$ is normal and if either (a) every X_λ is a ***κ-compact space*** (i.e., each open cover of cardinality $\leqslant \kappa$ has a finite subcover) of ***character*** $\leqslant \kappa$ or (b) every X_λ is a **strongly κ-compact space** (i.e., ***closure*** of each set of cardinality $\leqslant \kappa$ is compact) of tightness $\leqslant \kappa$, then the σ-product is normal and κ-paracompact. By this theorem, the σ-product of copies of ω_1 is normal. On the other hand, K. Chiba [5] showed that there exists a non-normal σ-product such that every finite subproduct is normal (and countably paracompact).

V.G. Pestov [24] proved: Assume that every finite product of spaces X_n, $n = 1, 2, \dots,$ is normal and countably paracompact. Then the σ-product of $\{X_n\colon n = 1, 2, \dots\}$ is normal and countably paracompact. Kombarov [16] proved: Let σ be a σ-product of an uncountable number of spaces, each space having at least two points, and $x \in \sigma$. Then $\sigma \setminus \{x\}$ is not normal. In particular, such a σ-product is not hereditarily normal. Later, Kombarov [17] improved this result to deduce that such a σ-product is not hereditarily pseudonormal and is not hereditarily countably paracompact. A space X is **pseudonormal** if any two disjoint closed sets one of which is countable are separated by open sets in X. Obviously, any normal space is pseudonormal.

Several papers have investigated results for σ-products of the following type:

(∗) Let $\mathcal{P}$ be a topological property. Let σ be a σ-product of spaces. If each finite subproduct of σ has property $\mathcal{P}$, then σ has $\mathcal{P}$.

First, Kombarov [14] proved that (∗) holds for $\mathcal{P}$ being paracompactness and Lindelöfness. After that, K. Chiba [4, etc.] proved that (∗) holds for $\mathcal{P}$ being the $\mathcal{B}$-property, collectionwise normality, shrinking, ***submeta-Lindelöfness*** if σ is normal, and H. Teng [30] proved that (∗) holds for $\mathcal{P}$ being metacompactness. Recently, M. Sakai and Y. Yajima [29] proved that (∗) holds for $\mathcal{P}$ being subparacompactness and submetacompactness.

On the other hand, K. Chiba [5] proved that (∗) does not hold for $\mathcal{P}$ being strong paracompactness, ***orthocompactness*** and star-Lindelöfness. A space X is called **strongly paracompact** if every open cover of X has a ***star-finite*** open refinement. A space X is called **star-Lindelöf** if for every open cover $\mathcal{U}$ of X, there is a countable subset $A \subset X$ such that $\mathrm{St}(A, \mathcal{U}) = X$.

(b) Σ-products and σ-products with the box topology. Σ-products and σ-products with box topology have been studied, too. L.B. Lawrence [21] proved that if $2^\omega = 2^{\omega_1}$, then the Σ-product, that is,

$$\Sigma = \bigl\{x \in \Box^{\omega_1}(\omega + 1)\colon \{\alpha \in \omega_1\colon x_\alpha \neq \omega\} \text{ is countable}\bigr\},$$

is non-normal (cf. Σ with the Tychonoff topology is normal). The followings are not known:

(1) Is Σ non-normal in ZFC?
(2) Does CH imply that Σ is paracompact?

For σ-products, (∗) holds for $\mathcal{P}$ being paracompactness (this was proved by van Douwen [10] in case of countable number of spaces and by P. Nyikos and L. Platkiewicz [23] in case of uncountable number of spaces). Refer to [KV, Chapter 9] for unmentioned definitions of covering properties.

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b-4 Quotient Spaces and Decompositions

Let Y be a set. For a ***topological space*** X and a surjection $f: X \to Y$, let $\mathcal{T}(f) = \{U \subset Y\colon f^{-1}(U)$ is open in $X\}$. Then $\mathcal{T}(f)$ is called the **quotient topology** (or **identification topology**) for Y determined by f. ($\mathcal{T}(f)$ is the largest topology on Y such that the map f is ***continuous***.) Let X and Y be topological spaces. Let $f: X \to Y$ be a surjection. Then f is called a **quotient map** (or **identification map**) if the topology in Y is exactly $\mathcal{T}(f)$; that is, U is open in Y if and only if $f^{-1}(U)$ is open in X, here we can replace "open" by "closed". The space Y is called the **quotient space** (or **quotient image**) of X by f. Also, f is called an **open map** (respectively **closed map**) if the image of each ***open*** (respectively ***closed***) subset of X is open (respectively closed) in Y. Every open (or closed) continuous map is a quotient map. Every one-to-one quotient map is a ***homeomorphism***.

Let X, Y, Z be topological spaces. Let $f: X \to Y$, $g: Y \to Z$ be surjections, and let $h = g \circ f$. Then the following basic facts hold:

(1) (a) If f is a quotient map, and h is continuous, then g is continuous.
 (b) If f and g are quotient maps, then h is a quotient map.
 (c) If h is a quotient map, and f and g are continuous, then g is a quotient map.

(2) Let f be continuous. For $A \subset X$ with $f(A) = Y$, if the restriction $f|A$ of f to A is a quotient map, then f is a quotient map.

(3) Let f be a quotient map. For $B \subset Y$, if B is open or closed in Y, or f is open or closed, then $f|(f^{-1}(B))$ is a quotient map; that is, the ***relative topology*** on B is exactly $\mathcal{T}(f|(f^{-1}(B)))$.

Let X be a set. Let $\mathcal{D}$ be a **decomposition** (or **partition**) of X; that is, $\mathcal{D}$ is a cover of X such that any two distinct members are disjoint. Let

$$R = \{(x, y)\colon x, y \in D \text{ for some } D \in \mathcal{D}\}.$$

Then R is an **equivalence relation** on X; that is, R is a subset of $X \times X$ having the following properties: For every $x \in X$, $(x, x) \in R$; if $(x, y) \in R$, then $(y, x) \in R$; and if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. Conversely, let R be an equivalence relation on X, and let

$$R[x] = \{y \in X\colon (x, y) \in R\}.$$

Then $\mathcal{D} = \{R[x]\colon x \in X\}$ is a decomposition of X. For a topological space X and a decomposition $\mathcal{D}$ of X, let $P: X \to \mathcal{D}$ be the **projection** (i.e., P maps each point $x \in X$ to the unique member of $\mathcal{D}$ containing x). Let $X(\mathcal{D})$ be the space $\mathcal{D}$ having the quotient topology determined by P (i.e., $\mathcal{D}' \subset \mathcal{D}$ is open in $X(\mathcal{D})$ if and only if $P^{-1}(\mathcal{D}')$ is open in X). The space $X(\mathcal{D})$ is called the **decomposition space** of X by $\mathcal{D}$. Namely, the decomposition space $X(\mathcal{D})$ is obtained from X by identifying the points belonging to the same member of $\mathcal{D}$, and a subset of $X(\mathcal{D})$ is open if and only if its inverse image by the projection P is open in X. Here, we can replace "open" by "closed". The space $X(\mathcal{D})$ is a ***T_1-space*** if and only if every member of $\mathcal{D}$ is closed in X.

Let X be a topological space, and R be an equivalence relation on X. Let X/R be the set of all equivalence classes of R, and let $P: X \to X/R$ be the natural map defined by $P(x) = R[x]$. Let X/R be the space having the quotient topology determined by the map P. The space X/R is called the **quotient space** of X by R. For example, let $\mathbb{R}$ be the real line with the usual topology, and let $R_1 = \{(p, q)\colon p - q$ is an integer$\}$, and $R_2 = \{(p, q)\colon p - q$ is rational$\}$. Then $\mathbb{R}/R_1$ is ***homeomorphic*** to a circle S^1, and $\mathbb{R}/R_2$ is an ***indiscrete space***. Let X be a topological space. For $A \subset X$, let

$$R_A = \{(a, b)\colon a, b \in A\} \cup \{(x, x)\colon x \in X - A\}$$

and $\mathcal{D}_A = \{\{x\}, A\colon x \in X - A\}$. Then $X/R_A = X(\mathcal{D}_A)$. X/R_A is the space X with A identified to a point, $[A]$, and it is written X/A. For example, $\mathbb{R}/\mathbb{Z}$, $\mathbb{Z}$ is the integers in $\mathbb{R}$, is not ***metrizable***. Let X be a topological space. Let $\mathcal{D}$ be a decomposition of X. Then the decomposition space $X(\mathcal{D})$ is the quotient space X/R, where $R = \{(x, y)\colon x, y \in D$ for some $D \in \mathcal{D}\}$. Conversely, let R be an equivalent relation on X. Then the quotient space X/R is the decomposition space $X(\mathcal{D})$, where $\mathcal{D} = \{R[x]\colon x \in X\}$.

Let X be a topological space. A decomposition $\mathcal{D}$ of X is **upper semi-continuous** if for each $D \in \mathcal{D}$ and each open set U containing D in X, there exists an open set V such that $D \subset V \subset U$ and V is the union of members of $\mathcal{D}$; equivalently, for any closed subset F of X, $\bigcup\{D \in \mathcal{D}\colon D \cap F \neq \emptyset\}$ is closed in X. Also, a decomposition $\mathcal{D}$ of X is **lower semi-continuous** if for any open subset U of X, $\bigcup\{D \in \mathcal{D}\colon D \cap U \neq \emptyset\}$ is open in X. The projection $P: X \to X(\mathcal{D})$ is a closed map (respectively open map) if and only if the decomposition $\mathcal{D}$ is upper semi-continuous (respectively lower semi-continuous).

For topological spaces X and Y, and a quotient map $f: X \to Y$, let $\mathcal{D} = \{f^{-1}(y)\colon y \in Y\}$ and $R = \{(x, x')\colon f(x) = f(x')\}$. Then Y is homeomorphic to $X(\mathcal{D}) = X/R$. Also, f is closed if and only if $\mathcal{D}$ is upper semi-continuous. For a ***metric space*** and an upper semi-continuous decomposition $\mathcal{D}$ of X, $X(\mathcal{D})$ is metrizable if and only if the

boundary $\operatorname{Fr} D$ of D is ***compact*** in X for every $D \in \mathcal{D}$ (in other words, the Hanai–Morita–Stone theorem in [E]). In particular, for a metric space X, X/A is metrizable if and only if A is a closed subset of X with $\operatorname{Fr} A$ compact.

Let us recall "weak topologies", which are very important topologies closely related to "quotient topologies". Let $\mathcal{A} = \{X_\alpha: \alpha \in A\}$ be a family of topological spaces. Let $X = \bigcup\{X_\alpha: \alpha \in A\}$, and let

$$\mathcal{T}(\mathcal{A}) = \{U \subset X: U \cap X_\alpha \text{ is open in } X_\alpha \text{ for each } \alpha \in A\}.$$

Then $\mathcal{T}(\mathcal{A})$ is called the **weak topology** in X with respect to (or determined by) the family $\mathcal{A}$ (cf. [1], etc.). ($\mathcal{T}(\mathcal{A})$ is the largest topology in X such that each inclusion map from X_α into X is continuous. Note that $F \subset X$ is closed in X if and only if each $F \cap X_\alpha$ is closed in X_α.) Let X be a topological space, and let $\mathcal{C}$ be a cover of X consisting of subsets, here each subset is a subspace of X, but it is not necessarily open or closed in X. Then X is said to have the weak topology with respect to (or determined by) the cover $\mathcal{C}$ if the topology of X is exactly $\mathcal{T}(\mathcal{C})$. For a topological space X and a cover $\mathcal{C}$ of X, following [3], let us use "X is **determined by** $\mathcal{C}$" instead of "X has the weak topology with respect to $\mathcal{C}$". (The term "weak topology" is used in a different way and meaning by functional analysts, etc.) Every topological space is determined by any ***open cover*** of it. Let $\{X_\alpha: \alpha \in A\}$ be a family of topological spaces such that the topologies of X_α and X_β agree on $X_\alpha \cap X_\beta$ for any $\alpha, \beta \in A$. Let $X = \bigcup\{X_\alpha: \alpha \in A\}$ be the space having the weak topology with respect to the family $\{X_\alpha: \alpha \in A\}$. If $X_\alpha \cap X_\beta$ is closed (respectively open) in X_α for any $\alpha, \beta \in A$, then each X_α is a closed (respectively open) subset of X retaining its original topology as a subspace of X, thus, X is determined by the ***closed cover*** (respectively open cover) $\{X_\alpha: \alpha \in A\}$. When the spaces X_α are pairwise disjoint, the space X is called the **topological sum** (or **direct sum**) of $\{X_\alpha: \alpha \in A\}$ (hence, X is determined by the closed and open disjoint cover $\{X_\alpha: \alpha \in A\}$). For a family $\{X_n: n \in N\}$, $N = \{1, 2, 3, \ldots\}$, of topological spaces such that each X_n is a subspace of X_{n+1}, let $X = \bigcup\{X_n: n \in N\}$ be the space having the weak topology with respect to $\{X_n: n \in N\}$ (hence, X is also determined by the cover $\{X_n: n \in N\}$). Then X is called the **inductive limit** (or **direct limit**) of $\{X_n: n \in N\}$, and it is denoted by $X = \varinjlim X_n$. For a topological space X determined by a cover $\{C_n: n \in N\}$, let $X_n = \bigcup\{C_m: m \leqslant n\}$ for each $n \in N$. Then $X = \varinjlim X_n$. Let X be a topological space, and let $\mathcal{F}$ be a closed cover of X. Then X is **dominated by** $\mathcal{F}$ [5] if the union of any subfamily $\mathcal{G}$ of $\mathcal{F}$ is closed in X, and the union is determined by $\mathcal{G}$. (Instead of "X is dominated by $\mathcal{F}$", we sometimes use "X has the **weak topology** (in the sense of K. Morita [9]); **hereditarily weak topology**; or **Whitehead weak topology**, with respect to $\mathcal{F}$".) For a topological space X and a closed cover $\mathcal{F}$ of X, if X is dominated by $\mathcal{F}$, then X is determined by the closed cover $\mathcal{F}$ which is ***closure-preserving***, but the converse need not hold even if $\mathcal{F}$ is countable. Every topological space is dominated by any ***hereditarily closure-preserving*** closed cover of it. For a topological space $X = \varinjlim X_n$, if all X_n are closed in X, then X is dominated by $\{X_n: n \in N\}$. As is well-known, every ***CW-complex*** (J.H.C. Whitehead, 1949) is dominated by a cover of compact metric subsets. Let X be a topological space dominated by a closed cover $\mathcal{F}$. Then X is ***normal*** (respectively ***paracompact Hausdorff***) if and only if each member of $\mathcal{F}$ is normal [5, 9] (respectively paracompact Hausdorff [5, 10]). In particular, if each member of $\mathcal{F}$ is metric, then X is an M_1**-space** (i.e., a topological space with a ***σ-closure-preserving base***) which is a countable union of metrizable closed subsets.

Let X be a topological space determined by a cover $\{C_\alpha: \alpha \in A\}$ (except (8)), and let Y and Z be topological spaces. Let $f: X \to Y$, and $g: Z \to X$ be surjections. Then the following basic facts on weak topologies hold:

(1) If A is open or closed in X, then A is determined by $\{A \cap C_\alpha: \alpha \in A\}$.
(2) For a cover $\mathcal{A}$ of X, if each C_α is contained in some member of $\mathcal{A}$, then X is determined by $\mathcal{A}$.
(3) If each C_α is determined by a cover $\mathcal{P}_\alpha$, then X is determined by $\bigcup\{\mathcal{P}_\alpha: \alpha \in A\}$.
(4) If each $f|C_\alpha$ is continuous, then f is continuous.
(5) (a) The map g is closed (respectively open) if and only if each $g|(g^{-1}(C_\alpha))$ is closed (respectively open).
    (b) Let g be continuous. If each $g|(g^{-1}(C_\alpha))$ is a quotient map, then g is a quotient map.
(6) (a) If f is a quotient map, then Y is determined by $\{f(C_\alpha): \alpha \in A\}$. The converse holds if each $f|C_\alpha$ is a quotient map.
    (b) Let g be closed continuous. If Z is ***regular*** and X is T_1, then Z is determined by $\{g^{-1}(C_\alpha): \alpha \in A\}$.
(7) Let T be the topological sum of $\{C_\alpha \times \{\alpha\}: \alpha \in A\}$. Define $h: T \to X$ by $h((x, \alpha)) = x$. Then h is a quotient map, and X is homeomorphic to the decomposition space $T(\mathcal{D})$, where $\mathcal{D} = \{h^{-1}(x): x \in X\}$.
(8) Let X be a T_1-space having a cover $\mathcal{P}$. Let $\mathcal{P}^* = \{\bigcup\mathcal{F}: \mathcal{F} \subset \mathcal{P} \text{ is finite}\}$. Then each ***countably compact*** subset of X is contained in some member of $\mathcal{P}^*$ if X is either determined by $\mathcal{P}^*$ such that $\mathcal{P}$ is point-countable (i.e., each $x \in X$ is in at most countably many $P \in \mathcal{P}$) [3], or X is dominated by $\mathcal{P}$ (with $\mathcal{P}$ closed).

We recall k-spaces and related spaces as classical quotient spaces. We assume that all topological spaces are Hausdorff for the remaining parts. A topological space X is a k**-space** (respectively **sequential space** [2]) if X is determined by the cover of all compact subsets (respectively all compact metric subsets) of X. Here, in the parenthetic part, we can replace "compact metric subsets" by "convergent sequences". (A convergent sequence means the union of the sequence and its limit point.) Also, we can omit "all", and replace "the cover" by "some cover". A topological space is a k_ω**-space** [7] if it is determined by a countable cover of compact subsets. A topological space is a **Lašnev space** if it is a closed continuous image of a metric space. A topological space X is a **Fréchet space** (or a **Fréchet–Urysohn**

space) if, whenever $x \in \bar{A}$, then there exists a sequence in A converging to x. A topological space X has **countable tightness** (or it is **countably tight**) if, whenever $x \in \bar{A}$, then $x \in \overline{B}$ for some countable $B \subset A$. A topological space has countable tightness if and only if it is determined by a cover of countable subsets (cf. [8]). ***First-countable*** spaces, or Lašnev spaces are Fréchet. Fréchet spaces are sequential. Sequential spaces are k-spaces having countable tightness. (k-spaces whose points are ***G_δ-sets*** are sequential.) ***Locally compact*** spaces and k_ω-spaces are also k-spaces. Every topological space determined by k-spaces is a k-space. Thus, every **locally k-space** (i.e., each point has a ***neighbourhood*** which is a k-space) is a k-space. We can replace "k-space" by "sequential space (or space of countable tightness)". As is well known, every k-space is precisely a quotient space of a (paracompact) locally compact space, and every sequential space is precisely a quotient space of a (locally compact) metric space. Also, every k_ω-space is precisely a quotient space of a ***Lindelöf*** locally compact space. For other basic properties of k-spaces and related spaces, see [E] (or [1]).

We recall canonical quotient spaces. Let α be an infinite cardinal. For each $\beta < \alpha$, let L_β be an infinite convergent sequence having a limit point x_β, and put $L_0 = \{p_n: n \in N\} \cup \{x_0\}$. Let T_α be the topological sum of $\{L_\beta: \beta < \alpha\}$. Define decomposition spaces S_α of T_α, and A_ω of T_ω as follows: For a decomposition

$$\mathcal{D}_\alpha = \big\{\{x\}, D_\alpha: x \in T_\alpha - D_\alpha\big\},$$

$D_\alpha = \{x_\beta: \beta < \alpha\}$, of T_α, let $S_\alpha = T_\alpha(\mathcal{D}_\alpha)$; that is, S_α is the quotient space obtained from T_α by identifying all the limit points x_β to a single point. Also, for a decomposition

$$\mathcal{D} = \big\{\{x\}, \{x_n,\ p_n\}: x \in T_\omega - E,\ n \in N\big\},$$

$E = \{x_n, p_n: n \in N\}$, of T_ω, let $A_\omega = T_\omega(\mathcal{D})$; that is, A_ω is the quotient space obtained from T_ω by identifying the limit point $x_n \in L_n$ with $p_n \in L_0$ for each $n \in N$. (S_ω is homeomorphic to the quotient space A_ω/L_0.) The space S_ω is called the **sequential fan**. The space A_ω is called the **Arens' space**, and it is denoted by S_2. (For $n < \omega$, S_n is defined inductively in a different way from the spaces S_α, and S_2 is the space of R. Arens (1950).) The space S_α is a Lašnev space dominated by a cover of α many convergent sequences. While, S_2 is a sequential space determined by a cover of countably many convergent sequences, but it is not a Fréchet space. Let X be a topological space containing an infinite convergent sequence with a limit point p. For each $n \in N$, let

$$Y_n = X^n \times \{p\} \times \{p\} \times \cdots$$

be a subspace of the countable product X^ω. Then the space $Y = \varinjlim Y_n$ contains a closed copy of S_ω, and S_2. (A (closed) copy of a topological space S means a (closed) subset which is homeomorphic to S.) In particular, the space $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ is not even Fréchet.

The canonical quotient spaces S_α (in particular, S_ω, S_{ω_1}, S_{2^ω}) and S_2 have played essential roles in the theory of quotient spaces, products of k-spaces, k-networks, etc. Let us give some of these topics. For details, see [MN, Chapter 8], [12–14], for example.

A topological space X is called **strongly Fréchet** (F. Siwiec, 1971) (or **countably bi-sequential** [8]) if, whenever $\{A_n: n \in N\}$ is a decreasing sequence with $x \in \overline{A_n - \{x\}}$ for every $n \in N$, then there exist $x_n \in A_n$ such that the sequence $\{x_n: n \in N\}$ converges to x. Strongly Fréchet spaces are Fréchet. Strongly Fréchet spaces have been used in some metrization theorems, characterizations for certain quotient images of metric spaces, and products of Fréchet spaces (or k-spaces); see [8], or [MN, Chapter 8], etc. The following hold in view of [11]:

(1) A regular Fréchet space is strongly Fréchet if and only if it contains no (closed) copy of S_ω.
(2) Let X be a regular k-space in which every point is a ***G_δ-set***, or a sequential ***hereditarily normal*** space. Then X is Fréchet if and only if it contains no (closed) copy of S_2.

A cover $\mathcal{P}$ of a topological space X is called a **k-network** (P. O'Meara, 1971) if, for any compact set K and any open set U with $K \subset U$, there exists a finite union S of members of $\mathcal{P}$ such that $K \subset S \subset U$. When $S = P \in \mathcal{P}$, such a cover $\mathcal{P}$ is called a **pseudobase** (E. Michael, 1966); and when K is any singleton, as is well-known, $\mathcal{P}$ is called a **network** (or **net**). Every base for a topological space is a k-network. In 1966, E. Michael proved that every quotient space of a ***separable*** metric space is characterized as a k-space with a countable pseudobase (equivalently, a countable k-network); see [8]. (Every quotient space of a separable metric locally compact space is characterized as a k_ω-space with a countable (k-) network.) k-networks have played important roles in characterizations for certain quotient images of metric spaces, generalized metric spaces and their metrizations, function spaces, and products of k-spaces, etc. For these topics, see [MN, Chapter 8], [3, 13, 14], for example. We note that a topological space X is a k-space with a point-countable k-network if one of the following cases holds:

(i) X is dominated by a closed cover of metric subsets.
(ii) X is Lašnev.
(iii) X is a quotient space of a metric space M by a quotient map f with each $f^{-1}(x)$ separable.

On the other hand, for a regular k-space X with a point-countable k-network, the following (a), (b), and (c) are equivalent [4]:

(a) X is first-countable (equivalently, X has a point-countable base [3]).
(b) X contains no (closed) copy of S_ω, and no S_2.
(c) X^ω is a k-space.

Thus, for case (i), (ii), or (iii) with X regular and M locally separable, X is metrizable if (b) or (c) holds. (For case (ii), we need not "X contains no (closed) copy of S_2" in (b).)

The result for case (iii) is, in other words, the following: For a ***locally separable*** metric space X, and a decomposition $\mathcal{D}$ of X with each $D \in \mathcal{D}$ separable, let the decomposition space $X(\mathcal{D})$ be regular. Then $X(\mathcal{D})$ is metrizable if and only if $\mathcal{D}$ satisfies the following: For $D_0, D_n, D_{nm} \in \mathcal{D}$ $(n, m \in N)$, here the case $D_n = D_0$ for all $n \in N$ (omitting L_0) is also admitted, if there exists a discrete family $\{L_0, L_n: n \in N\}$ of convergent sequences in X, where $L_0 = \{p_n: n \in N\} \cup \{p_0\}$ with the limit point p_0, $L_n = \{x_{nm}: m \in N\} \cup \{x_n\}$ with the limit point x_n, $p_0 \in D_0$, $p_n, x_n \in D_n$, and $x_{nm} \in D_{nm}$, then there exists a sequence $\{D_{nm_n}: n \in N\}$ such that $\bigcup\{D_{nm_n}: n \in N\}$ is not closed in X. Also, $X(\mathcal{D})$ is metrizable if and only if $(X(\mathcal{D}))^\omega$ is a k-space.

Finally, let us review products related to quotient spaces and quotient maps. In 1948, J.H.C. Whitehead proved the following essential theorem which is often used in the theory of products: For a locally compact space X, $\mathrm{id}_X \times f$, id_X is the identity map on X, is a quotient map for every quotient map f (the Whitehead theorem in [E]). Using this, in 1954, D.E. Cohen showed that, if X is locally compact, then the product space $X \times Y$ is a k-space for every k-space Y [E, 3.3.27]. However, as is well known, neither $\mathbb{Q} \times S_\omega$, $\mathbb{Q}$ is the rationals in $\mathbb{R}$, nor $S_\omega \times S_{2^\omega}$ is a k-space, here the latter example is essentially due to C.H. Dowker (1952) (cf. [1]). (We can replace "S_ω" by "S_2" in these examples.) The above examples also show that even if f is an identity map and g is a closed continuous map, $f \times g$ need not be a quotient map. In 1968, E. Michael [6] proved that the following (a), (b), and (c) are equivalent for a regular space X [E, 3.12.14]:

(a) X is locally compact (respectively ***locally countably compact***; that is, each point has a neighbourhood whose ***closure*** is countably compact).
(b) $\mathrm{id}_X \times g$ is a quotient map for every quotient map g (respectively quotient map g defined on a sequential space).
(c) $X \times Y$ is a k-space (respectively sequential space) for every k-space (respectively sequential space) Y.

In 1976, Y. Tanaka proved the following result holds for regular spaces X and Y: Let X be a first-countable space. Let Y be a Fréchet space, or a k-space in which every point is a G_δ-set. Then $X \times Y$ is a k-space if and only if X is locally countably compact, or Y is strongly Fréchet. (This result remains true when Y is a sequential hereditarily normal space. See [12] for generalizations of this result, and see [13] for products of k-spaces having point-countable k-networks.)

The following are basic results on products:

(1) For topological spaces X_i $(i = 1, 2)$ determined by covers $\mathcal{C}_i$, the following hold:
   (a) If $\mathcal{C}_1$ is a ***locally finite*** closed cover by locally compact subsets, then $X_1 \times X_2$ is determined by $\mathcal{C}_1 \times \mathcal{C}_2$ $(= \{C_1 \times C_2: C_i \in \mathcal{C}_i\})$.
   (b) If $\mathcal{C}_i$ are closed (or increasing) countable covers by locally compact subsets, then $X_1 \times X_2$ is determined by $\mathcal{C}_1 \times \mathcal{C}_2$ (hence, $X_1 \times X_2$ is a k-space).
   (c) Let $\mathcal{C}_1$ be a cover by locally compact subsets (respectively regular, locally countably compact sequential subsets), and let X_2 be a k-space (respectively sequential space). Then $X_1 \times X_2$ is determined by $\mathcal{C}_1 \times \mathcal{C}_2$ if and only if it is a k-space (respectively sequential space).

(2) For quotient maps $f_i: X_i \to Y_i$ $(i = 1, 2)$, if X_1 and $Y_1 \times Y_2$ are k-spaces, then $f_1 \times f_2$ is quotient ([7] or [E, 3.3.28]).

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b-5 Adjunction Spaces

The concept of adjunction spaces was first introduced by Borsuk [3] in 1935 for ***compact metric spaces***, and used by Hanner [4] and others to work out the theory of ***retraction*** or to construct counterexamples in this and other places. Let X, Y be ***topological space***s, simply called spaces from here. Let H be a ***closed*** subset of X and $f: H \to Y$ be a ***continuous*** map. We define the ***quotient space*** Z of the ***topological sum*** $X \oplus Y$ by the classification

$$\mathcal{D} = \{\{p\}: p \in X \setminus H\} \cup \{f^{-1}(p) \cup \{p\}: p \in Y\}.$$

In this case, we call Z the **adjunction space** of X and Y obtained from $f: H \to Y$, and denote it by $Z = X \cup_f Y$. Let $g: X \oplus Y \to Z$ be the ***quotient map***. Since $g|X \setminus H: X \setminus H \to g(X \setminus H)$ and $g|Y: Y \to g(Y)$ are ***homeomorphisms***, $X \setminus H$ and Y are considered ***open*** and closed ***subspaces*** of Z, respectively. Thus Z is the union of an open subspace $X \setminus H$ and a closed subspace Y, as Figure 1 shows.

From the definition, $U \subset Z$ is open if and only if both $U \cap Y$ and $k^{-1}(U)$ are open in Y and X, respectively, where $k = g|X: X \to Z$.

In the special case, when Y is a singleton, then the adjunction space $X \cup_f Y$, where $f: H \to Y$ is constant, is the quotient space obtained from X by identifying all points of H with the single point, and it is usually denoted by X/H. In this case, the quotient map $g: X \to X/H$ is a ***closed map***. Moreover, if H is a compact space, then g is in fact a ***perfect map***. In general, the quotient map $g: X \oplus Y \to X \cup_f Y$ is closed if and only if f is closed.

From the definition, the following holds: Let $\{X_\alpha: \alpha \in A\}$, $\{Y_\alpha: \alpha \in A\}$ be collections of spaces such that $X_\alpha \cap Y_\alpha = \emptyset$, $\alpha \in A$. For each α, let $f_\alpha: H_\alpha \to Y_\alpha$ be a continuous map of a closed subspace H_α of X_α into Y_α. Then $(\bigoplus_\alpha X_\alpha) \cup_f (\bigoplus_\alpha Y_\alpha)$ is homeomorphic with the topological sum of the adjunction spaces $X_\alpha \cup_{f_\alpha} Y_\alpha$, $\alpha \in A$, where $f: \bigoplus_\alpha H_\alpha \to \bigoplus_\alpha Y_\alpha$ is a map such that $f|H_\alpha = f_\alpha$, $\alpha \in A$ [E, 2.40(a)]. Let X_i, Y_i, $i = 1, 2$, be spaces such that $X_i \cap Y_i = \emptyset$, $i = 1, 2$. Let $f_i: H_i \to Y_i$ be continuous maps of closed subspace H_i of X_i into Y_i for $i = 1, 2$. Then the ***product space*** $(X_1 \cup_{f_1} Y_1) \times (X_2 \cup_{f_2} Y_2)$ is homeomorphic to the adjunction space $(X_1 \times X_2) \cup_f (Y_1 \times Y_2)$, where $f: H_1 \times H_2 \to Y_1 \times Y_2$ is the map defined by $f_1 \times f_2$ [E, 2.4D(b)].

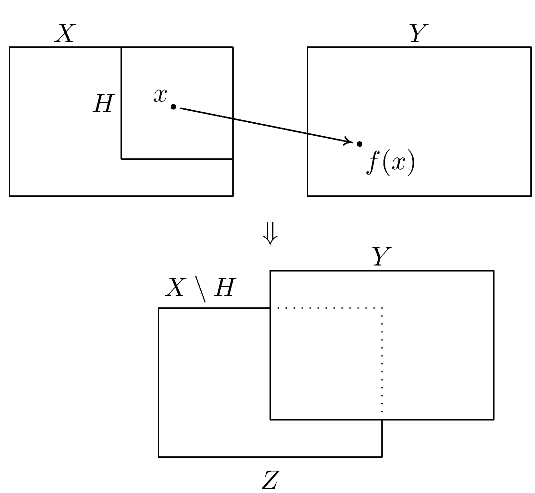

Fig. 1. Z.

After reviewing what has been done till now, the main topic on adjunction spaces could be stated as follows: (∗) Let $\mathcal{C}$ be a class of spaces. If both X and Y belong to $\mathcal{C}$, then does the adjunction space $X \cup_f Y$ belong to the same class? It is well known that if $\mathcal{C}$ is one of classes of ***perfectly normal*** spaces, ***normal*** spaces, ***collectionwise normal*** spaces, ***paracompact*** spaces, then this is the case for $\mathcal{C}$.

For the class of ***metrizable spaces***, this is not the case because X/A need not metrizable for a metrizable space X and a non-compact closed subset A of X. For this problem, Borges [2] showed that for metrizable spaces X and Y, $Z = X \cup_f Y$ is metrizable if Z is a space of ***point-countable type***. Similarly, Y.M. Liu and L.Y. Liu [5] showed $Z = X \cup_f Y$ with X, Y metrizable if and only if it satisfies the ***first-countable*** axiom.

There are some positive results to the class of generalized metric spaces. If $\mathcal{C}$ is a class of ***stratifiable*** spaces, then Borges [1] showed that (∗) is true. Also, Mizokami [8] showed the validity of (∗) for the class of M_3-***μ-spaces***, or equivalently M_3-***spaces*** with M-structures.

However, for the class of M_1-***spaces***, we do not know whether (∗) holds or not. It is not known even whether X/A is an M_1-space for an M_1-space X and a closed subset A of X. In addition, this problem is equivalent to the M_3 vs. M_1 problem. Let $\mathcal{P}$ be a class of M_1-spaces whose every closed subset has a ***closure-preserving*** open ***neighbourhood base***. If $X \in \mathcal{P}$ and Y is an M_1-space, then $Z = X \cup_f Y$ is an M_1-space, and if both X, $Y \in \mathcal{P}$, then so does Z [7].

As is well known, stratifiable spaces are factorized as ***monotonically normal***, ***semi-metrizable*** spaces. It is easy to see that the adjunction space of ***semistratifiable*** spaces is semistratifiable. For monotonically normal spaces, this is also the case, as shown by Sannou [13].

For G_δ-***diagonal*** properties, Mizokami [10] has shown that if X, Y are T_1-***spaces*** with a G_δ-diagonal and H is a closed G_δ-***set*** of X and $f: H \to Y$ is a continuous map, then $X \cup_f Y$ has a G_δ-diagonal.

Nagami defined L-***spaces*** [11] and **free** L**-space**s [12] in terms of anti-covers as nice generalizations of ***Lašnev spaces***. Concerning the adjunction spaces of these spaces, Miwa first showed that $X \cup_f Y$ is a free L-space if X is metrizable and Y is a free L-space [6]. This is refined in [9] as follows: $X \cup_f Y$ is a free L-space even if X is an L-space.

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b-6 Hyperspaces

All spaces are assumed to be ***Hausdorff topological space***s. In order to define the various ***topologies*** on the collection of ***closed*** subsets of a space X, we consider the following collections of sets:

(1) $\mathrm{CL}(X) = \{E: E \text{ is closed in } X\}$,
(2) $2^X = \{E: E \text{ is closed in } X \text{ and } E \neq \emptyset\}$,
(3) $\mathcal{K}(X) = \{K \in 2^X: K$ is ***compact***$\}$,
(4) $\mathcal{F}(X) = \{F: F \text{ is a non-empty finite set of } X\}$.

When these collections are topologized, they are called **hyperspaces** of X. The first step toward topologizing 2^X was taken by F. Hausdorff [8], who defined a ***metric*** ρ_H on 2^X, later called the **Hausdorff metric**, in the case when X is a ***bounded metric*** space as follows: For $A, B \in 2^X$,

$$\rho_H(A, B) = \max\Big\{\sup_{a\in A} \rho(a, B), \sup_{b\in B}(b, A)\Big\},$$

where ρ is a metric on X.

The so-called **Vietoris topology** was introduced by L. Vietoris [24] and later its basic neighbourhoods were identified by E.A. Michael in [14], under the name of **finite topology**. This topology has a ***subbase*** consisting of all sets of the form $\{U^-, V^+: U, V$ are ***open*** in $X\}$, where

$$U^- = \{A \in 2^X: A \cap U \neq \emptyset\},$$
$$V^+ = \{A \in 2^X: A \subset V\}.$$

This topology has the base consisting of all sets of the form

$$\langle U_1, \dots, U_k\rangle = \{A \in 2^X: A \subset U_1 \cup \dots \cup U_k \text{ and } U_i \cap A \neq \emptyset \text{ for } i = 1, \dots, k\},$$

where $\{U_1, \dots, U_k\}$ is a finite family of open subsets of X. $\mathcal{K}(X)$ and $\mathcal{F}(X)$ are considered as ***subspaces*** of 2^X. From here on, we denote the Vietoris topology by τ_V. In analyzing hyperspaces with this topology, the following basic relations are often useful:

(1) $\langle U_1, \dots, U_m\rangle \cap \langle V_1, \dots, V_n\rangle = \langle U_1 \cap V, \dots, U_m \cap V, U \cap V_1, \dots, U \cap V_n\rangle$, where $U = \bigcup_{i=1}^m U_i$ and $V = \bigcup_{i=1}^n V_i$.
(2) $\langle U_1, \dots, U_m\rangle \subset \langle V_1, \dots, V_n\rangle \rightleftharpoons \bigcup_{i=1}^m U_i \subset \bigcup_{i=1}^n V_i$ and for each i, there exists j such that $U_j \subset V_i$.
(3) $\mathrm{Cl}_{\tau_V}(\langle U_1, \dots, U_m\rangle) = \langle \overline{U_1}, \dots, \overline{U_m}\rangle$.
(4) If $\mathcal{U}$ is an open ***cover*** of X, or a ***base*** for X, then the collection

$$\langle \mathcal{U}\rangle = \{\langle U_1, \dots, U_k\rangle: U_1, \dots, U_k \in \mathcal{U}, k = 1, 2, \dots\}$$

is an open cover of $\mathcal{K}(X)$, or a base for $\mathcal{K}(X)$, respectively.
(5) $\widehat{X} = \{\{x\}: x \in X\}$ is a closed subspace of 2^X and $\mathcal{F}(X)$ is dense in 2^X.
(6) For a map $f: X \to Y$, let us define set-valued maps as follows:

$$f^*: \mathcal{K}(X) \to \mathcal{K}(Y) \text{ by } f^*(E) = f(E),$$
$$f^{-1*}: Y \to 2^X \text{ by } f^{-1*}(y) = f^{-1}(y),$$
$$f^{-1**}: \mathrm{CL}(Y) \to 2^X \text{ by } f^{-1**}(E) = f^{-1}(E).$$

Then the following hold: (i) f^* is ***continuous*** (a ***homeomorphism***) if and only if f is continuous (a homeomorphism), (ii) f^{-1*} is continuous if and only if f is ***open*** and ***closed***, (iii) f^{-1**} is continuous if and only if f^{-1*} is continuous. (These statements are mainly due to [14].)

The main themes in studying hyperspaces of a space X are summarized in the following two points:

(1) to investigate what properties of X are carried over to 2^X, $\mathcal{K}(X)$ or $\mathcal{F}(X)$,
(2) to determine whether 2^X, $\mathcal{K}(X)$, $\mathcal{F}(X)$ belong to $\mathcal{C}$ or not if X belongs to $\mathcal{C}$ for a class $\mathcal{C}$ of ***generalized metric spaces*** or spaces with some special properties.

Basic properties such as compactness, ***connectedness***, separation axioms were studied by Michael [14]. Some of these results are as follows: X is ***regular*** if and only if 2^X is Hausdorff; X is ***completely regular*** if and only if 2^X is a ***Stone space***; X is ***normal*** if and only if 2^X is completely regular if and only if 2^X is regular; X is compact if and only if 2^X is compact; X is compact ***metrizable*** if and only if 2^X is compact metrizable if and only if 2^X is metrizable; in particular, for compact metric spaces both the Vietoris topology and the Hausdorff metric topology are compatible. For $\mathcal{K}(X)$, the following properties of both X and $\mathcal{K}(X)$ are equivalent: Being Hausdorff, being a Stone space, being compact Hausdorff, being metrizable, being ***zero-dimensional***, being ***totally disconnected***, being ***second-countable***.

But this is not the case for being ***first-countable***. In fact, there exists a compact first-countable space X such that $\mathcal{K}(X)$ is not first-countable. R.E. Smithon [20] gave the following necessary and sufficient condition for $\mathcal{K}(X)$ to be first-countable: for each $K \in \mathcal{K}(X)$, there exists a countable collection $\mathcal{V}$ of open sets of X such that whenever $\{U_1, \dots, U_n\}$ is a minimal finite open cover of K there exists a minimal finite cover $\{V_1, \dots, V_m\} \subset \mathcal{V}$ of K such that $\bigcup_{j=1}^m V_j \subset \bigcup_{i=1}^n U_i$ and for each i there exists j such that $V_j \subset U_i$.

As a cardinal invariant, Čoban proved that for a space X, $\chi(\mathcal{K}(X)) \leqslant \alpha \geqslant \aleph_0$ if and only if $d(K) \leqslant \alpha$ and $\chi(K, X) \leqslant \alpha$ for each $K \in \mathcal{K}(X)$, where χ and d are the cardinal functions called the ***character*** and the ***density*** of the space. Čoban also proved that for a compact space X, $\chi(2^X) \leqslant \alpha \geqslant \aleph_0$ if and only if $hd(X) \leqslant \alpha$ and $\chi(A, X) \leqslant \alpha$ for every $A \in 2^X$, where hd is the cardinal function called *hereditary density* [E, 3.12.26(d)].

Normality and ***paracompactness*** do not behave well with respect to hyperspaces. V.M. Ivanova showed that even if X is a countable ***discrete space***, 2^X need not be normal [E, 2.7.20(f)]. J. Keesling [11] showed that if 2^X is normal, then X is ***countably compact*** and that the following are equivalent: (a) X is compact, (b) 2^X is compact, (c) 2^X is Lindelöf and (d) 2^X is paracompact. He also proved that under the assumption of the ***Continuum Hypothesis*** normality of 2^X is equivalent to compactness of X, [12]. Later N.H. Velichko [23] proved that 2^X is normal if and only if X is compact.

As for ***inverse limits***, S. Sirota proved that if $\mathcal{S} = \{X_\alpha, \pi^\alpha_\beta, \Delta\}$ is an ***inverse system*** of spaces, then

$$\widetilde{\mathcal{S}} = \{\mathcal{K}(X_\alpha), (\pi^\alpha_\beta)^*, \Delta\},$$

where $(\pi^\alpha_\beta)^* : \mathcal{K}(X_\alpha) \to \mathcal{K}(X_\beta)$ is defined above, is also an inverse system and $\varprojlim \widetilde{\mathcal{S}}$ is homeomorphic to the space $\mathcal{K}(\varprojlim \mathcal{S})$, and if $\mathcal{S}$ is an inverse system of compact spaces, then

$$\widetilde{\mathcal{S}} = \{\mathrm{CL}(X_\alpha), (\pi^\alpha_\beta)^*, \Delta\}$$

is also an inverse system such that $\varprojlim \widetilde{\mathcal{S}}$ is homeomorphic to $\mathrm{CL}(\varprojlim \mathcal{S})$ [E, 3.12.27(f)]. Using this result, it is proved that $\mathcal{K}(X)$ is ***Čech-complete*** if and only if X is Čech-complete, [E, 3.12.27(g)] and that $\mathcal{K}(X)$ is ***realcompact*** if and only if X is realcompact, [E, 3.12.27(h)]. For ***completeness*** of metric spaces, K. Kuratowski [13] showed that if a metric space (X, ρ) is complete, then $(\mathcal{K}(X), \rho_H)$ is complete. K. Morita [17] proved that if Φ is a ***uniformity*** compatible with the topology of a completely regular space X, then $\langle\Phi\rangle = \{\langle\mathcal{U}\rangle : \mathcal{U} \in \Phi\}$ is a uniformity of $\mathcal{K}(X)$ compatible with the topology of $\mathcal{K}(X)$, where $\langle\mathcal{U}\rangle$ is the same as above for an open cover $\mathcal{U}$ of X, and that $\mathcal{K}(X)$ is complete with respect to $\langle\Phi\rangle$ if X is complete with respect to Φ.

Recalling the bad behavior of normality and paracompactness to 2^X, stated above, it is better to restrict our interest to $\mathcal{K}(X)$ and $\mathcal{F}(X)$ for the study of the case of generalized metric spaces.

With respect to positive results, there are only a limited number of cases that apply to $\mathcal{K}(X)$. If $\mathcal{C}$ is one of the classes of metrizable spaces, ***Moore spaces***, $\aleph_0$-***space***s in the sense of Michael, $\aleph$-***space***s in the sense of P. O'Meara or paracompact $\aleph$-spaces, then both $\mathcal{K}(X)$ and $\mathcal{F}(X)$ belongs to $\mathcal{C}$ whenever X belongs to $\mathcal{C}$. If X is a paracompact p-***space*** in the sense of A.V. Arhangel'skiĭ, then $\mathcal{K}(X)$ is a paracompact p-space, but $\mathcal{F}(X)$ need not be so. The former part follows from the fact that f^*, in the above, is ***perfect*** if $f : X \to Y$ is perfect [4].

There are spaces indicating bad behavior of $\mathcal{K}(X)$ and $\mathcal{F}(X)$ in terms of the preservation of generalized metric spaces as follows:

(1) There exists an M_3-***space*** X such that $\mathcal{K}(X)$ is neither normal nor a k-***space*** or a σ-***space*** [3].
(2) There exists a compact space X such that $\mathcal{F}(X)$ is not a $w\Delta$-*space* [15].
(3) There exists a countably compact space X such that $\mathcal{F}_2(X) = \{F \in \mathcal{F}(X) : |F| \leqslant 2\}$ is not a $w\Delta$-space [15].
(4) There exists a space X with a G_δ-***diagonal*** such that $\mathcal{K}(X)$ has no G_δ-diagonal, [15].
(5) There exists an M_3-space X such that $\mathcal{K}(X)$ is not a Σ-***space*** [15].
(6) There exists a paracompact Σ-space X such that $\mathcal{F}(X)$ is not a Σ-space [15].
(7) For the ***Sorgenfrey line*** S, $\mathcal{F}_2(S)$ is not a d-paracompact in the sense of Brandenburg [16].
(8) There exists a countable ***Lašnev*** space X such that $\mathcal{F}(X)$ is not a ***free*** L-***space*** in the sense of Nagami [18].
(9) There exists a space X such that all finite power of X are MN (= ***monotonically normal***), but X is not ***linearly stratifiable*** such that $\mathcal{F}(X)$ is MN but not M_3 [7].

Being an M_3-space, being a paracompact σ-space, being a σ-space, and having a G_δ-diagonal of X are preserved to $\mathcal{F}(X)$, but not to $\mathcal{K}(X)$. Being a Lašnev space, being an L-***space*** in the sense of K. Nagami, being paracompact, being a $w\Delta$-space, being an M-***space*** in the sense of Morita, being a Σ-space, and being countably compact of X are preserved by neither $\mathcal{K}(X)$ nor $\mathcal{F}(X)$.

For M_3-ness of $\mathcal{K}(X)$, it is proved in [7] that $\mathcal{K}(X)$ is M_3 if an M_3-space X contains two disjoint convergent sequences and $\mathcal{K}(X)$ is ***separable*** MN.

There are other things to be noted. If X is a compact space, then assigning to each $f \in \mathcal{C}_k(X, Y)$, the mapping space with ***compact-open topology***, the ***graph*** $G(f) \subset X \times Y$ defines a homeomorphic embedding of $\mathcal{C}_k(X, Y)$ in $\mathcal{K}(X \times Y)$, and assigning to each pair $(f, A) \in \mathcal{C}_k(X, Y) \times \mathcal{K}(X)$ the image $f(A) \in \mathcal{K}(Y)$ defines a continuous map [E, 3.12.27].

In 1962, J.M. Fell [6] introduced another topology on hyperspaces, called the **Fell topology**. This topology has its subbase consisting of all sets of the form V^-, where V is an open set of X, plus all sets of the form $(X \setminus K)^+$, where $K \in \mathcal{K}(X)$. In other words, Fell topology is the topology with a base consisting of all sets of the form $\langle U_1, \ldots, U_n\rangle$, where $\{U_1, \ldots, U_n\}$ is a finite collection of open sets of X such that $X \setminus \bigcup_{i=1}^n U_i \in \mathcal{K}(X)$. From here, we denote the topology by τ_F.

As for separation axioms of τ_F G. Beer and R. Tamaki [2] proved that the following are equivalent:

(a) $(2^X, \tau_F)$ is Hausdorff,
(b) $(2^X, \tau_F)$ is regular,
(c) $(2^X, \tau_F)$ is completely regular,
(d) X is ***locally compact***.

L. Holá, S. Levi and J. Pelant [10] proved that the following are equivalent:

(a) $(2^X, \tau_F)$ is normal,
(b) $(2^X, \tau_F)$ is paracompact,
(c) $(2^X, \tau_F)$ is Lindelöf,
(d) $(2^X, \tau_F)$ is ***σ-compact*** and regular,
(e) X is locally compact and Lindelöf.

According to Beer [1, Theorem 5.1.5], the following are equivalent:

(a) $(2^X, \tau_F)$ is metrizable,
(b) $(2^X, \tau_F)$ is a ***Polish space***,
(c) $(\mathrm{CL}(X), \tau_F)$ is compact metrizable,
(d) X is locally compact and second-countable.

Fell [6] himself proved the following:

(1) $(\mathrm{CL}(X), \tau_F)$ is compact for any space X.
(2) If X is locally compact, then $(\mathrm{CL}(X), \tau_F)$ is compact Hausdorff.
(3) If we let $\mathcal{H}(X) = \mathrm{Cl}_{\tau_F}\{\{x\}: x \in X\} \subset \mathrm{CL}(X)$, then $\mathcal{H}(X)$ is the Hausdorff ***compactification*** of X.
(4) If X is a compact space, then $\mathcal{H}(X)$ and X are the same under the identification of x and $\{x\}$.
(5) If X is locally compact, but not compact, then $\mathcal{H}(X)$ is the ***one-point compactification*** of X with $\emptyset$ the point at infinity.

The concept of hit-and-miss topology stems from generalizing both of the topologies of Vietoris and Fell, and it has been studied by H. Poppe, Beer, Tamaki, G. Di Maio, and Holá. Let Δ be a non-empty subcollection of 2^X for a space X. By the **hit-and-miss topology** τ_Δ determined by Δ, we mean the topology having the subbase consisting of all sets of the form V^- with V open in X and all sets of the form $(X \setminus B)^+$ with $B \in \Delta$. It is needless to say that Vietoris topology is the one determined by $\Delta = 2^X$ and Fell topology is by $\Delta = \mathcal{K}(X)$. In terms of Δ, the **upper Δ-topology** τ_Δ^+ is defined as the topology generated by 2^X and the collection of all sets of the form $(X \setminus B)^+$ with $B \in \Delta$. Similarly, the **lower Vietoris topology** τ_V^- is one generated by the collection of all sets of the form V^- with V open in X. It is easy to see that τ_Δ is the join of τ_V^- and τ_Δ^+. Concerning the separation axiom of τ_Δ, Holá and Levi [9] proved the following:

(1) $(2^X, \tau_\Delta)$ is Hausdorff if and only if whenever V is a neighbourhood of x, there exist $D_1, \dots, D_n \in \Delta$ with

$$x \in \mathrm{Int}\bigcup_{i=1}^n D_i \subset \bigcup_{i=1}^n D_i \subset V.$$

(2) $(2^X, \tau_\Delta)$ is first-countable if and only if $(2^X, \tau_\Delta^+)$ and $(2^X, \tau_V^-)$ are first-countable.
(3) Let Δ contain the singletons. Then the following are equivalent: (a) $(2^X, \tau_\Delta)$ is metrizable, (b) there exists a countable family $\Delta' \subset \Delta$ such that for every $B \in \Delta$ and every open neighbourhood V of B, there exist $B_1, \dots, B_n \in \Delta'$ with

$$B \subset \mathrm{Int}\bigcup_{i=1}^n B_i \subset \bigcup_{i=1}^n B_i \subset V.$$

(c) $(2^X, \tau_\Delta)$ is second-countable and ***uniformizable***.

As for other properties and relation to the other topologies, refer to Beer [1].

C. Pixley and P. Roy [19] constructed a non-separable ***ccc*** (= ***countable chain condition***) Moore space by introducing a new topology to $\mathcal{F}(\mathbb{R})$. E.K. van Douwen [5] extended the idea of Pixley and Roy to create a new topology on 2^X for any space X, which is called **Pixley–Roy topology**, denoted by τ_{PR}. For subsets A and U of X, we define

$$[A, U] = \{E \in 2^X: A \subset E \subset U\}.$$

Then τ_{PR} is the topology with a base $\{[A, U]: A \in 2^X$ and U is an open neighbourhood of $A\}$. Obviously τ_{PR} is finer than τ_V. The following are due to van Douwen [5]:

(1) If X is a first-countable space, then $(\mathcal{F}(X), \tau_{\mathrm{PR}})$ is a Moore space.
(2) For any space, $(\mathcal{F}(X), \tau_{\mathrm{PR}})$ is hereditarily ***metacompact***.

H. Tanaka [22] showed that for a metric space M, the following are equivalent: (a) $(\mathcal{F}(M), \tau_{\mathrm{PR}})$ is normal, (b) $(\mathcal{F}(M), \tau_{\mathrm{PR}})$ is countably paracompact, (c) M is an almost strong q-set (a set M is called an **almost strong q-set** if for each natural number n every subset A of M^n with the property $\tau(A) = A$ for each permutation τ of $\{1, \dots, n\}$ is an ***F_σ-set*** of M^n). He also considered in another paper [21] metrizability and being an ***M_1-space*** of $(\mathcal{F}(X), \tau_{\mathrm{PR}})$.

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b-7 Cleavable (Splittable) Spaces

Let $\mathcal{P}$ be a class of ***topological space***s. A space X is **cleavable over** $\mathcal{P}$, or **splittable over** $\mathcal{P}$, or $\mathcal{P}$**-cleavable**, or $\mathcal{P}$**-splittable** provided that for every set $A \subseteq X$ there exist a space $Z \in \mathcal{P}$ and a ***continuous map*** $f_A : X \to Z$ (depending on A) such that $f_A(A) \cap f_A(X \setminus A) = \emptyset$, or equivalently, $A = f_A^{-1}(f_A(A))$. If Z is a space, then a space cleavable over the class $\{Z\}$ consisting of a single space Z is called **cleavable over** Z or **splittable over** Z. A space X is called **cleavable** or **splittable** if it is cleavable over the class of ***separable metric spaces*** (or equivalently, if it is cleavable over the ***Hilbert cube*** I^{ω}).

The general notion of a $\mathcal{P}$-cleavable space is due to Arhangel'skiĭ [1], while cleavable spaces were introduced (under the name "splittable spaces") and extensively studied in [7].

The original motivation for introducing splittable spaces in [7] came from $C_p(X)$-theory. We say that a space X has the **property of countable functional approximation** if for every real-valued function $f : X \to \mathbb{R}$ defined on X there exists a countable family of real-valued continuous functions $\{f_n : n \in \omega\}$ defined on X such that for any $\varepsilon > 0$ and every finite set $F \subseteq X$ one can find an $n \in \omega$ for which $|f(x) - f_n(x)| < \varepsilon$ for all $x \in F$. In other words, X has the property of countable functional approximation if and only if

$$\mathbb{R}^X = \big[C(X)\big]_{\omega} = \bigcup \big\{\overline{E} : E \subseteq C(X),\ |E| \leqslant \omega\big\},$$

where $\mathbb{R}^X$ is the set of all real-valued functions on X equipped with the ***Tychonoff product topology***, $C(X)$ is its ***subspace*** consisting of all continuous real-valued functions defined on X and the bar denotes the ***closure*** in $\mathbb{R}^X$. This property was introduced in [11] and studied extensively in [7]. One of the main results in [7] is that a ***Tychonoff space*** has the property of countable functional approximation if and only if it is cleavable.

The concept of cleavability generalizes the notion of one-to-one continuous map. Indeed, if $f : X \to Z$ is a one-to-one continuous map from X into Z, then X is obviously cleavable over Z because the same map f can be taken as f_A for all subsets A of X simultaneously. It is therefore natural to expect that some topological results about spaces X that admit a one-to-one continuous map into a space from class $\mathcal{P}$ may remain valid under a weaker assumption that X is cleavable over $\mathcal{P}$.

Let us give one particular example of this general conjecture. Since a one-to-one continuous map from a ***compact*** space X onto a ***Hausdorff space*** is a ***homeomorphism***, one may then expect that, for some "natural" subclasses $\mathcal{P}$ of the class of all Hausdorff spaces, a compact space cleavable over $\mathcal{P}$ must be ***homeomorphic*** to a subspace of some space from class $\mathcal{P}$. An example of positive result in this direction: a compact space cleavable over the reals $\mathbb{R}$ is homeomorphic to a subset of $\mathbb{R}$ [3].

When the class $\mathcal{P}$ is ***hereditary*** with respect to subspaces, we get the following particular case of the above conjecture: for many natural classes $\mathcal{P}$ of spaces it should be true that if a compact space X is cleavable over $\mathcal{P}$, then X belongs to $\mathcal{P}$ [2]. This conjecture holds for the following classes $\mathcal{P}$: spaces of cardinality $\leqslant \tau$ [2], ***countably tight*** spaces [1], ***sequential spaces*** [1], ***Fréchet–Urysohn*** spaces (Kočinac), ***bisequential*** spaces [2], ***first-countable*** spaces [2], ***metrizable*** spaces [7], spaces with G_δ***-diagonal***s [1], ***symmetrizable*** T_2-spaces [1]. It is not known if this conjecture holds when $\mathcal{P}$ is the class of ***linearly ordered topological space***s (LOTS) but the answer is positive under the additional assumption of ***connectedness***: If a ***continuum*** X is splittable over the class of all linearly ordered topological spaces, then X itself is a linearly ordered topological space [10]. Furthermore, if a continuum X of cardinality $\leqslant 2^{\omega}$ is splittable over a linearly ordered topological space L, then X can be embedded into L [10].

Some of the above results hold even for more general spaces close to being compact. If a Hausdorff ***countably compact*** space X is cleavable over the class of all sequential (Fréchet–Urysohn) Hausdorff spaces, then X is sequential (Fréchet–Urysohn) [2]. If a k***-space*** X is cleavable over the class of all spaces of countable ***tightness***, then the tightness of X is countable [2]. If a k-space X is cleavable over the class of all Fréchet–Urysohn spaces, then X is sequential [2]. If a ***pseudocompact*** space X is cleavable over the class of all metric spaces, then X itself is metrizable [7].

For a fixed natural number n, let $\mathcal{P}_{\mathrm{dim}}(n)$ be the class of all compact spaces Y with $\dim Y \leqslant n$, $\mathcal{P}_{\mathrm{ind}}(n)$ be the class of all compact spaces Y with $\operatorname{ind} Y \leqslant n$, and $\mathcal{P}_{\mathrm{Ind}}(n)$ be the class of all compact spaces Y with $\operatorname{Ind} Y \leqslant n$. If $\mathcal{P}$ is any of the above three classes, it is not known if a compact space cleavable over $\mathcal{P}$ must belong to $\mathcal{P}$ or not [5]. A positive answer to this question for the class $\mathcal{P}_{\mathrm{dim}}(n)$ is known to be consistent with ZFC [5]. Furthermore, the answer for all three classes is positive in case $n = 0$, as can be seen from the following more general result: If a space X is cleavable over the class of ***totally disconnected*** Hausdorff spaces then X itself is totally disconnected [5]. If a pseudocompact space X is cleavable over the class of all metrizable spaces Y such that $\dim Y \leqslant n$, then $\dim X \leqslant n$ (Tkachuk).

Let us now summarize principal results of Arhangel'skiĭ and Shakhmatov [7] about the classical cleavability (over $\mathbb{R}^{\omega}$).

Every cleavable space has a countable ***pseudocharacter*** but there exists a ***normal ccc*** **strongly** σ**-discrete** (i.e., a union of a countable family of its ***closed discrete*** subspaces) cleavable space that does not admit a one-to-one continuous map onto a bisequential (in particular, first-countable) space. (A space with somewhat weaker properties was first constructed in [11].) A space with a single non-***isolated point*** is cleavable if and only if its pseudocharacter is countable.

A pseudocompact space is cleavable if and only if it is metrizable. A ***Lindelöf*** space is cleavable if and only if it has a weaker separable ***metric topology***. In particular, Lindelöf spaces with a G_δ-diagonal, spaces with a countable ***network*** are cleavable. A Lindelöf ***scattered*** space is cleavable if and only if it is countable. A Lindelöf ***Σ-space*** is cleavable if and only if it has a countable network.

A ***Čech-complete*** cleavable ***paracompact*** space is metrizable but there exists a nonmetrizable ***locally compact*** cleavable space. Nevertheless, a Čech-complete cleavable space always has a ***dense*** subspace metrizable by a ***complete metric***. A locally compact metric space need not be cleavable. A cleavable complete metric space has cardinality 2^ω. A complete metric space without isolated points is cleavable if and only if it has cardinality 2^ω. A cleavable paracompact ***p-space*** is metrizable.

If a space X is covered by an increasing sequence of its subspaces $\{X_n: n \in \omega\}$ such that each X_n is cleavable and ***C^*-embedded*** in X, then X itself is cleavable. In particular, a normal space that can be represented as a union of an increasing sequence of its closed cleavable subspaces is cleavable. Therefore, a normal space covered by a countable family of its closed discrete subspaces is cleavable. This yields cleavability of ***perfectly normal metacompact*** scattered spaces (in particular, metrizable scattered spaces).

The class of cleavable spaces is not closed under ***products***: the product $I \times D(\tau)$ of the closed interval I and the discrete space $D(\tau)$ of size τ is cleavable if and only if $\tau \leqslant 2^\omega$ (note that both I and $D(\tau)$ are trivially cleavable).

If X is a cleavable space, then $2^{|X|} < 2^\omega \cdot 2^{d(X)}$, where $d(X)$ is the ***density*** of X. If X is a cleavable space and $|X| \geqslant 2^{2^\omega}$, then $|X| < 2^{d(X)}$. Under the ***Generalized Continuum Hypothesis*** GCH, a cleavable space X with $X \geqslant 2^{2^\omega}$ satisfies the equality $|X| = d(X)$. It is an open question if the same equality can be proved in ZFC. In a cleavable space the closure of every countable subset has size not bigger that 2^ω. Under GCH, the closure of every subset of size $\leqslant 2^\omega$ in a cleavable space has size $\leqslant 2^\omega$. Again, it is an open question if GCH is necessary in this result.

It is unclear if every cleavable space is ***Dieudonné complete***. However, if a space X is cleavable and the cardinality of X is an ***Ulam nonmeasurable*** cardinal, then X is ***realcompact*** [4].

Cleavability over finite dimensional Euclidean spaces $\mathbb{R}^n$ is of special interest. If a compact space X is cleavable over $\mathbb{R}^n$, then $\dim X \leqslant n$ [12]. If a separable n-dimensional ***manifold*** is cleavable over $\mathbb{R}^n$, then X is homeomorphic to an ***open*** subspace of $\mathbb{R}^n$; in particular, a compact n-dimensional manifold is not cleavable over $\mathbb{R}^n$ [5]. This implies that the n-dimensional sphere S^n is not cleavable over $\mathbb{R}^n$, and therefore $\mathbb{R}^{n+1}$ is not cleavable over $\mathbb{R}^n$ [5]. If a separable space X is cleavable over $\mathbb{R}^n$, then there exists a one-to-one continuous map of X into $\mathbb{R}^{n+1}$; in particular, a compact space cleavable over $\mathbb{R}^n$ can be embedded into $\mathbb{R}^{n+1}$ (Yaschenko). It is well known that the ***n-dimensional skeleton*** X_n of a $(2n+1)$-dimensional ***simplex*** cannot be embedded into $\mathbb{R}^{2n}$, and therefore X_n is not cleavable over $\mathbb{R}^{2n-1}$ by Yaschenko's result. This should be compared with a positive theorem of Tkachuk: An n-dimensional compact ***polyhedron*** is cleavable over $\mathbb{R}^{2n}$ [12]. The ***complete graph*** with five vertices is a one-dimensional compact polyhedron that cannot to be embedded into the plane $\mathbb{R}^2$ but is cleavable over $\mathbb{R}^2$. It is not known if every compact n-dimensional manifold is cleavable over $\mathbb{R}^{2n}$ [5].

Arhangel'skiĭ also studied spaces cleavable over the real line $\mathbb{R}$ and the rationals $\mathbb{Q}$ [3, 4]. A specific result that does not hold for cleavability over the plane $\mathbb{R}^2$ was mentioned earlier: a compact space cleavable over $\mathbb{R}$ is homeomorphic to a subspace of $\mathbb{R}$. No characterization of spaces cleavable over $\mathbb{R}$ is known but Arhangel'skiĭ [4] found a characterization of spaces cleavable over $\mathbb{Q}$: A space X is cleavable over $\mathbb{Q}$ if and only if it is a **strict Q-space**, i.e., if for every subset A of X there exist increasing families $\{A_n: n \in \omega\}$ and $\{B_n: n \in \omega\}$ of closed sets in X such that $A = \bigcup\{A_n: n \in \omega\}$, $X \setminus A = \bigcup\{B_n: n \in \omega\}$ and for each pair A_n, B_n there exists a ***clopen*** subset C_n of X with $A_n \subseteq C_n$, $B_n \subseteq X \setminus C_n$. Every strict Q-space is trivially a ***Q-set space***, i.e., all its subsets are ***F_σ-sets***, and so a space cleavable over $\mathbb{Q}$ is a Q-set space. A Q-set space with $\operatorname{Ind} X = 0$ is a strict Q-space, and thus cleavable over $\mathbb{Q}$. A space is cleavable over $\mathbb{Q}$ if and only if it is cleavable over the class of countable spaces [4].

A space X is said to have a **small diagonal** if for every uncountable set $A \subset X^2 \setminus \Delta$, there is an open neighbourhood U of Δ in X^2 such that $A \setminus U$ is uncountable, where $\Delta = \{(x, x): x \in X\}$. A cleavable space has a small diagonal [6]. A major problem left open in [7] was whether a cleavable space must have a G_δ-diagonal. This question was answered negatively by Balogh [8] who gave an example of a paracompact Q-space X without a G_δ-diagonal such that $\operatorname{Ind} X = 0$. By the results mentioned in the previous paragraph this X is cleavable over $\mathbb{Q}$ (and thus over $\mathbb{R}^\omega$ as well).

A topological space X is metrizable if and only if the product of X with any metric space is cleavable over the class of all metric spaces [9].

Every space can be represented as an image of a paracompact cleavable space under an ***open map*** [7]. Open ***perfect maps*** preserve cleavability, while perfect or open and closed maps do not [13].

The concept of cleavability in a most general setting is defined as follows. Let $\mathcal{P}$ be a class of topological spaces and let $\mathcal{M}$ be a class of maps. Arkhangel'skiĭ [1] calls a space X **$\mathcal{M}$-cleavable over $\mathcal{P}$** (also called **$\mathcal{M}$-splittable over $\mathcal{P}$**, **$(\mathcal{M}, \mathcal{P})$-cleavable**, **$(\mathcal{M}, \mathcal{P})$-splittable**) provided that for every set $A \subseteq X$ there exist a space $Z \in \mathcal{P}$ and a map $f_A: X \to Z$ (depending on A) such that $f \in \mathcal{M}$ and $f_A(A) \cap f_A(X \setminus A) = \emptyset$. The most interesting classes $\mathcal{M}$ of maps considered are closed maps and perfect maps. This line of investigation is being carried out by A.V. Arhangel'skiĭ,

A. Bella, M. Bonanzinga, F. Cammaroto, L.D. Kočinac, G. Tironi, A. Yakivchik and others.

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b-8 Inverse Systems and Direct Systems

1. Ordered sets

A **preordering** (or **quasi-order**) on a set Λ is a binary relation $\leqslant$ on Λ which is **reflexive**, i.e., $\lambda \leqslant \lambda$ for each $\lambda \in \Lambda$, and **transitive**, i.e., $\lambda \leqslant \lambda'$ and $\lambda' \leqslant \lambda''$ implies $\lambda \leqslant \lambda''$. The preordering $\leqslant$ is an **ordering** when it is **antisymmetric**, i.e., $\lambda \leqslant \lambda'$ and $\lambda' \leqslant \lambda$ imply $\lambda = \lambda'$. A preordered set $(\Lambda, \leqslant)$ is **directed** provided, for each $\lambda', \lambda'' \in \Lambda$, there is a $\lambda \in \Lambda$ such that $\lambda', \lambda'' \leqslant \lambda$. We say $(\Lambda, \leqslant)$ is **cofinite preordered set** provided it is an ordering and each element $\lambda \in \Lambda$ admits only finitely many predecessors, $\lambda_1, \lambda_2, \ldots, \lambda_n \leqslant \lambda$.

The set $\mathbb{N} = (\mathbb{N}, \leqslant)$ of all positive integers with the usual order $\leqslant$ and a singleton set $\{*\}$ are cofinite directed ordered sets.

A function $f : M \to \Lambda$ between preordered sets $M = (M, \leqslant)$ and $\Lambda = (\Lambda, \leqslant)$ is **increasing** provided $f(\mu) \leqslant f(\mu')$ for $\mu \leqslant \mu'$ in M.

2. Direct systems and limits

A **direct system** $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ of ***topological spaces*** consists of a directed ordered set $\Lambda = (\Lambda, \leqslant)$, topological spaces X_λ for $\lambda \in \Lambda$, and ***continuous*** maps $p_{\lambda\lambda'} : X_\lambda \to X_{\lambda'}$ for $\lambda \leqslant \lambda'$ (the **bonding maps**) satisfying the conditions $p_{\lambda\lambda} = \mathrm{id}$ for $\lambda \in \Lambda$ and $p_{\lambda'\lambda''} p_{\lambda\lambda'} = p_{\lambda\lambda''}$ for $\lambda \leqslant \lambda' \leqslant \lambda''$.

A **direct sequence** $\{X_i, p_{i,i+1}\}$ of topological spaces is a direct system $\{X_i, p_{ii'}, \mathbb{N}\}$ where

$$p_{ii'} = p_{i+k,i'} \circ \cdots \circ p_{i+1,i+2} \circ p_{i,i+1}$$

for $i' = i + k + 1$ and $p_{ii'} = \mathrm{id}$ for $i' = i$.

For any direct system $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$, consider the ***direct sum*** $\sum X_\lambda$. Two elements $x_\lambda \in X_\lambda$ and $x_{\lambda'} \in X_{\lambda'}$ are said to be **equivalent** if $p_{\lambda\lambda''}(x_\lambda) = p_{\lambda'\lambda''}(x_{\lambda'})$ for some $\lambda'' \geqslant \lambda, \lambda'$. This relation R defines an ***equivalence relation*** on $\sum X_\lambda$, and the ***quotient space*** $X = \sum X_\lambda / R$ is called the **direct limit** (or limit, for short) of $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$. Then the natural projections $p_\lambda : X_\lambda \to X$ for $\lambda \in \Lambda$ satisfy $p_\lambda = p_{\lambda'} p_{\lambda\lambda'}$ for $\lambda \leqslant \lambda'$.

In a similar way, direct systems of sets, (Abelian) groups are defined, and their limits exist.

3. Inverse systems and limits

An **inverse system** $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ of topological spaces consists of a directed ordered set $\Lambda = (\Lambda, \leqslant)$, topological spaces X_λ for $\lambda \in \Lambda$, and continuous maps $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ for $\lambda \leqslant \lambda'$ (the **bonding maps**) satisfying the conditions $p_{\lambda\lambda} = \mathrm{id}$ for $\lambda \in \Lambda$ and $p_{\lambda\lambda'} p_{\lambda'\lambda''} = p_{\lambda\lambda''}$ for $\lambda \leqslant \lambda' \leqslant \lambda''$.

An **inverse sequence** $\{X_i, p_{i,i+1}\}$ of topological spaces is an inverse system $\{X_i, p_{ii'}, \mathbb{N}\}$ where

$$p_{ii'} = p_{i,i+1} \circ p_{i+1,i+2} \circ \cdots \circ p_{i+k,i'}$$

for $i' = i + k + 1$ and $p_{ii'} = \mathrm{id}$ for $i' = i$.

The **inverse limit** (or limit for short) of $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is the ***subspace*** of the ***product*** $\prod_{\lambda \in \Lambda} X_\lambda$ consisting of all points $x = (x_\lambda)$ such that $p_{\lambda\lambda'}(x_{\lambda'}) = x_\lambda$ for $\lambda < \lambda'$. Then the ***projections*** $p_\lambda : X \to X_\lambda$ for $\lambda \in \Lambda$ satisfy $p_\lambda = p_{\lambda\lambda'} p_{\lambda'}$ for $\lambda < \lambda'$.

The limit X of an inverse system of (non-empty) ***compact Hausdorff*** spaces X_λ is a (non-empty) compact space. However, this is not true for noncompact spaces. For example, the inverse sequence consisting of the open intervals $(0, 1/n)$ $(n = 1, 2, \ldots)$ and inclusions has an empty limit.

In a similar way, inverse systems of sets, (Abelian) groups, compact spaces, ***topological group***s are defined, and their limits exist as a set, (Abelian) group, compact space, topological space, respectively. However, the limit of an inverse system of compact ***polyhedra*** does not exist as a compact polyhedron in general. For example, there is a decreasing sequence of compact polyhedra whose intersection X is not a polyhedron. Such a sequence is viewed as an inverse sequence $\{X_i, p_{i,i+1}\}$ with each $p_{i,i+1}$ being an inclusion and its limit being X. Moreover, if we consider an inverse system $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ of topological spaces with each $p_{\lambda\lambda'}$ being ***homotopy classes***, its limit may not exist at all ([14, p. 56], [4]).

In general, we can define an inverse system and its limit for a ***category*** $\mathcal{C}$ as follows: An inverse system $\boldsymbol{X} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ in the category $\mathcal{C}$ indexed by Λ consists of a directed ordered set $\Lambda = (\Lambda, \leqslant)$, of ***objects*** X_λ for $\lambda \in \Lambda$, and of ***morphisms*** $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ for $\lambda \leqslant \lambda'$ satisfying the conditions: $p_{\lambda\lambda} = \mathrm{id}$ for $\lambda \in \Lambda$ and $p_{\lambda\lambda'} p_{\lambda'\lambda''} = p_{\lambda\lambda''}$ for $\lambda \leqslant \lambda' \leqslant \lambda''$.

Before we define limits in $\mathcal{C}$, we will introduce the notion of pro-category.

A **system morphism** $\boldsymbol{f} = \{f, f_\mu\} : \boldsymbol{X} \to \boldsymbol{Y}$ between inverse systems $\boldsymbol{X} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ and $\boldsymbol{Y} = \{Y_\mu, q_{\mu\mu'}, M\}$ consists of a function $f : M \to \Lambda$, of morphisms $f_\mu : X_{f(\mu)} \to Y_\mu$ in the category $\mathcal{C}$ satisfying the condition

For any $\mu, \mu' \in M$ with $\mu \leqslant \mu'$, there is a $\lambda \in \Lambda$ such that $f(\mu), f(\mu') \leqslant \lambda$ and
$$f_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{f(\mu')\lambda}.$$

If $\boldsymbol{Z} = \{Z_\nu, r_{\nu\nu'}, N\}$ is an inverse system in $\mathcal{C}$ and if $\boldsymbol{g} = \{g, g_\nu\} : \boldsymbol{Y} \to \boldsymbol{Z}$ is a system morphism, then we define the **composition of morphisms** $\boldsymbol{h} = \{h, h_\nu\} : \boldsymbol{X} \to \boldsymbol{Z}$ of $\boldsymbol{f}$ and $\boldsymbol{g}$ as follows: $h = fg : N \to \Lambda$ and $h_\nu = g_\nu f_{g(\nu)} :$

$X_{h(\nu)} \to Z_\nu$ for each $\nu \in N$. Thus the defined $\boldsymbol{h}$ is a system morphism, and the associativity of compositions holds. We can also define the **identity morphism** $\mathbf{1}_X = \{1, 1_{X_\lambda}\} : \boldsymbol{X} \to \boldsymbol{X}$ where $1 = 1_\Lambda : \Lambda \to \Lambda$ and $1_{X_\lambda} : X_\lambda \to X_\lambda$ are the identities. Thus, all inverse systems and system morphisms form a category inv-$\mathcal{C}$.

Now we define an equivalence relation $\equiv$ on system morphisms as follows: Let $\boldsymbol{f}' = \{f', f'_\mu\} : \boldsymbol{X} \to \boldsymbol{Y}$ be a system morphism. We say that $\boldsymbol{f}$ is **congruent** to $\boldsymbol{f}'$, in notation, $\boldsymbol{f} \equiv \boldsymbol{f}'$, provided that for each $\mu \in M$ there is a $\lambda \in \Lambda$ with $f(\mu), f'(\mu) \leqslant \lambda$ and $f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}$. The **equivalence class** of the system morphism $\boldsymbol{f}$ is denoted by $[\boldsymbol{f}]$. It is easy to see that composition in inv-$\mathcal{C}$ induces a well-defined composition of equivalence classes. Thus, all inverse systems and all equivalence classes of system morphisms form a category pro-$\mathcal{C}$, which is called the **pro-category** of $\mathcal{C}$.

In particular, if $\boldsymbol{f} : (X) \to \boldsymbol{Y}$ is a morphism of pro-$\mathcal{C}$ from a **rudimentary inverse system**, i.e., an inverse system indexed by a singleton (X), to an inverse system $\{Y_\mu, q_{\mu\mu'}, M\}$, then it satisfies the property:

(PROJ) $f_\mu = q_{\mu\mu'} f_{\mu'}$ for $\mu \leqslant \mu'$.

A **limit of an inverse system $\boldsymbol{X}$** is a morphism $\boldsymbol{p} : X \to \boldsymbol{X}$ in pro-$\mathcal{C}$ for some object X of $\mathcal{C}$ with the universal property:

(UP) For any morphism $\boldsymbol{p}' : X' \to \boldsymbol{X}$ in pro-$\mathcal{C}$, there exists a unique morphism $f : X' \to X$ of $\mathcal{C}$ such that $\boldsymbol{p} f = \boldsymbol{p}'$, that is, $p_\lambda f = p'_\lambda$ for each $\lambda \in \Lambda$.

Consequently, if a limit exists, it is unique up to natural isomorphism. We refer to an object X, denoted by $\lim \boldsymbol{X}$, and a morphism $\boldsymbol{p} : X \to \boldsymbol{X}$ as a **limit** and a **natural projection**, respectively.

Every inverse system in $\mathcal{C}$ has a limit provided the following two conditions are satisfied:

(L1) Every collection of objects has a product in $\mathcal{C}$; and
(L2) Every pair of morphisms $f_0, f_1 : X \to Y$ has an **equalizer**, i.e., a morphism $j : Y \to Z$ such that
  (1) $jf_0 = jf_1$; and
  (2) If $j' : Y \to Z'$ is another morphism such that $j' f_0 = j' f_1$, then there is a morphism $h : Z \to Z'$ such that $hj = j'$.

In this case, any system morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ induces a morphism f, denoted by $\lim \boldsymbol{f} : \lim \boldsymbol{X} \to \lim \boldsymbol{Y}$ in $\mathcal{C}$, such that $q_\mu f = f_\mu p_{f(\mu)}$ for any μ, which defines a functor $\lim : \text{pro-}\mathcal{C} \to \mathcal{C}$.

Let Top denote the category of topological spaces and continuous maps. The following are the ***full subcategories*** of Top that are often used in general topology:

Notation	Objects
$\mathsf{Top}_{3.5}$	***Tychonoff space***s
$\mathsf{CTop}_{3.5}$	***Dieudonné complete*** Tychonoff spaces
CH	compact Hausdorff spaces
CM	compact ***metric space***s
ANR	***Absolute Neighborhood Retract***s
Pol	polyhedra with the ***CW-topology***

4. Approximate limits and approximate resolutions

Compact spaces X are often studied by expressing X as the limit of an inverse system of compact polyhedra, whose idea goes back to the work of P.S. Alexandroff [1] (see also [5, 6, 9]). Indeed, every compact Hausdorff space X admits an inverse system $\boldsymbol{X} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ and an inverse limit $\boldsymbol{p} : X \to \boldsymbol{X}$. Moreover, the limit of any inverse system of non-empty compact Hausdorff spaces is a non-empty compact space. However, the theory of inverse limits of non-metric compact spaces has some defects. A classical result of H. Freudenthal [6] asserts that every compact metric space X with $\dim X = n$ is the limit of an inverse sequence of finite polyhedra of ***dimension*** $\leqslant n$. However, this result cannot be generalized to compact Hausdorff spaces as pointed out in [10, 23].

S. Mardešić and L. R. Rubin [13] then introduced the notion of approximate inverse limits, and many theorems for compact metric spaces have been extended to compact Hausdorff spaces. For maps f, g from a topological space X to a metric space (X, d) and $\varepsilon > 0$, we write $d(f, g) \leqslant \varepsilon$ if $d(f(x), g(x)) \leqslant \varepsilon$ for each $x \in X$. An **approximate inverse system** $\boldsymbol{X} = \{X_\lambda, \varepsilon_\lambda, p_{\lambda\lambda'}, \Lambda\}$ of compact metric spaces consists of a directed ordered set $(\Lambda, \leqslant)$ with no maximal element, a compact metric space X_λ with the metric d_λ and a real number $\varepsilon_\lambda > 0$ for each $\lambda \in \Lambda$, a continuous map $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ for each pair $\lambda \leqslant \lambda'$, satisfying the following conditions (A0), (A1), (A2) and (A3):

(A0) $(\forall \lambda \in \Lambda)(p_{\lambda\lambda} = \mathrm{id})$.
(A1) $(\forall \lambda, \forall \lambda', \forall \lambda'' \in \Lambda)$
  $(\lambda \leqslant \lambda' \leqslant \lambda'' \Longrightarrow d_\lambda(p_{\lambda\lambda'} p_{\lambda'\lambda''}, p_{\lambda\lambda''}) \leqslant \varepsilon_\lambda)$.
(A2) $(\forall \lambda \in \Lambda)(\forall \delta > 0)(\exists \lambda_0 \geqslant \lambda)\ (\forall \lambda', \forall \lambda'' \in \Lambda)$
  $(\lambda_0 \leqslant \lambda' \leqslant \lambda'' \Longrightarrow d_\lambda(p_{\lambda\lambda'} p_{\lambda'\lambda''}, p_{\lambda\lambda''}) \leqslant \delta)$.
(A3) $(\forall \lambda \in \Lambda)(\forall \delta > 0)(\exists \lambda_0 \geqslant \lambda)(\forall \lambda' \geqslant \lambda_0)$
  $(\forall x, \forall x' \in X_{\lambda'})$
  $(d_{\lambda'}(x, x') \leqslant \varepsilon_{\lambda'} \Longrightarrow d_\lambda(p_{\lambda\lambda'}(x), p_{\lambda\lambda'}(x')) \leqslant \delta)$.

The **approximate limit** (or **limit**, for short) of an approximate inverse system $\boldsymbol{X} = \{X_\lambda, \varepsilon_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is defined to be the subspace of the product $\prod_{\lambda \in \Lambda} X_\lambda$ consisting of all points $x = (x_\lambda)$ satisfying that:

(AL) $(\forall \lambda \in \Lambda)(\forall \delta > 0)(\exists \lambda_0 \geqslant \lambda)(\forall \lambda' \in \Lambda)$
  $(\lambda' \geqslant \lambda_0 \Longrightarrow d(x_\lambda, p_{\lambda\lambda'}(x_{\lambda'})) \leqslant \delta)$.

S. Mardešić and L.R. Rubin [13] proved that a compact Hausdorff space X has $\dim X \leqslant n$ if and only if it is the limit of an approximate inverse system of finite polyhedra, whose dimension $\leqslant n$.

On the other hand, without compactness, the concept of inverse limits leads to many pathological phenomena, and hence, inverse limits are not often used for non-compact spaces. Indeed, there is an inverse sequence of 0-***dimensional paracompact*** spaces whose limit is a ***normal space*** with positive ***covering dimension*** [3]. S. Mardešić [11] then introduced the notion of resolution of topological spaces to deal with non-compact spaces, extending the properties of compact inverse limits. Let $\mathcal{N}(X)$ denote the set of all ***normal covers*** of X. For two maps $f, g : X \to Y$ and $\mathcal{U} \in \mathcal{N}(Y)$, we write $(f, g) \leqslant \mathcal{U}$ if for each $x \in X$ there exists $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subseteq U$. Let X be a topological space and $\boldsymbol{X} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an inverse system of topological spaces. Then a **map of system** $\boldsymbol{f} = \{f_\lambda, \Lambda\} : X \to \boldsymbol{X}$ is defined to be a collection of continuous maps $f_\lambda : X \to X_\lambda$, $\lambda \in \Lambda$, such that $p_{\lambda\lambda'} f_{\lambda'} = f_\lambda$ for each $\lambda, \lambda' \in \Lambda$ with $\lambda \leqslant \lambda'$. A **resolution** of X is a map of system $\boldsymbol{f} = (f_\lambda, \Lambda) : X \to \boldsymbol{X}$ which satisfies the following conditions (R1) and (R2):

(R1) If P is a polyhedron, $\mathcal{U} \in \mathcal{N}(P)$ and $f : X \to P$ is a continuous map, then there exist $\lambda \in \Lambda$ and a continuous map $g : X_\lambda \to P$ such that $(g f_\lambda, f) \leqslant \mathcal{U}$.

(R2) If P is a polyhedron, $\mathcal{U} \in \mathcal{N}(P)$, then there exists $\mathcal{V} \in \mathcal{N}(P)$ with the following property: For each $\lambda \in \Lambda$ and each pair of two continuous maps $h, h' : X_\lambda \to P$, if $(h f_\lambda, h' f_\lambda) \leqslant \mathcal{V}$, then there exists $\lambda' \geqslant \lambda$ such that $(h p_{\lambda\lambda'}, h' p_{\lambda\lambda'}) \leqslant \mathcal{U}$.

Combining the notions of approximate limits and resolutions, S. Mardešić and T. Watanabe [16] introduced the notion of approximate resolutions to overcome the defects of inverse limits of non-compact and non-metric spaces. An **approximate inverse system** $\boldsymbol{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ of topological spaces consists of a directed ordered set $(\Lambda, \leqslant)$ with no maximal element, a topological space X_λ and $\mathcal{U}_\lambda \in \mathcal{N}(X_\lambda)$ for each $\lambda \in \Lambda$, a continuous map $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ for each pair $\lambda \leqslant \lambda'$, satisfying the following conditions (B0), (B1), (B2) and (B3):

(B0) $(\forall \lambda \in \Lambda)(p_{\lambda\lambda} = \mathrm{id})$.

(B1) $(\forall \lambda, \forall \lambda', \forall \lambda'' \in \Lambda)$
$(\lambda \leqslant \lambda' \leqslant \lambda'' \Longrightarrow (p_{\lambda\lambda'} p_{\lambda'\lambda''}, p_{\lambda\lambda''}) \leqslant \mathcal{U}_\lambda)$.

(B2) $(\forall \lambda \in \Lambda)(\forall \mathcal{V} \in \mathcal{N}(X_\lambda))(\exists \lambda_0 \geqslant \lambda)\ (\forall \lambda', \forall \lambda'' \in \Lambda)$
$(\lambda_0 \leqslant \lambda' \leqslant \lambda'' \Longrightarrow (p_{\lambda\lambda'} p_{\lambda'\lambda''}, p_{\lambda\lambda''}) \leqslant \mathcal{V})$.

(B3) $(\forall \lambda \in \Lambda)(\forall \mathcal{V} \in \mathcal{N}(X_\lambda))(\exists \lambda_0 \geqslant \lambda)\ (\forall \lambda' \geqslant \lambda_0)$
($\mathcal{U}_{\lambda'}$ is a refinement of $\{p_{\lambda\lambda'}^{-1}(U) : U \in \mathcal{V}\}$).

Let X be a topological space and let $\boldsymbol{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an approximate inverse system of topological spaces. Then an **approximate map** $\boldsymbol{f} = \{f_\lambda, \Lambda\} : X \to \boldsymbol{X}$ is defined to be a collection of continuous maps $f_\lambda : X \to X_\lambda$, $\lambda \in \Lambda$, satisfying:

(AM) $(\forall \lambda \in \Lambda)(\forall \mathcal{U} \in \mathcal{N}(X_\lambda))(\exists \lambda_0 \geqslant \lambda)(\forall \lambda' \in \Lambda)$
$(\lambda' \geqslant \lambda_0 \Longrightarrow (p_{\lambda\lambda'} f_{\lambda'}, f_\lambda) \leqslant \mathcal{U})$.

An **approximate resolution** of X is an approximate map $\boldsymbol{f} = \{f_\lambda, \Lambda\} : X \to \boldsymbol{X}$ that satisfies the conditions (R1) and (R2) above. T. Watanabe [28] proved that a topological space X has $\dim X \leqslant n$ if and only if it admits an approximate resolution consisting of polyhedra with dimension $\leqslant n$.

The above generalizations of inverse limits are summarized as in the following diagram:

Inverse limits for CM	$\longrightarrow$	Resolutions for Top
$\downarrow$		$\downarrow$
Approximate limits for CH	$\longrightarrow$	Approximate resolutions for Top

5. Further development

Maps between topological spaces can be also studied by use of inverse systems and their modifications. A map $f : X \to Y$ between compact Hausdorff spaces admits compact polyhedral inverse systems

$$\boldsymbol{X} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\} \quad \text{and} \quad \boldsymbol{Y} = \{Y_\mu, q_{\mu\mu'}, M\}$$

and system morphism $\boldsymbol{f} = \{g, f_\mu\} : \boldsymbol{X} \to \boldsymbol{Y}$ such that $\lim \boldsymbol{f} = f$ [8], which means that

(M) whenever $\mu \leqslant \mu'$, there is a $\lambda > g(\mu), g(\mu')$ such that $f_\mu p_{g(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{g(\mu')\lambda}$, and

(LM) $f_\mu p_{g(\mu)} = q_\mu f$ for $\mu \in M$.

An analogous fact for resolutions was obtained in [11]. However, if we choose the polyhedral inverse systems $\boldsymbol{X}$ and $\boldsymbol{Y}$ in Pol in advance, it may not be possible to find f_μ with the strict commutativity conditions (M) and (LM) as pointed out in [10, 17, 25, 26]. Indeed, consider any map f from a ***Cantor set*** $\mathbb{C}$ onto the unit interval I, where the Cantor set $\mathbb{C}$ is the limit of an inverse sequence $\boldsymbol{X}$ of finite sets. Then there is no system morphism $\boldsymbol{f} : \boldsymbol{X} \to I$, where I is a rudimentary inverse system, such that $\lim \boldsymbol{f} = f$ since all terms of $\boldsymbol{X}$ are finite. If these conditions are replaced by approximate commutativity conditions, this will become possible. This was done for resolutions by T. Watanabe [26], for compact approximate limits by S. Mardešić and J. Segal [15], and for approximate resolutions by S. Mardešić and T. Watanabe [16].

In general, maps between approximate resolutions with some reasonable conditions can be defined by using approximate commutativity conditions. Moreover, one can obtain a category whose objects are those approximate resolutions and whose morphisms are reasonably defined equivalence classes of those maps. This category gives a useful tool in studying topological spaces and continuous maps since it is equivalent to the category $\mathsf{CTop}_{3.5}$

More generally, inverse systems and their generalizations give basic tools in many applications. S. Mardešić [11] used resolutions to extend shape theory over topological spaces. In fact, inverse systems, approximate inverse limits and approximate resolutions can be used to define shape theory in a categorical way for compact metric spaces, compact Hausdorff spaces, topological spaces, respectively. Similarly, such inverse system approach can also be used to define strong shape theory [2, 7, 12]. Moreover, ***Čech homology*** and ***Steenrod homology*** theories are also studied by using the pro-category pro-HPol of the homotopy category of polyhedra [27, 12]. The theory of approximate resolutions for ***uniform spaces*** and its applications in uniform spaces were obtained in [18]. The theory of approximate resolutions was also used to study fixed-point theory [24], and more recently, T. Miyata and T. Watanabe applied it to fractal geometry and obtained interesting results [19, 21].

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b-9 Covering Properties

Let S be a set. A family $\mathcal{A}$ of subsets of S is called a **cover** (**covering**) if the union $\bigcup\{A\colon A \in \mathcal{A}\}$ is S. A cover $\mathcal{C}$ of S is called **finite** (**countable**) if the cardinality of $\mathcal{C}$ is finite (countable). For two covers $\mathcal{B}$ and $\mathcal{C}$ of S, $\mathcal{B}$ is a **refinement** of $\mathcal{C}$ (or $\mathcal{B}$ **refines** $\mathcal{C}$) if each member of $\mathcal{B}$ is contained in some member of $\mathcal{C}$. In particular, $\mathcal{B}$ is called a **subcover** of $\mathcal{C}$ if $\mathcal{B}$ is a subfamily of $\mathcal{C}$. A cover $\mathcal{C}$ of S has a finite (countable) refinement iff it has a finite (countable) subcover.

Let X be a ***space*** (exactly, a ***topological space***). A cover $\mathcal{U}$ of X is called an **open cover** (a **closed cover**) if each member of $\mathcal{U}$ is open (closed) in X. A family $\mathcal{A}$ of subsets of X is a ***locally finite family*** if each point of X has an ***open neighbourhood*** which meets at most finitely many members of $\mathcal{A}$. Recall that a space X is called **paracompact** (respectively, ***compact***, ***Lindelöf***) if every open cover of X has a locally finite (respectively, finite, countable) open refinement.

1. Variations and implications

Paracompactness seems to be the most important covering property of spaces. Throughout the study of it, various covering properties have been introduced and studied in terms of several variations of local finiteness.

Let X be a space and $\mathcal{A}$ a family of subsets of X. We say that $\mathcal{A}$ is a **discrete family** if each point of X has an open neighbourhood which meets at most one member of $\mathcal{A}$. We say that $\mathcal{A}$ is **point-finite** (respectively, **star-finite**) if each point of X is in (respectively, each member of $\mathcal{A}$ meets) at most finitely many members of $\mathcal{A}$. We say that $\mathcal{A}$ is **closure-preserving** if

$$\overline{\bigcup\{A\colon A \in \mathcal{B}\}} = \bigcup\{\bar{A}\colon A \in \mathcal{A}'\}$$

for every $\mathcal{A}' \subset \mathcal{A}$, and that $\mathcal{A}$ is **interior-preserving** if $\{X \setminus A\colon A \in \mathcal{A}\}$ is closure-preserving. In particular, for an open cover $\mathcal{U}$ of X, the following implications are true:

$$\begin{array}{ccc} & \text{finite} \Longrightarrow \text{countable} & \\ & \Downarrow & \\ \text{discrete} \Longrightarrow & \text{star-finite} & \\ & \Downarrow & \\ & \text{locally finite} \Longrightarrow \text{closure-preserving} & \\ & \Downarrow & \\ & \text{point-finite} \Longrightarrow \text{interior-preserving} & \end{array}$$

Moreover, $\mathcal{A}$ is called **σ-locally finite** (respectively, **σ-discrete**, **σ-closure-preserving**) if it is represented as a union of countably many locally finite (respectively, discrete, closure-preserving) subfamilies.

These concepts play quite important roles in the investigation of paracompact spaces. Hereafter, ***regular*** spaces and ***normal*** spaces are always assumed to be ***T_1-spaces***.

2. Normal covers

Let X be a space. For two open covers $\mathcal{U}$ and $\mathcal{V}$ of X, we say that $\mathcal{V}$ is a **star refinement** of $\mathcal{U}$ if for each $V \in \mathcal{V}$ there is a $U_V \in \mathcal{U}$ such that $\mathrm{St}(V, \mathcal{V}) \subset U_V$, where

$$\mathrm{St}(V, \mathcal{V}) = \bigcup\{W \in \mathcal{V}\colon W \cap V \neq \emptyset\}.$$

Moreover, we say that $\mathcal{V}$ is a **point-star refinement** (a **barycentric refinement**, a **delta-refinement** or a **Δ-refinement**) of $\mathcal{U}$ if for each $x \in X$ there is a $U_x \in \mathcal{U}$ such that $\mathrm{St}(x, \mathcal{V}) \subset U_x$, where $\mathrm{St}(x, \mathcal{V}) = \bigcup\{V \in \mathcal{V}\colon x \in V\}$.

An open cover $\mathcal{U}$ of X is a **normal cover** if there is a sequence $\mathcal{V}_0, \mathcal{V}_1, \ldots$ of open covers (called a **normal sequence**) of X such that $\mathcal{V}_0 = \mathcal{U}$ and each $\mathcal{V}_n$ is a star refinement of $\mathcal{V}_{n-1}$. Here, we can replace "star refinement" with "point-star refinement" (see [E, 5.1.15], [N, II.5 B]). Tukey [15] showed that a T_1-space X is normal iff every finite (binary) open cover of X is normal, and that every open cover of a ***metric space*** is normal [E, 5.1.A (C)] etc.

A **cozero set** in X is a set of the form $\{x \in X\colon f(x) > 0\}$ for some ***continuous*** function $f: X \to [0, 1]$. Each open ***F_σ-set*** in a normal space is exactly a cozero set. An open cover $\mathcal{U}$ of X is called a **cozero cover** if each member of $\mathcal{U}$ is a cozero set in X.

A.H. Stone and E.A. Michael gave an essential characterization of normal covers as follows (see [12, 13]). The following are equivalent for a space X and an open cover $\mathcal{U}$ of X:

(a) $\mathcal{U}$ is normal.
(b) There is a metric space M and a continuous map f from X onto M such that $\mathcal{U}$ is refined by $\{f^{-1}(W)\colon W \in \mathcal{W}\}$ for some open covering $\mathcal{W}$ of M.
(c) $\mathcal{U}$ has a locally finite and σ-discrete cozero refinement.
(d) $\mathcal{U}$ has a locally finite cozero refinement.
(e) $\mathcal{U}$ has a σ-discrete cozero refinement.
(f) $\mathcal{U}$ has a σ-locally finite cozero refinement.

This immediately yields A.H. Stone's remarkable result: Every metric space is paracompact [E, 5.1.3] etc. Thus paracompact spaces turned out to be a significant generalization of metric spaces and compact spaces.

Let $\{f_\alpha\colon \alpha \in \Lambda\}$ be a family of continuous functions from a space X into the unit interval $[0,1]$. We say that $\{f_\alpha\colon \alpha \in \Lambda\}$ is a **partition of unity** on X if

$$\sum_{\alpha\in\Lambda} f_\alpha(x) = 1 \quad \text{for each } x \in X,$$

where the equality implies that $\{\alpha \in \Lambda\colon f_\alpha(x) \neq 0\}$ is at most countable; let us say, $\{\alpha_n\colon n \in \omega\}$ and $\sum_{n\in\omega} f_{\alpha_n}(x)$ converges to 1 (note that the arrangement of $\{\alpha_n\colon n \in \omega\}$ does not matter). We say that a partition of unity $\{f_\alpha\colon \alpha \in \Lambda\}$ of X is a **locally finite partition of unity** if the cover $\{f_\alpha^{-1}((0,1])\colon \alpha \in \Lambda\}$ of X is locally finite. We also say that a partition of unity $\{f_\alpha\colon \alpha \in \Lambda\}$ of X is **subordinated to** a cover $\mathcal{C}$ of X if the cover $\{f_\alpha^{-1}((0,1])\colon \alpha \in \Lambda\}$ of X is a refinement of $\mathcal{C}$. Partitions of unity are very important not only in topology but also in analysis and differential geometry. E.A. Michael essentially gave a characterization of normal covers in terms of these concepts (see [12]): An open cover $\mathcal{U}$ of a space X is normal iff $\mathcal{U}$ has a (locally finite) partition of unity subordinated to it. Thus, the characterizations of normal covers above yield several useful characterizations of paracompactness (see E.A. Michael's article in this Encyclopedia).

Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets in a space X. We say that $\mathcal{B}$ is **cushioned in** $\mathcal{A}$ if one can assign to each $B \in \mathcal{B}$ some $A(B) \in \mathcal{A}$ such that

$$\overline{\bigcup\{B\colon B \in \mathcal{B}'\}} \subset \bigcup\{A(B)\colon B \in \mathcal{B}'\}$$

for every $\mathcal{B}' \subset \mathcal{B}$. Moreover, $\mathcal{B}$ is called **σ-cushioned in** $\mathcal{A}$ if $\mathcal{B}$ is represented as a union of countably many cushioned subfamilies in $\mathcal{A}$. This was defined by Michael [9] after his definition of the closure-preserving property. Observe that for an open cover $\mathcal{U}$ of X, every closure-preserving closed refinement of $\mathcal{U}$ and every open star refinement of $\mathcal{U}$ are cushioned in $\mathcal{U}$. Recall that the important generalized metric spaces of ***M_1-spaces*** and ***stratifiable space***s are defined by the σ-closure-preserving and σ-cushioned properties of bases, respectively (see P.M. Gartside's article in this Encyclopedia). Normal covers on normal spaces were also characterized in terms of this property in [18]: An open cover $\mathcal{U}$ of a normal space X is normal iff it has an open refinement $\mathcal{V}$ which is a σ-cushioned refinement of itself (that is, $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$ and each $\mathcal{V}_n$ is cushioned in $\mathcal{V}$).

There are many metrization theorems. But, the earliest metrization theorem, which is sometimes called the Alexandroff–Urysohn theorem, was stated in terms of normal covers: A ***T_0-space*** X is ***metrizable*** iff there is a normal sequence $\{\mathcal{G}_n\}$ of open covers of X such that $\{\mathrm{St}(x,\mathcal{G}_n)\colon n \in \omega\}$ is a ***neighbourhood base*** at x for each $x \in X$ (see [E, 5.4.9], [N, VI.1]). This was extended as Bing's Metrization Theorem (see [E, 5.4.1], [N, VI.4]): A space is metrizable iff it is a ***collectionwise normal*** and ***Moore space*** (see R. Hodel's article in this Encyclopedia).

3. Point-finite open covers

We begin with two classical results for point-finite open covers. The first is as follows: For a point-finite open cover $\{U_\alpha\colon \alpha \in \Lambda\}$ of a normal space X, there is an open (a cozero) cover $\{V_\alpha\colon \alpha \in \Lambda\}$ of X such that $\overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in \Lambda$ (see [E, 1.5.18], [N, III.2 C]). A cover $\mathcal{C}$ of a space X is called **irreducible** if $\bigcup\mathcal{C}' \neq X$ for every $\mathcal{C}' \subsetneq \mathcal{C}$. The second one was given by R. Arens and J. Dugundji: Every point-finite cover of a space contains an irreducible subcover [E, 5.3.1].

Let X be a space. For two open covers $\mathcal{U}$ and $\mathcal{V}$ of X, we say that $\mathcal{V}$ is **pointwise W-refinement** of $\mathcal{U}$ if for each $x \in X$, there is a finite subfamily $\mathcal{F}(x)$ of $\mathcal{U}$ such that each $V \in \mathcal{V}$ with $x \in V$ is contained in some $U \in \mathcal{F}(x)$. Obviously, every point-star refinement of $\mathcal{U}$ is a pointwise W-refinement. J.M. Worrell gave an analogue of normal covers for the point-finite version of open covers (slightly modifying the proof of [5, 2.2]): An open cover $\mathcal{U}$ of X has a point-finite open refinement iff there is a sequence $\{\mathcal{U}_n\}$ of open covers of X such that $\mathcal{U}_0 = \mathcal{U}$ and each $\mathcal{U}_{n+1}$ is a pointwise W-refinement of $\mathcal{U}_n$.

Let X be a space. A family $\mathcal{F}$ of closed sets in X is closure-preserving iff $\bigcup\mathcal{F}'$ is closed in X for every $\mathcal{F}' \subset \mathcal{F}$. This was defined by E.A. Michael to characterize paracompactness [E, 5.1.G] and a notable fact concerning closure-preserving covers is that they are preserved by ***closed map***s. Analogously, a family $\mathcal{U}$ of open sets in X is interior-preserving iff $\bigcap\mathcal{U}'$ is open in X for every $\mathcal{U}' \subset \mathcal{U}$. As shown in the diagram of implications above, every locally finite family of closed sets in X is closure-preserving, and every point-finite family of open sets in X is interior-preserving.

Note that if an open cover $\mathcal{U}$ of a space X has a locally finite closed refinement, then it has an open (locally finite) point-star refinement [E, 5.1.13 and 5.1.14]. This fact is expanded by H.J.K. Junnila [5] as follows: An interior-preserving open cover $\mathcal{U}$ of a space X has a closure-preserving closed refinement iff $\mathcal{U}$ has an interior-preserving open point-star refinement. Moreover, an interior-preserving open cover $\mathcal{U}$ of a space X has an interior-preserving open pointwise W-refinement iff $\mathcal{U}^F$ has a closure-preserving closed refinement, where $\mathcal{U}^F$ denotes the family of all finite unions of members of $\mathcal{U}$. Recall that a space X is called ***metacompact*** if every open cover of X has a point-finite open refinement. These two results by H.J.K. Junnila yield some useful characterizations of metacompactness (see Junnila's article in this Encyclopedia). There are good surveys [7], [KV, Chapter 9] and [MN, Chapter 5] on this topic.

A cover $\mathcal{C}$ of X is called **semi-open** if $x \in \mathrm{Int}\,\mathrm{St}(x,\mathcal{C})$ for each $x \in X$. In terms of this definition, there are two analogous results: An open cover $\mathcal{U}$ of a space X has a cushioned refinement iff it has a semi-open point-star refinement [6]. An interior-preserving open cover $\mathcal{U}$ of a space X has an open pointwise W-refinement iff $\mathcal{U}^F$ has a cushioned refinement [4]. However, these two results do not yield a nice characterization of metacompactness (see also H.J.K. Junnila' article in this Encyclopedia).

4. Star-finite covers

Let X be a space and let A, B be a pair of disjoint subsets of X. We say that a subset L of X is a **partition** between A and B if there are disjoint open sets U and V in X such that $A \subset U$, $B \subset V$ and $X \setminus L = U \cup V$. Clearly, the partition L is closed in X. It is also clear that a T_1-space X is normal iff for each pair of disjoint closed sets in X, there is a partition between them. The existence of such a partition is assured by certain open covers. In fact, the following result is usually stated as a lemma, however it is technically very useful not only to show the normality of spaces but also to show the partitions in ***dimension theory***: Let X be a space and let E, F be a pair of disjoint closed sets in X. If there is a σ-locally finite open cover $\mathcal{V}$ of X such that $\overline{V} \cap E = \emptyset$ or $\overline{V} \cap F = \emptyset$ for each $V \in \mathcal{V}$, then there is a partition L between E and F such that

$$L \subset \bigcup \{\partial V \colon V \in \mathcal{V}\},$$

where ∂V denotes the ***boundary*** of V (see [En, 2.3.16], [E, 1.5.15]).

We say that a space X is **strongly paracompact** (or has the **star-finite property**) if every open cover of X has a star-finite open refinement. Obviously, every strongly paracompact space is paracompact. C.H. Dowker and K. Morita proved that every countable cozero cover of a space has a star-finite cozero refinement (see the proof of [KV, Chapter 9, 3.11]). It follows from this that every regular Lindelöf space is strongly paracompact (see [E, 5.3.11], [N, V.4 B]), because it is normal by the technical result above. On the other hand, there is a metric space which is not strongly paracompact [E, 5.3 F(e)]. Every uncountable discrete space is strongly paracompact but not Lindelöf.

Let S be a set. We say that a family $\mathcal{A}$ of subsets in S is **star-countable** if each $A \in \mathcal{A}$ meets at most countably many members of $\mathcal{A}$. Note that a family $\mathcal{A}$ of subsets in S is star-countable iff it is decomposed into $\{\mathcal{A}_\alpha \colon \alpha \in \Lambda\}$ such that each $\mathcal{A}_\alpha$ is countable and

$$\Bigl(\bigcup \mathcal{A}_\alpha\Bigr) \cap \Bigl(\bigcup \mathcal{A}_\beta\Bigr) = \emptyset \quad \text{if } \alpha \neq \beta$$

[E, 5.3.8 and 5.3.9]. Ju.M. Smirnov showed from this fact that a space X is strongly paracompact iff every open cover of X has a star-countable open refinement [E, 5.3.10], and that a ***connected*** regular space X is strongly paracompact iff it is Lindelöf [N, V.4 C]. Thus, we can see that the notions of star-countability, star-finiteness and countability are essentially not so different.

We can characterize paracompactness in terms of a generalization of star-finiteness: A family $\mathcal{A} = \{A_\alpha \colon \alpha \in \Lambda\}$ of subsets of a set S is called **order star-finite** if there is a well-order $<$ of the index set Λ such that, for each $\alpha \in \Lambda$, A_α meets at most finitely many A_β with $\beta < \alpha$. It was given in [17] that a regular space X is paracompact iff every open cover of X has an order star-finite open refinement.

5. Closure-preserving covers

Note that a T_2-***space*** is paracompact and ***locally compact*** iff it has a locally finite (and star-finite) closed cover by compact sets. So every paracompact locally compact T_2-space can be represented as the union of disjoint family of closed-open Lindelöf subspaces [E, 5.1.27]. Hence it is strongly paracompact. Thus it is likely that a space which has a closure-preserving cover by compact sets is paracompact. However, Bing's example modified by E.A. Michael is a normal space which has a closure-preserving cover by finite sets, but not paracompact [MN, Chapter 13, 3.22]. Still, every space with a (σ-) closure-preserving cover by compact sets is metacompact (***submetacompact***), and every metacompact (submetacompact) and locally compact space has a (σ-) closure-preserving cover by compact sets (see [MN, Chapter 13, 3.8 and 3.15]).

Let $\mathbb{K}$ be a class of spaces such that $X \in \mathbb{K}$ implies $F \in \mathbb{K}$ for each closed subspace F of X. For such a class $\mathbb{K}$ of spaces and a space X, R. Telgársky introduced a **topological game** denoted by $G(\mathbb{K}, X)$ as follows [MN, Chapter 13]: There are two players I (the pursuer) and II (the evader). They alternately choose consecutive terms of a sequence $\langle E_1, F_1, E_2, F_2, \ldots \rangle$ of closed subsets in X, called a **play** of $G(\mathbb{K}, X)$, satisfying the following conditions; for each $n \in \omega$,

(1) E_n is the choice of Player I with $E_n \in \mathbb{K}$,
(2) F_n is the choice of Player II,
(3) $E_n \cup F_n \subset F_{n-1}$, where $F_0 = X$,
(4) $E_n \cap F_n = \emptyset$.

When each player chooses their term, he/she knows $\mathbb{K}$, X and their previous choices. Player I wins this play if $\bigcap_{n\in\omega} F_n = \emptyset$ (Player II has no place to run away), otherwise Player II wins. H.B. Potoczny gave the essential relation between closure-preserving covers by compact sets and topological games [MN, Chapter 13, 3.4]: If a space X has a (σ-) closure-preserving closed cover by compact sets, then Player I has a winning strategy in $G(\mathbb{DC}, X)$, where $\mathbb{DC}$ denotes the class of all discrete unions of compact subspaces. Telgársky [14] gave a good survey for other various topological games similar to the game $G(\mathbb{K}, X)$ here.

These topological games are rather important in the case when the Player I has a winning strategy in $G(\mathbb{DC}, X)$. Using this, R. Telgársky proved the following: Let X and Y be paracompact T_2-spaces. If X has a σ-closure-preserving cover by compact sets, then $X \times Y$ is paracompact [MN, Chapter 13, 4.2]. There has been no nice characterization for the class of all paracompact spaces X such that $X \times Y$ is paracompact (or normal) for every paracompact factor Y [KV, Chapter 18, Problem 5]. So this seems to be the best for the paracompactness of products. This result is also true for ***subparacompactness***, metacompactness and submetacompactness instead of paracompactness [MN, Chapter 13, 4.6] and [3].

The topological game $G(\mathbb{K}, X)$ gave another application to closure-preserving sum theorems in dimension theory: If a normal space X has a closure-preserving closed cover $\mathcal{F}$ such that F is ***countably compact*** and $\dim F \leqslant n$ for each $F \in \mathcal{F}$, then $\dim X \leqslant n$ [MN, Chapter 13, 6.6]. This result is also true for the ***large inductive dimension*** "Ind" instead of the ***covering dimension*** "dim" under an additional condition on X [MN, Chapter 13, 6.13]. Moreover, the product theorem of the inequality

$$\dim(X \times Y) \leqslant \dim X + \dim Y$$

was obtained by applying the same sort of games (also see [MN, Chapter 13, 6.23]).

6. Dominating covers

A topological property $\mathcal{P}$ is called **additive** if, for any family $\{X_\alpha\}_{\alpha \in \Lambda}$ of spaces with property $\mathcal{P}$, the ***topological sum*** $\bigoplus_{\alpha \in \Lambda} X_\alpha$ also has property $\mathcal{P}$. Let $\mathcal{P}$ be a topological property which is additive and is preserved by ***perfect maps***. For the property $\mathcal{P}$, we can count so many kinds of fundamental topological properties such as metrizability, paracompactness, normality and so on. If a space X has a locally finite closed cover $\mathcal{F}$, each member of which has property $\mathcal{P}$, then X has property $\mathcal{P}$ [E, 3.7.22].

Every CW-complex is defined by a dominating cover, which is a generalization of a locally finite closed cover, as stated below. Let X be a space. For a cover $\mathcal{C}$ of X, we say that X has the **weak topology** with respect to $\mathcal{C}$ (or X is **determined** by $\mathcal{C}$) if $A \subset X$ is closed in X whenever $A \cap C$ is relatively closed in C for each $C \in \mathcal{C}$. Here, we can replace "closed" with "open". Obviously, every space has the weak topology with respect to every open cover of it. Recall that a space X is called a ***k-space*** (***sequential space***) if it has the weak topology with respect to the cover consisting of all compact (compact metric) subsets.

Let X be a space and $\mathcal{F}$ a closed cover of X. We say that X is **dominated** by $\mathcal{F}$ if $\mathcal{F}$ is closure-preserving and for every subfamily $\mathcal{G}$ of $\mathcal{F}$, the union $\bigcup \mathcal{G}$ has the weak topology with respect to $\mathcal{G}$. In other words, X is dominated by $\mathcal{F}$ iff $A \subset X$ is closed in X whenever there is some $\mathcal{G} \subset \mathcal{F}$ such that $A \subset \bigcup \mathcal{G}$ and $A \cap F$ is closed in F (in X) for each $F \in \mathcal{G}$. Obviously, every space is dominated by every locally finite closed cover of it.

Michael [8] and Morita [10, 11] proved that a T_2-space X is paracompact (respectively, normal, ***perfectly normal***, normal and ***countably paracompact***, normal with $\dim X \leqslant n$) iff it is dominated by a closed cover $\mathcal{F}$ such that each $F \in \mathcal{F}$ is paracompact (respectively, normal, perfectly normal, normal and countably paracompact, normal with $\dim F \leqslant n$). Moreover, a regular space X is stratifiable (respectively, a ***σ-space***, ***semistratifiable***) iff X is dominated by its closed cover, each member of which is stratifiable (respectively, a σ-space, semistratifiable) [N, VI.8 E]. However, it is not known whether it is true or not for M_1-spaces.

Let X be a space and e be a subset of X. We say that e is a **cell** of X if there is a continuous map φ from I^n onto the closure $\overline{e}$ of e such that

$$\varphi(I^n \setminus \operatorname{Int} I^n) = \overline{e} \setminus e$$

and $\varphi \restriction \operatorname{Int} I^n$ is a homeomorphism onto e, where $I = [0, 1]$ and n is called the **dimension** of e. We say that a T_2-space X is a **cell complex** if it has a ***decomposition*** $\mathcal{D}$ consisting of cells of X such that for each cell $e \in \mathcal{D}$ with dimension n, $\overline{e} \setminus e$ is contained in the union of all cells in $\mathcal{D}$ with dimension $\leqslant n - 1$. We denote by $e \in X$ the fact that e is a cell in the decomposition $\mathcal{D}$. A **finite cell-complex** is one with a finite number of cells. A subspace A of a cell-complex X is called a **subcomplex** of X if $\overline{e} \subset A$ for each $e \in X$ meeting A. Every subcomplex A of X is a cell-complex itself with respect to its decomposition $\{e \in X\colon e \text{ meets } A\}$.

A cell-complex X is called a **CW-complex** if the following two conditions (C) and (W) are satisfied;

(C) for each $e \in X$, there is a finite subcomplex K of X with $e \in K$, and

(W) X has the weak topology with respect to the closed cover $\{\overline{e}\colon e \in X\}$.

Every subcomplex of a CW-complex X is closed in X and is a CW-complex. A subspace C of a CW-complex X is compact (and metrizable) iff there is a finite subcomplex of X containing C. Note that every CW-complex is dominated by the cover consisting of all finite subcomplexes of X. J.G. Ceder [1] proved that every CW-complex is an M_1-space. However, there is a countable CW-complex which is not ***Lašnev*** (hence, not metrizable) [MN, Chapter 8, 2.8]. The concept of CW-complexes were introduced by Whitehead [16], and one of the most important points of CW-complexes is that homotopy theory can be developed on this class in the most natural way (see A. Koyama's article in this Encyclopedia).

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b-10 Locally (P)-Spaces

Let (P) denote a topological property. Then, in order to define locally (P)-spaces, we usually adopt two ways:

(A) a ***topological space*** X is said to be a **locally (P)-space** if each point of X has a ***neighbourhood*** with the property (P), and
(B) a topological space X is said to be a **locally (P)-space** if each point x of X has a ***neighbourhood base*** $\mathcal{B}_x$, where each member of $\mathcal{B}_x$ has the property (P).

In most cases, locally (P)-spaces of type (A) are included in those of type (B). Generally speaking, (A) is used more often than (B); ***locally compact*** spaces, **locally countably compact** spaces, **locally metrizable** spaces, **locally separable** spaces, **locally countable** spaces, etc., are such ones. For spaces with local (P) of type (B), we can mention locally connected spaces and related ones which often appear in a geometric part of general topology. We observe that a space satisfying (P) is always a locally (P)-space of type (A), but need not be of type (B); one can find an example of a ***connected***, but not locally connected space.

We review here mainly local compactness and local connectedness. Both notions have important and applicable properties to various areas; other spaces as stated above are very often used only in each individual case, some of which, therefore, seem difficult to be reviewed.

Recall that X is locally compact if each point of X has a ***compact*** neighbourhood. Assuming the ***Hausdorff*** separation axiom (T_2), local compactness behaves well. Every locally compact Hausdorff space is ***completely regular***. For every locally compact Hausdorff space X which is not ***compact***, we can take a point p^∞ not belonging to X and introduce a topology to the set $Z = X \cup \{p^\infty\}$ so that the space Z is a compact Hausdorff space and X is ***dense*** in Z; Z is known as so-called the **Alexandroff one-point compactification** of X. In every locally compact space X every ***closed subspace*** is locally compact, and if X is Hausdorff, every ***open*** subspace is locally compact. These results permit us to describe local compactness as follows: a Hausdorff space X is locally compact if and only if X is an open dense subspace of a compact Hausdorff space. Local compactness is preserved by ***open continuous*** images and, between Hausdorff spaces, by images and also preimages under ***perfect maps***. Among Hausdorff spaces local compactness is productive for finite products, but not for infinite ones. Indeed, the product space of locally compact Hausdorff spaces is locally compact if and only if all but a finite number of factor spaces are compact (all these results can be seen in [E, Chapter 3]).

Local compactness has been characterized in the realm of product spaces; a Hausdorff space X is locally compact if and only if $X \times Y$ is a ***k-space*** for any Hausdorff k-space Y if and only if the map

$$1_X \times f : X \times Y \to X \times Z$$

is a ***quotient*** map for any quotient map $f : Y \to Z$ of every Hausdorff spaces Y and Z (Whitehead [19] and Michael [7]). The next result is also a useful one established by T. Isiwata [6], K. Morita [9] and H. Ohta [11]. A ***Tychonoff space*** X is locally compact and ***Dieudonné complete*** if and only if

$$\gamma(X \times Y) = \gamma X \times \gamma Y$$

for every Tychonoff space Y, where γX denotes the ***Dieudonné completion*** of X. A similar result for the ***Hewitt realcompactification*** of a product space also holds (see J. Schommer's article in this Encyclopedia). We review another result due to S. Oka [12]. A Tychonoff space X is locally compact if and only if $\tau(X \times Y) = X \times \tau(Y)$ for any topological space Y, where τ denotes the ***epireflective*** functor from the category of all topological spaces and continuous maps onto the ***full subcategory*** of Tychonoff spaces. There are further studies due to T. Ishii [5]. In [12, 5] the functor τ is called the **Tychonoff functor**.

Paracompact Hausdorff or ***metrizable*** spaces with local compactness often yield good results. For example, in [8] K. Morita proved that if X is a locally compact paracompact Hausdorff space and $f : X \to Y$ is a ***closed*** continuous onto map, then the subspace $Y_1 = \{y \in Y\colon f^{-1}(y)$ is not compact$\}$ is ***discrete*** and ***closed*** in Y. Subsequently, analogous or extended results were obtained for X in other related classes of spaces. On the other hand, Morita's result was applied in the same paper [8] to extend closed maps between locally compact paracompact spaces to maps between their ***Freudenthal compactifications***. K. Nowiński [10] extended this result to the case of closed maps between locally compact ***weakly paracompact*** (= ***metacompact***) spaces.

Locally compact spaces are generalized to C-scattered spaces due to R. Telgárski [16]. Let X be a Hausdorff space. For an ordinal α, we inductively define a subspace $X^{(\alpha)}$ of X as follows: $X^{(0)} = X$,

$$X^{(\alpha+1)} = \bigl\{x \in X^{(\alpha)}\colon x \text{ has no compact neighbourhood in } X^{(\alpha)}\bigr\},$$

and $X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$ for a limit ordinal λ. Then, X is said to be **C-scattered** if $X^{(\gamma)} = \emptyset$ for some γ. Clearly, X is locally compact if and only if $X^{(1)} = \emptyset$. The space Y given in the result of K. Morita above satisfies $Y^{(2)} = \emptyset$. R. Telgárski [16] showed several interesting results on ***normality*** of product spaces of C-scattered paracompact spaces with paracompact spaces. The idea of defining $X^{(\alpha)}$ is based on A.H. Stone [14], and such an idea has been utilized in various studies in general topology.

As for locally metrizable spaces, a result is that every locally metrizable paracompact Hausdorff space is metrizable. This is implied by the metrizability of Hausdorff spaces having a ***locally finite closed cover*** consisting of metrizable sets.

For locally separable spaces, there is a theorem of P. Alexandroff [1] that every locally separable metrizable space X is expressed as $X = \bigoplus_{\alpha \in \Omega} A_\alpha$, the ***topological sum*** of A_α's, where each A_α is ***separable***. Concerning covering properties it is known that every weakly paracompact locally separable space is ***strongly paracompact*** (D.R. Traylor [17], R.E. Hodel [4]).

A topological space X is said to be a **locally connected space** if each point of X has a neighbourhood base consisting of connected sets. We review some basic facts. Local connectedness is hereditary with respect to open subsets, but not to closed subsets in general; it is known that a ***retract*** of a locally connected space is locally connected. A space X is locally connected if and only if the ***components*** of all open subspaces of X are open. Local connectedness is preserved by images under ***quotient maps***. The product space of locally connected spaces is locally connected if and only if all but a finite number of factor spaces are connected (see [E, Chapter 6]).

Locally connectedness is a classically important notion especially in the geometric part of general topology. Indeed, there is a famous theorem of Hahn–Mazurkiewicz that a Hausdorff space is a continuous image of the closed unit interval I if and only if X is a locally connected, compact and connected metric space. There is another notion which is closely related to local connectedness. A space X is **locally pathwise connected** if each point of X has a neighbourhood base consisting of ***pathwise connected*** sets. Basic facts for locally pathwise connected spaces are similar to those as above for locally connected spaces. Every locally pathwise connected space is locally connected, but the converse need not hold. Locally connected or locally pathwise connected spaces together with connected or ***arcwise connected*** spaces are fundamentally important for the study of Continuum Theory.

Locally connected spaces have been used to obtain some set-theoretic results. By B. Banaschewski [2], and M. Henriksen and J.R. Isbell [3] the ***Stone–Čech compactification*** βX of a Tychonoff space X is locally connected if and only if X is locally connected and ***pseudocompact***. In [13] G.M. Reed and P. Zenor proved that every locally connected, locally compact ***normal Moore space*** is metrizable. This result is related to the problem on metrizability of normal Moore spaces. As is mentioned in F.D. Tall [15] (see also, S. Watson [18]), it is unknown whether a locally connected, locally compact normal space is ***collectionwise normal***.

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b-11 Rim(P)-Spaces

In this paper, all ***topological space***s under consideration are assumed to be ***Hausdorff*** and all maps are assumed to be ***continuous*** and onto. For the terms which are not defined in this paper we refer the reader to [E], [11, 28, 17].

Let P be a ***topological property*** and X a topological space. Then X is said to be **rim-**P if X admits a ***base*** of ***open sets*** with ***boundaries*** having the property P. Some of the spaces that are studied with natural rim-properties are **rim-finite space**s, **rim-countable space**s, **rim-metrizable space**s, **rim-scattered space**s and **rim-compact space**s. Note that if X is a 0-***dimensional*** space, then it has a base of open sets with empty boundaries. Hence, such spaces are rim-P for any property P. Therefore, it is interesting to study rim-properties of spaces which are not 0-dimensional. In the literature, the term **locally peripherally** P is sometimes used instead of rim-P.

In 1880, C. Jordan defined a **continuous curve** as a continuous image of the ***closed*** interval of real numbers. Ten years later, in 1890, Peano constructed a map of the unit interval [0, 1] onto the unit square [0, 1] × [0, 1]. Another development was the **Hahn–Mazurkiewicz Theorem** (1914) characterizing ***locally connected metric*** continua as continuous images of the unit interval; such spaces are now commonly referred to as **Peano continua**. By a **continuum**, we mean a compact ***connected*** space. About 1925, P.S. Alexandroff and F. Hausdorff characterized Hausdorff continuous images of the ***Cantor set*** as the class of compact metric spaces. Following these results and others, see [E], it became necessary to give a precise definition of dimension. This led to study of notions of the ***small inductive dimension***, ***covering dimension*** and ***large inductive dimension***. The notion of small inductive dimension is based on a separation property. The small inductive dimension ind (called also the Menger-Urysohn dimension) is defined as follows: ind $X = -1$ if and only if $X = \emptyset$, and ind $X \leqslant n$, where $n = 0, 1, \ldots,$ if X has a base $\mathcal{U}$ of open sets such that the ***boundary*** of each $U \in \mathcal{U}$ has small inductive dimension $\leqslant n - 1$. In the case of ***separable*** metric spaces, the above three notions of dimension coincide. In general, they may not. Throughout the article, by a 0-dimensional space we mean a space with ind $\leqslant 0$.

The Hahn–Mazurkiewicz Theorem also changed the notion of curve from the topological point of view. It became necessary to define this notion more precisely. Using the small inductive dimension, a **curve** is defined to be a 1-dimensional (metric) continuum (see [28] or [11]). The simplest example of a 1-dimensional continuum is an ***arc***, i.e., an ***ordered continuum*** X with exactly two ***non-cut points***, and every basic open set in X has at most a two point boundary. Following these developments in the early 20th century, the study of those (metric) continua which are close to arcs, hence those continua with various rim-properties, and the classification of curves as well, became important research areas in Topology (see [28, p. 99], and [11]). Note that for many years, the terms continuous curve and ***Peano continuum*** were used to refer to locally connected metric continua.

Another study motivated by the Hahn–Mazurkiewicz Theorem and the Alexandroff–Hausdorff Theorem mentioned above, is the study of those compact Hausdorff spaces and continua which are continuous images of compact ordered spaces. By a **compact ordered space**, we mean a compact ***linearly ordered topological space***, in which the ***topology*** is the ***order topology***. In the metric case, the ***Cantor set*** is such an example.

In this article, the intention is to give a survey of results on rim-finite, rim-countable, rim-metrizable, rim-scattered spaces, mainly in the realm of compact spaces, and also rim-compact spaces and rim-separable spaces.

Since compact countable spaces are metrizable and scattered (recall that a space is said to be **scattered** if every non-empty subset of it has an ***isolated point***), we observe that, in the realm of compact spaces, the following implications are valid:

$$\begin{array}{l} \text{rim-finite} \\ \quad\Downarrow \\ \text{rim-countable} \Rightarrow \text{rim-scattered} \\ \quad\Downarrow \\ \text{rim-metrizable} \\ \quad\Downarrow \\ \text{rim-separable} \end{array}$$

In the literature, rim-finite spaces are sometimes called **regular** or **netlike** and rim-countable spaces are called **rational**, particularly in the case of metrizable spaces. Rim-compact spaces are also called **semi-compact**. When we consider compact spaces which are not 0-dimensional, it is natural to restrict our attention to compact and connected spaces, namely continua.

1. Metric continua

In the class of 1-dimensional (metric) continua, there is a wide spectrum. For instance, Knaster Bucket Handle is a 1-dimensional ***indecomposable continuum*** with basic opens sets having Cantor Set boundaries (see [11] or [17]). In this survey we are going to restrict our attention mainly to the class of ***decomposable*** continua, continua which can be written as a union of two distinct subcontinua.

In 1926, G.T. Whyburn [28] developed the theory of cyclic elements to study the structure of locally connected metric continua. A subset E of a locally connected continuum X is called a **cyclic element** if E is connected and maximal with respect to the property that no point ***separates*** E. For a locally connected continuum X, a property is said to be **cyclicly extensible** (respectively, **cyclicly reducible**) provided that when each cyclic element has this property, so does X (respectively, when the space X has this property, so does each cyclic element of X). Whyburn showed that the properties of being rim-finite, rim-countable, having small inductive dimension $\leqslant n$, and many others are cyclicly extensible and reducible (see [28]). The cyclic element theory is used by many mathematicians in the study of locally connected (metric) continua.

A space X is said to be **hereditarily locally connected**, if it is connected and each of its connected subsets is locally connected. Every rim-finite continuum is hereditarily locally connected and hereditarily locally connected continua are rim-countable (see [28, p. 91] for the metric case, for the non-metric case see Nikiel [18]), but there exists a hereditarily locally connected continuum which is not rim-finite (see [11, p. 283]). An easy generalization of arcs are dendrons. A **dendron** is a continuum in which every pair of distinct points can be ***separated*** by a third point. Each dendron is rim-finite. Note that a metric dendron is called **dendrite**. Dendrites are exactly the 1-dimensional compact AR.

Some other interesting subclasses of rim-finite continua are completely regular continua and totally regular continua. A metric continuum X is said to be **totally regular** (respectively, **completely regular**) if for each countable set $A \subset X$, there exists a base for X of open sets such that each element of the base has a finite boundary missing the set A (see [21] in the non-metric case) (respectively, if each subcontinuum has nonempty ***interior***). Each ***graph*** is a completely regular continuum; however, dendrons are not necessarily completely regular. The class of completely regular continua is contained in the class of totally regular continua. Note that metrizable totally regular continua are exactly the continua of finite linear measure studied by Eilenberg between 1938 and 1944, Eilenberg and Harrold in 1943, and later by Buskirk, Nikiel and Tymchatyn in 1992. The **linear measure** $\mu_\rho^*(A)$ of a subset A of a metric space (X, ρ) is defined by

$$\mu_\rho^*(A) = \sup_{\delta>0} \inf \Biggl\{ \sum_{i\in\mathbb{N}} \operatorname{diam}_\rho(A_i): \\ A \subseteq \bigcup_{i\in\mathbb{N}} A_i \subseteq X,\ \operatorname{diam}_\rho(A_i) \leqslant \delta\ \forall i \in \mathbb{N} \Biggr\}.$$

Fremlin [7] also studied spaces of finite linear measure in 1992; using infinite games, he proved that a ***Borel subset*** of a rim-finite continuum is ***arcwise connected*** if and only if it is connected. In the light of the above definitions, it is appropriate to state the following characterization of rim-finite metric continua due to Lelek [12]: A metric continuum X is rim-finite if and only if for each $\varepsilon > 0$, there exists an integer $n > 0$ such that any collection of pairwise disjoint subcontinua of X, each having diameter greater than ε, consists of at most n elements.

A natural technique to construct complicated topological spaces from simple ones is ***inverse systems***. However, the ***inverse limit*** of an inverse system of locally connected continua is not always locally connected unless some conditions are imposed on the ***bonding map***s. One such condition is that each bonding map be monotone. A map $f: X \to Y$ between topological spaces X and Y is said to be a **monotone map** if for each point $y \in Y$, $f^{-1}(y)$ is connected. It is known that the inverse limit of an inverse sequence of locally connected (1-dimensional) metric continua with monotone bonding maps is a locally connected (1-dimensional) metric continuum. Nikiel, Buskirk and Tymchatyn in 1992 showed that each totally regular (metric) continuum is ***homeomorphic*** to the inverse limit of an inverse sequence of graphs with monotone bonding maps (see also [21]). However, in general, the inverse limit of an inverse sequence of rim-finite continua with monotone bonding maps need not be rim-finite. Nikiel [19] proved that a locally connected 1-dimensional metric continuum X is homeomorphic to the inverse limit of an inverse sequence of rim-finite continua with monotone bonding maps. For instance, the 1-dimensional ***Menger Universal Curve***, which is a universal space for 1-dimensional compact metric spaces, can be obtained as an inverse limit using rim-finite continua and monotone bonding maps. (See the next section for the definition of universal space.) On the other hand, the Menger Universal Curve can also be constructed using an inverse sequence of graphs; however, in this case, the bonding maps cannot be monotone. These results indicate that the class of rim-finite (metric) continua contains a variety of interesting subclasses.

Lelek in [12] studied finitely Suslinian and Suslinian metric continua. A **Suslinian continuum** is one in which each subcollection of pairwise disjoint subcontinua is countable, and in a **finitely Suslinian continuum** each subcollection of pairwise disjoint continua forms a null-family. A family $\mathcal{F}$ of closed subsets of a compact space X is said to be a **null family**, if for each ***open cover*** $\mathcal{U}$ of X, the number of elements of $\mathcal{F}$ which are not contained in any element of $\mathcal{U}$ is finite. Each finitely Suslinian continuum is Suslinian. Lelek in [12] showed that each rim-finite continuum is finitely Suslinian, any finitely Suslinian continuum is hereditarily locally connected, and rim-countable continua are Suslinian. There exist examples of hereditarily locally connected metric continua which are not finitely Suslinian (see [11, p. 270]). There are also many rim-countable continua which are not locally connected (see [11] or [28]). Lelek [12] provided an example of a Suslinian metric continuum which is not rim-countable. However in the plane, hereditarily locally continua coincides with finitely Suslinian (see [11]). Note that each 1-dimensional Suslinian continuum is a **hereditarily decomposable continuum**, i.e., each subcontinuum is ***decomposable*** (see [12]).

Another problem involving rim-P spaces is the preservation of rim-properties under continuous maps. The Hahn–Mazurkiewicz Theorem and the theorem of Alexandroff indicate that rim-properties and dimension are not preserved under continuous maps in general. Hence, additional properties of maps are needed in order to preserve rim-properties. Two of such properties are: a map $f : X \to Y$ is said to be a **confluent map** (respectively, a **pseudo-confluent map**) if for any subcontinuum (respectively, ***irreducible continuum***) $K \subset Y$ we have $f(C) = K$ for every (respectively, some) ***component*** C of $f^{-1}(K)$. Note that each monotone map is confluent and each confluent map is pseudo-confluent. On the other hand, ***open maps*** (the image of every open set is open) are also confluent. Hence, confluent maps are generalizations of open and monotone maps.

Confluent images of dendroids (dendrites, arcs) are dendroids (dendrites, arcs, respectively), see [14, p. 69]. A continuum is said to be **unicoherent** provided that $A \cap B$ is connected whenever A and B are subcontinuum of X such that $A \cup B = X$. A **dendroid** is an arcwise connected hereditarily unicoherent continuum. However, pseudo-confluent image of a dendroid may not be a dendroid. In 1975, Lelek and Tymchatyn proved that pseudo-confluent maps preserve the following class of continua: Suslinian, hereditarily locally connected, finitely Suslinian, rim-finite. According to Grispolakis and Tymchatyn (1981), we can add to the above list the class of all graphs as well. On the other hand, Tymchatyn (1983) gave an example of a confluent map of a rim-countable continuum whose image is not rim-countable, and he also proved that any pseudo-confluent map $f : X \to Y$ preserves rim-countable continua provided Y is a locally connected continuum. For further discussion, we refer the reader to the 1979 paper of T. Maćkowiak [14] which contains an extensive survey on continuous maps of continua.

2. Rim-type, rational dimension and universal spaces

A notion which is used to classify rim-countable spaces and investigate properties of this class is rim-type (see [11, p. 290]). If A is a subset of a space X, let $A^{(0)} = A$ and denote by A' the set of all ***limit point***s of A. Then, for any ordinal α we define by transfinite induction the sets $A^{(\alpha)}$ as follows: if α is a successor of the ordinal β, define $A^{(\alpha)} = (A^{(\beta)})'$; if $\alpha > 0$ is a limit ordinal, then $A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$. A set A is said to be of **type** α if α is the least ordinal number such that $A^{(\alpha)} = \emptyset$. A space X is said to be of **rim-type** α if α is the least ordinal such that X admits a base of open sets with boundaries of type α. Note that each rim-finite space is of rim-type 1 and each rim-countable metric space is of rim-type α for some countable ordinal α (see also [11]). Tymchatyn (1975) proved that, for each countable ordinal α, there exists a hereditarily locally connected metric continuum of rim-type α, and also showed that a metric continuum X must be rim-finite given that each connected subset of X is arcwise connected. Z. Janiszewski in 1912 constructed an example of a rim-countable metric continuum of rim-type ω which contains no arcs. In contrast to these results, Grispolakis and Tymchatyn (1980) showed that every 1-dimensional continuum of finite rim-type contains arcs.

The notion of rational dimension was introduced by Menger and Nöbeling. A space X has rational dimension 0 if X is countable and nonempty, and X has **rational dimension** $\leqslant n$ if X has a base of open sets with boundaries of rational dimension $\leqslant n - 1$. It is known that a space X has a rational dimension $\leqslant n$ if and only if it can be written as a union of a countable set and a set with covering dimension $\leqslant n - 1$ (see [23]). Rim-countable spaces have rational dimension $\leqslant 1$. Nöbeling in 1934 showed that for each $n \geqslant 1$, there is a universal space in the class of separable metrizable spaces of rational dimension $\leqslant n$. Recall that a space X is said to be **universal space** for a class of spaces if X belongs to the class and contains a homeomorphic copy of every member of the class. Iliadis and Georgiu in 1998 also gave a construction of universal spaces for separable metrizable spaces of rational dimension $\leqslant n$. However, the class of rim-countable metric compact spaces (continua) contains no universal space. In fact, Iliadis [9] proved the following: For every metric space Y of rim-type $\leqslant \alpha$, there is a locally connected continuum X of rim-type $\leqslant \alpha$ such that X cannot be embedded in Y. Mayer and Tymchatyn [15] proved the existence of a universal space in the class of separable rim-countable spaces, and obtained the following results: For a countable ordinal $\alpha > 0$,

(i) there exists a universal space for the class of separable metric spaces of rim-type $\leqslant \alpha$;
(ii) there is a locally connected metric continuum of rim-type α which topologically contains every compact space of rim-type $< \alpha$.

When we consider universal spaces and containing spaces for rim-finite spaces, we see an interesting picture. Nöbeling (1931) proved that there is no universal space for compact rim-finite spaces. Iliadis (1980) constructed a universal completely regular continuum and Buskirk (1994) constructed a universal totally regular metric continuum. In 1986, Iliadis showed that there is a rim-finite T_1***-space*** (nonmetric) which contains every Hausdorff rim-finite space. Mayer and Tymchatyn [15] constructed a locally connected metric continuum of rim-type 2 which topologically contains every rim-finite space. We refer the reader to [8] and [23] for further discussion.

3. Rim-compact spaces, compactifications and rim-separable spaces

Another question that has been studied involves ***compactifications*** of spaces with various rim-properties. For a continuum X, each point p of X where X is fails to be rim-finite or rim-countable is contained in a nondegenerate continuum which consists of such points (see [28, p. 98], and also [11]).

A separable metric space is rim-countable if and only if it is the union of a countable set and a set which is at most 0-dimensional (see [11, p. 285]). These results indicate that it is natural to consider compactifications with 0-dimensional ***remainders*** if the rim-properties are to be preserved under compactification.

The class of rim-compact spaces is studied widely in relation to compactifications and dimension theory (see [1] and [5]). One of the main motivations to study rim-compact spaces is due to H. Freudenthal. Freudenthal in 1942 studied the compactification of rim-compact spaces with 0-dimensional remainders in the class of separable metric spaces. Later, various constructions of such compactifications were given. Morita in 1952 gave the following characterization: For a rim-compact space X, the **Freudenthal compactification** of X is the topologically unique compact space Y such that

(i) X is ***dense*** in Y,
(ii) any point $x \in Y$, has a ***neighbourhood*** V in Y whose boundary is contained in X, and
(iii) any two disjoint closed subsets of X with compact boundaries have disjoint ***closure***s in Y.

Note that the space Y is maximal among such compactifications and the small inductive dimension of the remainder $Y \setminus X$ is zero. It should be noted that the covering dimension or the big inductive dimension of the remainder may not be zero. In 1993, Aarts and Coplakova gave an example of a rim-compact space X such that every compactification of X has a remainder with covering dimension $\geqslant 1$.

In 1942, de Groot showed that a separable metric space X has a metric compactification with a zero-dimensional remainder if and only if X is rim-compact. Each rim-finite space has a rim-finite compactification (see [11, p. 290]). It is not difficult to see this fact: each rim-finite space is rim-compact, and hence it has a Freudenthal compactification, and since the remainder is 0-dimensional, the compactification is also rim-finite. Tymchatyn (1977) proved that the Freudenthal compactification of a rim-compact, hereditarily locally connected ***Tychonoff*** space is hereditarily locally connected; moreover, he also showed that the Freudenthal compactification of any connected, separable metric space is metrizable. On the other hand, the Knaster–Kuratowski example of a connected set with a ***dispersion point*** (see [E]) is rim-countable and rim-compact. Tymchatyn (1977) showed that this space has no rim-countable compactification. Later, Iliadis and Tymchatyn [10] established that a separable metric space of rim-type $\alpha + n$ (where n is nonnegative integer and α is either 0 or a limit ordinal) has a compactification of rim-type $\leqslant \alpha + 2n + \min\{\alpha, 1\}$.

A **0-space** is a Tychonoff space which has a compactification with 0-dimensional remainder. It is known that rim-compact (Tychonoff) spaces are 0-spaces. It is known that there are non rim-compact spaces with compactifications such that the remainder is 0-dimensional. B. Diamond in 1985 gave a characterization of 0-spaces and studied the relationships among the classes of rim-compact spaces, almost rim-compact spaces and 0-spaces (see [4]). A Tychonoff space X is **almost rim-compact** if X has a compactification Y such that each point in the remainder $Y \setminus X$ has a base of open sets of Y whose boundaries lie in X. Note that each rim-compact space is almost rim-compact and each almost rim-compact space is an 0-space (see also [3]).

F.B. Jones (1932) considered the question of separability in rim-separable spaces. He proved that each locally connected, connected, rim-separable metric space is separable. L.B. Treybig (1959) constructed an example of a connected rim-separable metric space which is not separable, and he also proved (1960) that in a connected rim-separable metric space, the points where the space fails to have separable neighbourhoods form a ***perfect set***. Motivated by Jones and Treybig's results, P. Roy [24] constructed a non-separable, connected and rim-compact metric space X containing a separable subset A such that X fails to be ***locally compact*** at each point of A, but locally connected and rim-connected at each point of A. This example of Roy answers a question of Isbell.

4. Non-metric case: Rim-metrizable compact spaces

It is natural to consider generalizations of the Hahn–Mazurkiewicz theorem and the Alexandroff theorem in the class of compact (Hausdorff) spaces. In 1960, Mardešić constructed an example of a locally connected continuum which is not a continuous image of an arc. Following this result, a considerable effort has been put into characterizing images of arcs (ordered continua) and compact ordered spaces. Nikiel (1989) gave a characterization of locally connected continua that are continuous images of compact ordered spaces. In 1973, Heath, Luzter and Zenor proved that continuous images of compact ordered spaces are ***monotonically normal***. Recently, M.E. Rudin [25] gave a complete characterization of this class by showing that each monotonically normal compact space is a continuous image of a compact ordered space.

The Nikiel characterization of continuous images of arcs is based on approximation by T-sets (see [21]). A closed subset A of a locally connected continuum is said to be a **T-set**, if each component in the closure of $X - A$ has exactly a two point boundary. Treybig [26] proved that any continuum X which is the continuous image of a compact ordered space is either metrizable or it contains a set K such that K separates X and contains at most two points. Later result of Treybig and Maehara (1984) led to the following result of Nikiel (see [21]): If a locally connected continuum X with no ***cut points*** is a continuous image of a compact ordered space, then each closed metrizable subset of X is contained in a T-set. Using this result, Nikiel (1989) showed that a locally connected continuum X with no cut points is a continuous image of an arc if and only if it can be approximated by sequence of T-sets $A_1, A_2, \ldots, A_n, \ldots$ satisfying the following conditions: (i) $\bigcup A_n$ is dense in X; (ii) A_1 is metrizable; (iii) $A_1 \subset A_2 \subset \cdots$; (iv) if J is a component of

$Y \setminus A_n$ for some n, then the set of separating points of the closure of J is contained in A_{n+1}; and (v) if J is a component of $Y \setminus A_n$ for some n and M is a nondegenerate cyclic element of the closure of J, then $M \cap A_{n+1}$ is a metrizable T-set which contains at least three points. On the other hand, Cornette (1974) extended Whyburn's cyclic extension and reduction theorem in the metric case to continuous images of arcs. He proved that a locally connected continuum X is a continuous image of an arc if and only if each cyclic element of X is a continuous image of an arc. Now combining Cornette's result with Nikiel's yields that a locally connected continuum X is a continuous image of an arc if and only if each cyclic element in X can be approximated by T-sets satisfying the conditions stated above. A shorter version of Nikiel's characterization of images of arcs reads as follows: a locally connected continuum X is a continuous image of an arc if and only if any three-point set in a cyclic element of X is contained in a metrizable T-set (see also [21]).

In addition to the problem of characterizing those spaces which are continuous images of arcs/compact ordered spaces, there is the problem of determining which classes of locally connected continua are continuous images of arcs. Nikiel [18] proved that each hereditarily locally connected continuum is a continuous image of an arc. By using a construction of Filippov (1969), Nikiel, Tuncali and Tymchatyn (1991) provided an example of a locally connected rim-countable continuum which is not a continuous image of an arc. Using T-set approximations, the following results were obtained by Nikiel, Tymchatyn and Tuncali [21]:

(i) the inverse limit of any inverse sequence of continuous images of arcs with monotone bonding maps also is a continuous image of an arc;
(ii) each 1-dimensional continuous image of an arc can be obtained as the inverse limit of an inverse sequence of rim-finite continua with monotone bonding maps.

These are generalizations of the results stated earlier in the case of metric locally connected continua, and they indicate that 1-dimensional continuous images of arcs behave like 1-dimensional metric locally connected continua. The reader is referred to [21] for further discussion on behavior of the images of arcs and subclasses under inverse limits.

In 1991, Nikiel, Tuncali and Tymchatyn [20] established that if X is a continuous image of an arc, then $\dim(X) = \operatorname{ind}(X) = \operatorname{Ind}(X) = \max\{1, \sup\{\dim(Y)\colon Y \subset X \text{ is closed and metrizable}\}\}$, where dim, ind and Ind refer to covering dimension, small inductive dimension, and large inductive dimension respectively. Treybig (1979) proved that any continuous image of an arc should be rim-finite provided that it does not contain a non-degenerate metric continuum.

A result of Mardešić [16] states that continuous images of compact ordered spaces are rim-metrizable. This was later strengthened by Grispolakis, Nikiel, Simone and Tymchatyn (1993) by showing that each irreducible **separator** (i.e., no proper subset is a separator, where a separator is a set whose removal leaves a space disconnected) in a continuous image of a compact ordered space is metrizable. Hence, it is also natural to ask what properties of images of compact ordered space can be generalized to the classes of rim-metrizable, rim-countable or rim-scattered spaces. Treybig (1964) proved that continuous images of compact ordered spaces do not contain a product of nonmetric compact infinite spaces. Tuncali [27] proved that analogues of Treybig's product theorem can be obtained in the class of rim-metrizable, rim-countable continua. A more precise formulation of this result reads as follows: If X is a rim-metrizable continuum or a rim-scattered locally connected continuum, then it does not contain a nonmetric product of a nondegenerate compact space and a perfect set. Here is a consequence of this theorem. The product $X \times Y$ of two compact spaces is rim-metrizable if and only if one of the following holds:

(a) both X and Y are metrizable;
(b) both X and Y are 0-dimensional;
(c) one of the spaces X and Y is rim-metrizable and the other one is metrizable and 0-dimensional.

In the case of rim-compact spaces, the last result can be compared with an earlier result of B. Diamond (1984): the product $X \times Y$ of two rim-compact spaces is rim-compact if and only if one of next conditions is satisfied

(i) both X and Y are locally compact;
(ii) both X and Y are 0-dimensional;
(iii) one of X, Y is locally compact and 0-dimensional.

Mardešić (1961) proved that ***light map***s of locally connected continua, as well as ***irreducible*** light maps of compact ordered spaces, preserve ***weight***. A similar fact is true for light maps of rim-metrizable continua (see [27]).

In recent years (1997–2001) Lončar studied σ directed inverse systems of locally connected continua and images of arcs with monotone bonding maps. Using Tuncali's weight preservation result [27], he also showed that each rim-metrizable continuum is the inverse limit of an inverse system of metrizable continua with monotone bonding maps, and applied this result to investigate ***hyperspaces*** of rim-metrizable continua. He obtained that a rim-metrizable dendroid with the property of Kelley is smooth [13]. A continuum X has the **property of Kelley** if for each $x \in X$, each $A \in C(X)$ (the space of all subcontinua of X with the ***Vietoris topology***), with $x \in A$, and each open neighbourhood V of A in $C(X)$ there exists an open neighbourhood W of x in X such that if $y \in W$, then there exists $B \in C(X)$ such that $y \in B \in V$. A continuum X is **smooth** at $p \in X$ if for each ***convergent net*** $\{x_n\colon n \in D\}$ in X and for each $K \in C(X)$ such that $p, x \in K$, where $x = \lim x_n$ in X, there exists a net $\{K_i\colon i \in E\}$ in $C(X)$ such that each K_i contains p and some x_n and $\lim K_i = K$ in $C(X)$. A **smooth continuum** is a continuum which is smooth at every point.

Any rim-scattered space which is a continuous image of an arc must be rim-countable (Nikiel, Tymchatyn and Tuncali [20]). On the other hand, Nikiel, Treybig and Tuncali [22] constructed a rim-metrizable locally connected continuum having continuous image that is not rim-metrizable. There exists also an example of rim-scattered locally connected continuum which is not rim-countable provided by

Drozdovskiĭ and Filippov [6]. This example has a base of open sets with metrizable 0-dimensional boundaries. In 1999, Drozdovskiĭ constructed a rim-countable locally connected continuum which is not arcwise connected. Constructions of these examples utilize inverse limits of transfinite inverse sequences of length ω_1 consisting of compact metric spaces and techniques of Filippov used in the construction of a ***perfectly normal*** compacta in 1969. Filippov constructed a perfectly normal compact space having a base of open sets with 0-dimensional metrizable boundaries. His construction involved replacing points of a ***Lusin set*** in the unit square with circles. In 1990, Gruenhage presented an example of a perfectly normal locally connected, rim-metrizable continuum which is not arcwise connected. Both of the last two constructions were done assuming CH (Continuum Hypothesis).

If an F_σ***-subset*** of a locally connected continuum has a 0-dimensional boundary, then its boundary is metrizable (see [16]). Using this result, one can prove that each locally connected perfectly normal continuum X with ind(X)=1 is rim-metrizable. The last fact and the above examples of Filippov and Gruenhage relate to the consistency question of M.E. Rudin: Is it consistent that a locally connected perfectly normal continuum is metrizable? (see [vMR, pp. 86–95]). It is known that the product $X \times [0, 1]$ of a locally connected perfectly normal continuum with [0,1] is perfectly normal and locally connected. However, the results of [27] imply that $X \times [0, 1]$ is rim-metrizable if and only if X is metrizable.

The reader is referred to [2] for further discussion of open problems concerning rim-metrizable continua.

In conclusion, there is a rich theory behind the study of topological spaces with various rim-properties. In this paper a sample of results is examined in order to give a sense of relationships between various classes of rim-P spaces and their connection to other problems and topics in Topology. There are many results and names which are not mentioned in this paper. There are many problems that have not yet been solved. Many of these problems are discussed in the papers listed in the references.

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b-12 Categorical Topology

Categorical topology applies results and methods of **Category Theory** to topological structures and, conversely, generalizes some results and methods from topological structures to more general categories. The word "category" comes here from the abstract Category Theory and not from the terms "sets of first or second category". The next description is restricted to some parts of categorical topology only and is far from being precise because otherwise one would need to define many categorical concepts, which is outside the scope of this publication.

A **category** $\mathcal{C}$ is defined as a class of **objects** $X, Y, \ldots$ (like ***topological space***s, groups, lattices) together with sets $\mathcal{C}(X, Y)$ of **morphisms** between objects X, Y (like ***continuous maps***, homomorphisms, ***monotone maps***) satisfying some natural conditions (composition of morphisms exists, it is associative and has units, i.e., identity morphisms on objects exist) – see, e.g., the book [1] for details. Thus, one has a category Set of sets and maps, Top of topological spaces with continuous maps, Unif of ***uniform spaces*** with ***uniformly continuous*** maps, TopGr of ***topological group***s with continuous homomorphisms. For ***metric spaces*** as objects one may choose for morphisms, e.g., continuous maps or uniformly continuous maps or ***Lipschitz maps*** or ***contraction***s.

A subclass $\mathcal{K}$ of objects of $\mathcal{C}$ together with subsets of morphism that form a category under the same composition and units is called a **subcategory** of $\mathcal{C}$ (in case the sets of morphisms of $\mathcal{K}$ are the same as in $\mathcal{C}$, one calls $\mathcal{K}$ a **full subcategory** of $\mathcal{C}$). Thus, the above subcategories of metric spaces with continuous maps composed of all metric spaces and uniformly continuous maps, or Lipschitz maps, or contractions, are nonfull subcategories; the category Haus of ***Hausdorff spaces*** and continuous maps is a full subcategory of Top.

Like for other mathematical structures, one has maps between categories preserving the structure. A map F that maps objects of a category $\mathcal{C}$ into objects of a category $\mathcal{K}$ and sets $\mathcal{C}(X, Y)$ into the sets $\mathcal{K}(F(X), F(Y))$ (or $\mathcal{K}(F(Y), F(X))$) is called a **covariant functor** (or a **contravariant functor**, respectively) provided it preserves the composition of morphisms and the identity morphisms. For instance, the so-called **forgetful functor**s from Top to Set or Unif to Set or TopGr to Gr, forgetting a structure and keeping the underlying sets (or groups) and maps are covariant; the functor assigning to sets their power sets and to maps $f : X \to Y$ the maps assigning to $A \subset Y$ its preimage $f^{-1}(A)$ is contravariant.

To describe a property by categorical concepts means to characterize it by using maps and their compositions, without using points. For instance, a one-to-one map f from a set X into a set Y (in symbols $f : X \to Y$) can be characterized by the property: if g, h are maps from a set Z into X such that $f \circ g = f \circ h$ then $g = h$. Such morphisms f are called **monomorphism**s. Dually (i.e., by reversing arrows) we get so called **epimorphisms** corresponding to maps f onto Y: if g, h are maps from Y into a set Z such that $g \circ f = h \circ f$ then $g = h$. Epimorphisms are not always maps onto; e.g., in the above category Haus the epimorphisms from X into Y are continuous maps onto ***dense subsets*** of Y.

A **product** of objects X_i, $i \in I$, is an object X together with morphisms $\mathrm{pr}_i : X \to X_i$ such that for any other object Y and morphisms $f_i : Y \to X_i$ there is a unique morphism $f : Y \to X$ with $\mathrm{pr}_i \circ f = f_i$ for all i. This definition gives the usual ***Cartesian product***s of structures (up to isomorphism, of course) in all usual categories. It is a special case of limits in categories (dual notion: colimit).

Another important notion from Category Theory is that of **reflection** of an object X of a category $\mathcal{C}$ in a subcategory $\mathcal{K}$: it is an object Y of $\mathcal{K}$ together with a morphism $r : X \to Y$ such that any other morphism $f : X \to Z$ into an object Z of $\mathcal{K}$ yields a unique morphism g of $\mathcal{K}$ with $f = g \circ r$. Examples in Top are ***completely regular*** modifications (reflections in the full subcategory of Top composed of all completely regular spaces), ***Čech–Stone compactifications*** (reflections in compact Hausdorff spaces), ***Hewitt–Nachbin realcompactifications*** (reflections in realcompact Hausdorff spaces), ***completions*** in metric spaces with uniformly continuous maps (reflections in complete metric spaces) – the reflective subcategory is usually clear from the name of the hull or the modification. The dual notion is called **coreflection**, e.g., ***locally connected*** modification in Top, ***sequential*** modification in Top, ***topologically fine uniformities*** in Unif. A map F assigning to X from $\mathcal{C}$ its reflection $Y = F(X)$ in $\mathcal{K}$ can be extended to morphisms making F a functor (called a **reflector**); the dual notion is **coreflector**. All the mentioned examples above have a common feature, namely that the morphism r is an epimorphism in $\mathcal{C}$; if this is the case, then $\mathcal{K}$ is said to be **epireflective** in $\mathcal{C}$ (and F an **epireflection**). Dually one has **monoreflective** subcategories – all the examples above are of that kind (in fact, they must be such). These special reflective and coreflective subcategories have simple characterizations (see below). Reflectors and coreflectors are special cases of so called adjoint functors that can describe many occurring relations between categories.

Systematic investigation of interactions between general topology and Category Theory started around 1960, thus about 10 years later than that between algebraic topology and Category Theory. Since that time the field grew up so that it got a name **Categorical Topology**. For main surveys of results and developments see [3–5]. There is no such survey from 90s, but some proceedings of conferences are a good source (e.g., [6]).

As mentioned above, interactions between category theory and topological structures are of two basic forms. Either general results and methods of Category Theory can be used to investigate special classes of topological structures, or topological results and methods are generalized to categories to get more general methods that can be applied backwards to other topological structures. As an example one can use the Čech's method of a construction of Čech–Stone compactification that was generalized to get reflections of a category $\mathcal{C}$ in a subcategory $\mathcal{K}$ (in the original case $\mathcal{C}$ is the category of ***Tychonoff space***s with continuous maps and $\mathcal{K}$ is the subcategory of ***compact*** spaces with continuous maps). To get reflections, one needed that products in $\mathcal{C}$ of spaces from $\mathcal{K}$ belong to $\mathcal{K}$, that some nice subspaces of spaces from $\mathcal{K}$ belong to $\mathcal{K}$, too (in the original case one uses closed subspaces) and, finally, that there is a representative set of epimorphisms from a given space into spaces from $\mathcal{K}$. So, "if $\mathcal{K}$ is productive and nicely hereditary in $\mathcal{C}$ and has the above representative sets, then every object of $\mathcal{C}$ has an epireflection in $\mathcal{K}$". From this characterization we know that various hulls and modifications exist without constructing them.

Unlike algebraic structures, topological ones on the same set can be compared (one has **finer** or **coarser** structures, i.e., identity maps on underlying sets are morphisms between non-coinciding objects). That makes some topological characterizations more complicated (e.g., subspaces); on the other hand, it allows to use the so called initial (other terms: weak, projective) structures and their duals (final, or strong or inductive structures, respectively). In Top, an initial structure (an **initial topology**) on a set X generated by maps f_i into topological spaces X_i is the **coarsest topology** (with the fewest open sets) on X making all f_i continuous. Likewise a final structure (a **final topology**) generated on a set X generated by maps f_i from spaces X_i into X is the **finest topology** (with the most open sets) that makes all f_i continuous. It is clear how to carry over that definition to Unif, for instance. In TopGr one must take groups instead of sets and homomorphisms instead of maps (for ***topological linear space***s one takes linear spaces and linear maps). That definition has a sense in any category of richer structures defined on some poorer structures. If all initial or final structures exist in a category $\mathcal{C}$ with respect to some "poorer" category $\mathcal{K}$, one says that $\mathcal{C}$ is a **topological category** over $\mathcal{K}$. Thus Top and Unif are topological categories over Set (there are much more such categories, e.g., ***convergence spaces***, **nearness spaces**[1] and their nice coreflective and reflective subclasses like completely regular spaces, sequential spaces), TopGr is topological over Gr, topological linear spaces are topological over the category of linear spaces and linear maps. Many general results for the class of topological spaces are valid in topological categories.

Since Top lacks some nice general properties (e.g., products do not preserve ***quotients***, Top is not **Cartesian closed**, i.e., $X^{Y\times Z} \neq X^{(Y^Z)}$ where the power A^B means a space of all morphisms from B into A), new topological categories were described that are "nicer" from the point of view of those properties. For instance, ***k-spaces*** form a Cartesian closed subcategory of Top, various convergence spaces form a Cartesian closed category containing Top.

There are many topological notions that were generalized to other categories. One may mention ***connectedness*** (some morphisms into given objects are of a given form, e.g., they are constant), ***closure operator***s (see the book [2]), or some constructions in compact spaces (see the book [7] on normal functors). Categorical methods are also used in finding representations of algebraic structures in topological ones; one of the Trnková's results (see [5] for this and other similar results) asserts that "every finite Abelian group can be represented by products of compact 0-***dimensional separable*** topological spaces".

[1]For a set X, $P(X)$ denotes the power set of X. Let $\xi \subset P(P(X))$. Then the pair (X, ξ) is called a nearness space if the following conditions (i)–(v) hold for every $\mathcal{A}, \mathcal{B} \in P(P(X))$: (i) if $\bigcap \mathcal{A} \neq \emptyset$, then $\mathcal{A} \in \xi$, (ii) if $\mathcal{B} \in \xi$ and if for each $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $B \subset A$, then $\mathcal{A} \in \xi$, (iii) if $\{A \cup B\colon A \in \mathcal{A}, B \in \mathcal{B}\} \in \xi$, then either $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$, (iv) if $\{\mathrm{cl}\, A\colon A \in \mathcal{A}\} \in \xi$, then $\mathcal{A} \in \xi$, where $x \in \mathrm{cl}\, A$ if and only if $\{\{x\}, A\} \in \xi$, (v) $\emptyset \in \xi$ and $P(X) \notin \xi$.

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b-13 Special Spaces

Many examples play an important role in the study of general topology. We introduce some of them which are of general interest. Let ω denote the first infinite cardinal and let $\mathfrak{c} = 2^{\omega}$. We shall denote by $\mathbb{R}$ the set of real numbers, $\mathbb{Q}$ the set of rational numbers and $\mathbb{N}$ the set of positive integers. The real line $\mathbb{R}$ is assumed to have the usual ***topology*** unless otherwise stated.

Alexandroff's double circle

The **Alexandroff double circle** is a classic example of a ***first-countable***, ***compact***, non-***metrizable*** space which is easy to describe [1]. Consider in the plane $\mathbb{R}^2$ two concentric circles $C_i = \{(i, \theta)\colon 0 \leqslant \theta < 2\pi\}$ for $i = 1, 2$, where points are represented by polar coordinates. Define $A = C_1 \cup C_2$ and let $p : C_1 \to C_2$ be the bijection defined by $p((1, \theta)) = (2, \theta)$ for each θ. For each $x \in C_1$ and each $n \in \mathbb{N}$, let $U(x, n)$ be the open arc of C_1 with center x and of length $1/n$. Topologize A by letting sets of the form

$$U(x, n) \cup p\big[U(x, n) \setminus \{x\}\big], \quad n \in \mathbb{N},$$

be basic ***neighbourhoods*** of a point $x \in C_1$ and declaring points of C_2 to be ***isolated***. The resulting space A is called the Alexandroff double circle. The space A is first-countable, compact, ***Hausdorff*** and ***hereditarily normal***, but is not ***perfectly normal*** since the ***open*** set C_2 is not an ***F_σ-set*** of A [E, 3.1.26]. Moreover, A does not satisfy the ***countable chain condition*** since C_2 is an uncountable set of isolated points. Thus, A is neither ***separable*** nor ***hereditarily Lindelöf***.

Engelking [4] generalized the construction of the space A by replacing the circle C_1 by an arbitrary space X as follows: For every space X, consider a disjoint copy X_1 of X and the union $A(X) = X \cup X_1$. Let $p : X \to X_1$ be the natural bijection. Topologize $A(X)$ by letting sets of the form $U \cup p[U \setminus \{x\}]$, where U is a neighbourhood of x in X, be basic neighbourhoods of a point $x \in X$ and declaring points of X_1 to be isolated. The space $A(X)$ is called the **Alexandroff duplicate** of X. Let $D(\kappa)$ be the ***discrete space*** of cardinality κ. Engelking [4] and Juhász [11] used the Alexandroff duplicate to prove that for every infinite cardinal κ, the ***product*** $D(\kappa)^{2^\kappa}$ includes $D(2^\kappa)$ as a ***closed subspace***, despite the fact that the ***density*** $d(D(\kappa)^{2^\kappa}) = \kappa$ by the **Hewitt–Marczewski–Pondiczery Theorem** [E, 2.3.15]. Since the projection $f : A(X) \to X$ is a ***perfect map***, if X is compact, so is $A(X)$, and if X is metrizable, then $A(X)$ is a ***paracompact M-space***. It is also known that $A(X)$ is Hausdorff, ***Tychonoff*** and ***normal*** if X has the corresponding property (see [4, 9, 11]).

Arens' space

Arens' space is an example of a countable space whose topology cannot be described by ***sequences*** alone. Let $X = \{(0, 0)\} \cup (\mathbb{N} \times \mathbb{N})$. For each $f \in \mathbb{N}^{\mathbb{N}}$ and each $k \in \mathbb{N}$, let

$$U(f, k) = \big\{(0, 0)\big\} \cup \bigcup_{i>k} \big\{(i, j) \in \mathbb{N} \times \mathbb{N}\colon j > f(i)\big\}.$$

Topologize X by letting sets of the form $U(f, k)$, where $f \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$, be basic neighbourhoods of the point $(0, 0)$ and declaring points of $\mathbb{N} \times \mathbb{N}$ to be isolated. The space X is called **Arens' space**. The subset $\mathbb{N} \times \mathbb{N}$ is not closed in X, but is ***sequentially closed*** since every convergent sequence in X is eventually constant [E, 1.6.20]. Thus, X is not a ***sequential space***. More precisely, X is not even a ***k-space*** since every compact set in X is finite [E, 3.3.24]. Franklin [7] showed that the space X can be embedded in a sequential space by defining the space

$$Y = \big\{(0, 0)\big\} \cup \big(\mathbb{N} \times \big(\mathbb{N} \cup \{0\}\big)\big)$$

with the following topology. A basic neighbourhood of the point $(0, 0)$ is a set of the form $U(f, k) \cup \{(i, 0)\colon i > k\}$ for $f \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$; a basic neighbourhood of the point $(i, 0)$, $i \in \mathbb{N}$, is a set of the form $\{(i, 0)\} \cup \{(i, j)\colon j > k\}$ for $k \in \mathbb{N}$; and each point of $\mathbb{N} \times \mathbb{N}$ is isolated. The space Y is a ***zero-dimensional***, sequential, ***T_1-space*** and includes Arens' space as a ***dense*** subspace [E, 1.6.19]. The latter fact implies that Y is not a ***Fréchet space***, since a subspace of a Fréchet space is Fréchet and every Fréchet space is sequential. Moreover, Y is Lindelöf and perfectly normal, because Y is countable.

Cantor set

The Cantor set is a well-known example of a ***nowhere dense***, closed, ***perfect*** subset (i.e., a closed set with empty interior and with no isolated points) of $\mathbb{R}$ which is useful in many areas of mathematics. Now, we construct the Cantor set. Start with the closed unit interval $I_0 = [0, 1]$. Let I_1 be the set obtained from I_0 by deleting its middle third $(1/3, 2/3)$, i.e., $I_1 = [0, 1/3] \cup [2/3, 1]$. Let I_2 be the set obtained from I_1 by deleting its middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$. In general, I_{n+1} is obtained from I_n by deleting the middle open interval, with length $1/3^{n+1}$, of each component of I_n. Then the intersection

$$\mathbb{C} = \bigcap_{n \in \mathbb{N}} I_n$$

is called the **Cantor set** or the **Cantor discontinuum**. The set $\mathbb{C}$ is a nowhere dense, closed, perfect set with Lebesgue measure zero. As a subspace of $\mathbb{R}$, it is not only a zero-dimensional, compact, metric space but also a ***universal space*** for zero-dimensional, separable, ***metric spaces***, i.e., every zero-dimensional, separable, metric space is homeomorphic to a subspace of $\mathbb{C}$ [E, 6.2.16]. Conversely, every ***complete*** metric space with no isolated point has a subspace which is ***homeomorphic*** to $\mathbb{C}$ [E, 4.5.5]. Further, every closed subspace of $\mathbb{C}$ is a ***retract*** of $\mathbb{C}$ and every compact metrizable space is a ***continuous*** image of $\mathbb{C}$ [E, 4.5.9]. It is known that the Cantor set is homeomorphic to the product D^ω of countably many copies of the two-point discrete space $D = \{0, 1\}$ [E, 3.1.28].

For an infinite cardinal κ, the product D^κ is called the **generalized Cantor discontinuum** or the **Cantor cube** of ***weight*** κ. The space D^κ is a universal space of zero-dimensional spaces of weight κ and every compact space of weight κ is a continuous image of a closed subspace of D^κ [E, 6.2.16 and 3.2.2]. More results related to the generalized Cantor discontinuum can be found in [E].

Hedgehog

Let κ be an infinite cardinal, S a set of cardinality κ, and let $I = [0, 1]$ be the closed unit interval. Define an equivalence relation E on $I \times S$ by $(x, \alpha)\ E\ (y, \beta)$ if either $x = 0 = y$ or $(x, \alpha) = (y, \beta)$. Let $H(\kappa)$ be the set of all equivalence classes of E; in other words, $H(\kappa)$ is the quotient set obtained from $I \times S$ by collapsing the subset $\{0\} \times S$ to a point. For each $x \in I$ and each $\alpha \in S$, $\langle x, \alpha\rangle$ denotes the element of $H(\kappa)$ corresponding to $(x, \alpha) \in I \times S$. There exist two natural topologies on $H(\kappa)$. The first one is the topology induced from the ***metric*** d on $H(\kappa)$ defined by

$$d\big(\langle x, \alpha\rangle, \langle y, \beta\rangle\big) = \begin{cases} |x - y| & \text{if } \alpha = \beta, \\ x + y & \text{if } \alpha \neq \beta. \end{cases}$$

The set $H(\kappa)$ with this topology is called the **hedgehog** of spininess κ and is often denoted by $J(\kappa)$ [E, 4.1.5]. The space $J(\kappa)$ is a complete, non-compact, metric space of weight κ. Hence, it is not ***totally bounded*** [E, 4.3.B and 4.3.29]. Moreover, the product $J(\kappa)^\omega$ is a universal space of metrizable spaces of weight κ; in particular, every ***completely metrizable*** space of weight κ is homeomorphic to a closed subspace of $J(\kappa)^\omega$ [E, 4.4.9 and 4.4.B]. It is known that $J(\kappa)$ is not ***strongly paracompact*** when $\kappa > \omega$ [E, 6.1.E]. Further, a T_1-space X is ***collectionwise normal*** if and only if for each cardinal κ and each closed subspace A of X, every continuous map $f: A \to J(\kappa)$ is continuously extendable over X [E, 5.5.1].

The second topology on $H(\kappa)$ is the ***quotient topology*** induced from the ***product topology*** of $I \times S$, where I has the usual topology and S has the discrete topology. The set $H(\kappa)$ with this topology is called the **fan** and is now denoted by $F(\kappa)$. The product space $I \times S$ is metrizable and the natural map $\varphi: I \times S \to F(\kappa)$ is closed, while the space $F(\kappa)$ is not first-countable, since the ***character*** of the point $\langle 0, \alpha\rangle$ is uncountable. The space $F(\kappa)$ is a typical example of a ***Lašnev space*** (i.e., the image of a metrizable space under a closed map) which is not metrizable.

Lexicographically ordered square

The lexicographically ordered square is another example of a first-countable, compact, non-metrizable space. Consider the linear order $<$ on the square I^2 defined by $(x_1, y_1) < (x_2, y_2)$ whenever either $x_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$. This order $<$ is called the **lexicographic order** and the ordered set $(I^2, <)$ with the ***order topology*** induced by $<$ is called the **lexicographically ordered square**. The space $(I^2, <)$ is compact and first-countable, but does not satisfy the countable chain condition, since $\{\{x\} \times \{y: 0 < y < 1\}: x \in I\}$ is an uncountable family of disjoint open sets. Hence $(I^2, <)$ is neither separable nor hereditarily Lindelöf. Moreover, $(I^2, <)$ is not perfectly normal since the open set $I \times \{y: 0 < y < 1\}$ is not an F_σ-set [E, 3.12.3]. The closed subspace

$$X = \big\{(x, 0): 0 < x \leqslant 1\big\} \cup \big\{(x, 1): 0 \leqslant x < 1\big\}$$

of $(I^2, <)$ is called the **two-arrow space**, which is also a classic example of a first-countable, compact, non-metrizable space [1]. Unlike the Alexandroff double circle or the space $(I^2, <)$, the space X is ***hereditarily separable*** and hereditarily Lindelöf and thus perfectly normal. It is, however, known that $X \times X$ is not hereditarily normal. Hence, X is not metrizable. For more details, see [E, 3.10.C].

Michael line

The Michael line is an example constructed by E. Michael to show that the product of a normal T_1-space with a metric space need not be normal. Let $\mathbb{P}$ be the set of irrational numbers, i.e., $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$. The **Michael line** M is the set $\mathbb{R}$ with the new topology generated by the ***base***

$$\mathcal{B} = \{U \cup K: U \text{ is open in } \mathbb{R} \text{ and } K \subset \mathbb{P}\}.$$

In other words, M is the space obtained from the real line $\mathbb{R}$ by making each point of $\mathbb{P}$ isolated. The space M is a paracompact, Hausdorff space, and hence, a normal T_1-space. If we consider $\mathbb{P}$ a subspace of $\mathbb{R}$, then the product $M \times \mathbb{P}$ is not normal, because the sets $\mathbb{Q} \times \mathbb{P}$ and $\{(x, x): x \in \mathbb{P}\}$ are disjoint closed in $M \times \mathbb{P}$ but cannot be separated by disjoint open sets [E, 5.1.32]. It is also known [14] that the product M^n is paracompact for all $n < \omega$, but M^ω is not normal. Thus, M^n is not homeomorphic to M^ω for every $n < \omega$. More precisely, Burke and Lutzer [2] proved that if $1 \leqslant m < n \leqslant \omega$, then M^m is not homeomorphic to M^n. The construction of the Michael line can be generalized as follows: Consider a space X and its subspace Y. The set X with the new topology generated by the base

$$\mathcal{B} = \{U \cup K: U \text{ is open in } X \text{ and } K \subset X \setminus Y\}$$

is denoted by X_Y. If X is a T_i-space with $i = 0, 1, 2, 3, 3\frac{1}{2}$, then so is X_Y for each $Y \subset X$ [E, 5.1.22]. On the other hand, the space X_Y cannot be normal if Y is a non-normal subspace of X. Thus X_Y need not be normal even if X is a compact Hausdorff space. The paracompactness of the Michael line follows from the general result that if X is hereditarily ***paracompact***, then so is X_Y for each $Y \subset X$ [E, 5.1.22]. It is known that there exists a subset $P \subset \mathbb{R}$ such that no uncountable closed set of $\mathbb{R}$ is contained either P or $\mathbb{R} \setminus P$. Such a subset P is called a **Bernstein set** or a **totally imperfect set**. If we use a Bernstein set P instead of $\mathbb{P}$ in the definition of the Michael line, then the space $M' = \mathbb{R}_{\mathbb{R}\setminus P}$ is ***Lindelöf*** and $M' \times P$ is not normal [E, 5.5.4]. It is an interesting open problem to find an example in ZFC of a Lindelöf, ***regular*** T_1-space X such that $X \times \mathbb{P}$ is not normal [KV, Chapter 18, Problem 7].

Bing's example G
Define a map $e : \mathbb{R} \to D^{\mathcal{P}(\mathbb{R})}$ from the real line into the Cantor cube indexed by its power set, as follows: $e(x)_A = 1$ iff $x \in A$. Denote the Cantor cube by X and let Y be the subset $e[\mathbb{R}]$ of X. Then Y is a relatively discrete subspace of X and hence a closed and discrete subset in the space $G = X_Y$ as defined above. The space G is known as **Bing's example G** and it is a normal but not ***collectionwise Hausdorff*** space. See [En, 5.1.23 and 5.5.3] for more of its properties.

Niemytzki Plane
The Niemytzki Plane is an example of a non-normal, Tychonoff, ***Moore space***. Let L be the closed upper half-plane, $L_1 = \{(x, 0): x \in \mathbb{R}\}$ and $L_2 = L \setminus L_1$. For every $p \in L$ and $\varepsilon > 0$, let $B(p, \varepsilon)$ be the set of all points of L inside the circle of radius ε and center at p, and define

$$U(p, \varepsilon) = B\big((x, \varepsilon), \varepsilon\big) \cup \{p\} \quad \text{for } p = (x, 0) \in L_1,$$

and $U(p, \varepsilon) = B(p, \varepsilon)$ for $p \in L_2$. The **Niemytzki Plane** is the set L with the topology generated by the base $\{U(p, \varepsilon): p \in L,\ \varepsilon > 0\}$ [E, 1.2.4]. The space L is a separable, Tychonoff, Moore space with the closed discrete subspace L_1 of cardinality $\mathfrak{c}$. Since the disjoint closed sets $A = \{(x, 0): x \in \mathbb{Q}\}$ and $L_1 \setminus A$ cannot be separated by disjoint open sets in L, L is not normal [E, 1.5.10]. This also follows from **Jones' Lemma** asserting that if a separable normal space contains a discrete closed set of cardinality κ, then $2^\kappa \leqslant \mathfrak{c}$ [E, 1.7.12 (c)]. Likewise, Fleissner [5] proved that no separable, ***countably paracompact*** space contains a discrete closed set of cardinality $\mathfrak{c}$. Hence, L is not countably paracompact. A subset $A \subset \mathbb{R}$ is called a ***Q*-set** if every subset of A is a G_δ-set in A. Take an uncountable Q-set A in $\mathbb{R}$, which exists under certain set-theoretic assumption such as ***Martin's Axiom*** and the negation of the ***Continuum Hypothesis*** [KV, Chapter 5, Theorem 4.2]. Then, the subspace

$$X = \big\{(x, 0): x \in A\big\} \cup (L \setminus L_1)$$

of L is known to be normal [KV, Chapter 15, Example F], but not collectionwise normal, since the discrete collection $\{\{(x, 0)\}: x \in A\}$ of closed sets cannot be separated by disjoint open sets in X. Hence, X is a separable, normal, non-metrizable, Moore space.

Spaces $\mathbb{N} \cup \mathcal{R}$
Spaces $\mathbb{N} \cup \mathcal{R}$ are examples of a ***pseudocompact***, Tychonoff space which is not ***countably compact*** [15], [KV, Chapter 3, Section 11]. Besides, they are non-metrizable, separable Moore spaces, like the Niemytzki plane. Two countable infinite sets are said to be **almost disjoint** if their intersection is finite. Take a maximal family $\mathcal{R}$ of pairwise almost disjoint, infinite subsets of $\mathbb{N}$, and topologize the union $\mathbb{N} \cup \mathcal{R}$ by letting sets of the form

$$U_n(A) = \big(A \setminus \{1, 2, \ldots, n\}\big) \cup \{A\}, \quad n \in \mathbb{N},$$

be basic neighbourhoods of $A \in \mathcal{R}$ and declaring points of $\mathbb{N}$ to be isolated. The resulting space $\mathbb{N} \cup \mathcal{R}$ is often called a **Ψ-space** and is now denoted by $\Psi(\mathcal{R})$. The space $\Psi(\mathcal{R})$ is a zero-dimensional, first-countable, ***locally compact***, Hausdorff space. By the maximality of $\mathcal{R}$, $\Psi(\mathcal{R})$ has no infinite discrete family of non-empty open sets, which implies that $\Psi(\mathcal{R})$ is pseudocompact. On the other hand, $\Psi(\mathcal{R})$ is not countably compact since $\mathcal{R}$ is discrete closed in $\Psi(\mathcal{R})$. Thus, $\Psi(\mathcal{R})$ is neither normal nor countably paracompact, because pseudocompactness coincides with countable compactness in the realm of normal spaces and countably paracompact spaces [E, 3.10.20 and 3.10.21]. Moreover, $\Psi(\mathcal{R})$ is a Moore space, and hence, every closed set is a ***G_δ-set***. In fact, if we put $\mathcal{U}_n = \{U_n(A): A \in \mathcal{R}\} \cup \{\{i\}: i \leqslant n\}$ for each $n \in \mathbb{N}$, then $\{\mathcal{U}_n: n \in \mathbb{N}\}$ is a ***development*** of $\Psi(\mathcal{R})$. Mrówka [17] proved that there exists $\mathcal{R}$ such that $\Psi(\mathcal{R})$ is ***almost compact***, i.e., $|\beta\Psi(\mathcal{R}) - \Psi(\mathcal{R})| = 1$, and Terasawa [19] proved that for every $n \in \mathbb{N} \cup \{\infty\}$, there exists $\mathcal{R}_n$ such that $\dim \Psi(\mathcal{R}_n) = n$, where dim stands for the ***covering dimension***.

Ordinal numbers
We identify a cardinal with its initial ordinal, and an ordinal with the set of all smaller ordinals. Let ω_1 be the first uncountable ordinal. For an ordinal α, let $\mathrm{cf}(\alpha)$ be the smallest ordinal β such that there exists a map $f : \beta \to \alpha$ with the property that $f[\beta]$ is unbounded in α. The ordinal $\mathrm{cf}(\alpha)$ is called the **cofinality** of α. Now, we consider every ordinal α a space with its order topology. Then, 0 and all successor ordinals in α are isolated points in α and all limit ordinals in α are limit points in α. As a ***linearly ordered topological space***, every ordinal is a collectionwise normal and countably paracompact T_1-space [E, 5.2.22]. Further, every ordinal α is a locally compact space with $\dim \alpha = 0$. The following statements (1)–(4) are also valid for every ordinal $\alpha > 0$:

(1) α is compact if and only if α is a successor;
(2) α is countably compact if and only if either α is a successor or $\mathrm{cf}(\alpha) > \omega$;
(3) α is metrizable if and only if $\alpha < \omega_1$; and
(4) α is first-countable if and only if $\alpha \leqslant \omega_1$.

Moreover, if $\mathrm{cf}(\alpha) > \omega$, then every real-valued continuous function f on α is constant on $\alpha \setminus \beta$ for some $\beta < \alpha$, which implies that f is continuously extendable over $\alpha + 1$. Hence, the ***Čech–Stone compactification*** of α is $\alpha + 1$ provided that $\mathrm{cf}(\alpha) > \omega$ [E, 3.1.27 and 3.6.10]. Let two ordinals α and β be fixed. Davis and Hart [3] proved that the product space $\alpha \times \beta$ is normal if and only if one of the following conditions (i)–(iii) holds: (i) $\mathrm{cf}(\alpha) \leqslant \omega$ and $\mathrm{cf}(\beta) \leqslant \omega$; (ii) $\alpha = \beta = \mathrm{cf}(\alpha)$; (iii) $\mathrm{cf}(\alpha) \leqslant \omega$ and $\alpha \leqslant \mathrm{cf}(\beta)$. In particular, the product $\omega_1 \times (\omega_1 + 1)$ is not normal. Kemoto and Smith [12] proved that all subspaces of $\alpha \times \beta$ are ***countably metacompact***. Fleissner, Kemoto and Terasawa [6] proved that $\dim(\alpha \times \beta) = 0$, while $\omega \times \mathfrak{c}$ contains a subspace S_n with $\dim S_n = n$ for each $n \in \mathbb{N} \cup \{\infty\}$. The space

$$T = \big((\omega_1 + 1) \times (\omega + 1)\big) \setminus \big\{(\omega_1, \omega)\big\}$$

is called the **Tychonoff plank**. The space T is pseudocompact since it has a countably compact, dense subspace $\omega_1 \times (\omega+1)$, while T itself is not countably compact since it contains an infinite, discrete, closed subset $\{\omega_1\} \times \omega$. Hence, T is neither countably paracompact nor normal. Moreover, $\beta(T) = (\omega_1 + 1) \times (\omega + 1)$, and hence, T is almost compact [E, 3.1.27].

For every limit ordinal $\alpha > 0$, let $L(\alpha)$ be the linearly ordered space constructed from the space α by placing between each ordinal $\beta < \alpha$ and $\beta + 1$ a copy of the open interval $\{x \in \mathbb{R}\colon 0 < x < 1\}$. In other words, $L(\alpha)$ is the linearly ordered space $\alpha \times \{x \in \mathbb{R}\colon 0 \leqslant x < 1\}$ with the lexicographic order. Then, $L(\alpha)$ is a ***connected*** space and the statements (1)–(4) above remain true if α is replaced by $L(\alpha)$. In particular, the space $L(\omega_1)$ is called the **long line** and the linearly ordered space $L(\omega_1) \cup \{\omega_1\}$ obtained by adjoining the maximum point ω_1 to $L(\omega_1)$ is called the **long segment**. For every $x \in L(\omega_1)$, the subspace $\{y \in L(\omega_1)\colon y \leqslant x\}$ of $L(\omega_1)$ is homeomorphic to the closed unit interval in $\mathbb{R}$, and the long segment is the Čech–Stone compactification of the long line [E, 3.12.19].

Sorgenfrey line

The Sorgenfrey line is an example constructed by R.H. Sorgenfrey to show that the square of paracompact, Hausdorff spaces need not be normal. Consider on the real line $\mathbb{R}$ the topology generated by the base

$$\mathcal{B} = \big\{[a, b)\colon a < b,\ a, b \in \mathbb{R}\big\},$$

where $[a, b) = \{x \in \mathbb{R}\colon a \leqslant x < b\}$. The set $\mathbb{R}$ with this topology is called the **Sorgenfrey line** and is now denoted by S [E, 1.2.2]. The space S is hereditarily separable, hereditarily Lindelöf [E, 3.8.14], and is a zero-dimensional Hausdorff space since each interval $[a, b)$ is open and closed in S. Thus, S is a paracompact, perfectly normal, T_1-space. On the other hand, the square S^2 is not normal, because the disjoint closed sets $A = \{(x, -x)\colon x \in \mathbb{Q}\}$ and $B = \{(x, -x)\colon x \in \mathbb{R} \setminus \mathbb{Q}\}$ cannot be separated by disjoint open sets in S^2 [E, 2.3.12]. Since the set $A \cup B$ is discrete closed in S^2, the latter fact also follows from Jones' Lemma stated above. Further, S^2 is not countably paracompact by the same reason as the case of the Niemytzki plane. The square S^2 is often called the **Sorgenfrey plane**. It is known [10, 13] that S^ω is ***perfect***, i.e., every closed subset of S^ω is a G_δ-set, and ***subparacompact***. Mrówka [16] and Terasawa [18] independently proved that $\dim S^\kappa = 0$ for every cardinal number κ, and Fora [8] proved that $\dim X \leqslant 0$ for every subspace X of S^ω. Moreover, Burke and Lutzer [2] proved that if $1 \leqslant m < n \leqslant \omega$, then S^m is not homeomorphic to S^n.

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C: Maps and general types of spaces defined by maps

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c-1 Continuous and Topological Mappings

1. Continuous mappings

In the present article X, Y and Z denote ***topological spaces***. Let $f: X \to Y$ be a map (= mapping) from X into Y and x a point of X. If for every ***open neighbourhood*** V of $f(x)$ in Y there is an open neighbourhood U of x such that $f(U) \subset V$, then the map f is said to be **continuous at** x. If f is continuous at every point of X, then it is called a **continuous map** (**mapping**), or we can say f is **continuous on** X. A continuous map from X into $\mathbb{R}$ ($\mathbb{C}$) is often called a real-valued (complex-valued) **continuous function** on X.[1] Real-valued continuous functions of one real variable are the best known examples of continuous maps (from $\mathbb{R}$ into $\mathbb{R}$). Another example is that every map from a ***discrete space*** into any space is a continuous map.

Suppose $f: X \to Y$ and $g: Y \to Z$ are continuous maps. Then the composite map $g \circ f$ defined by $g \circ f(x) = g(f(x))$, $x \in X$, is also a continuous map from X into Z. If X' is a ***subspace*** of X, then the restriction of f to X' denoted by $f|_{X'}$ is a continuous map from X' into Y. Let $f: X \to Z$ and $g: Y \to Z$ be both continuous maps; then the map $h: X \oplus Y \to Z$ defined by $h(x) = f(x)$ if $x \in X$ and $h(y) = g(y)$ if $y \in Y$ is a continuous map, where $X \oplus Y$ denotes the (***topological***) ***sum*** of X and Y. Suppose $f_\alpha: X \to Y_\alpha$ is a continuous map for each $\alpha \in A$. Then the map $f: X \to \prod_{\alpha \in A} Y_\alpha$ defined by $f(x) = (f_\alpha(x))_{\alpha \in A}$ is also a continuous map, where $\prod_{\alpha \in A} Y_\alpha$ denotes the ***product space*** (with ***Tychonoff product topology***) of the spaces Y_α, $\alpha \in A$.

If the domain of a continuous map is a ***topological product*** space then the map is said to be **jointly continuous**, as opposed to **separately continuous**, which in the case when $X = X_1 \times X_2$, say, means that $f(\cdot, x_2): X_2 \to Y$ and $f(x_1, \cdot): X_1 \to Y$ are continuous, for each individual x_1 and x_2.

Continuity of a map is characterized in various ways. The following conditions are equivalent for a map $f: X \to Y$.

(a) f is a continuous map,
(b) for every ***open set*** V in Y, $f^{-1}(V)$ is an open set in X,
(c) for every ***closed set*** G in Y, $f^{-1}(G)$ is a closed set in X,
(d) for a ***base*** (or ***subbase***) $\mathcal{B}$ of Y, and for every $B \in \mathcal{B}$, $f^{-1}(B)$ is an open set in X,
(e) for every subset A of X, $f(\operatorname{cl} A) \subset \operatorname{cl} f(A)$ in Y, where $\operatorname{cl} A$ denotes the ***closure*** of A.
(f) for every ***filter*** $\mathcal{F}$ in X ***converging*** to x, the filter $\mathcal{F}'$ in Y defined by $\mathcal{F}' = \{B \subset Y: B \supset f(F) \text{ for some } F \in \mathcal{F}\}$ converges to $f(x)$,
(g) for every ***net*** $\varphi: D \to X$ ***converging*** to x in X, the net $f \circ \varphi: D \to Y$ converges to $f(x)$ in Y,
(h) for every $x \in X$ and for every neighbourhood V of $f(x)$ in Y, $f^{-1}(V)$ is a neighbourhood of x in X.

(See [E] or [N] for the proofs.)

The last three conditions can be used to characterize continuity at a point, too. Especially, if X is ***first-countable***, then $f: X \to Y$ is continuous at $x \in X$ iff for every (point) sequence (x_i) converging to x in X, the (point) sequence $(f(x_i))$ converges to $f(x)$ in Y. This is a definition of continuity used in some elementary calculus books. Similarly we obtain the standard definition of continuity in calculus by modifying the original definition in case that (X, ρ) and (Y, σ) are both ***metric spaces***. Namely $f: X \to Y$ is continuous at $x \in X$ iff for every $\varepsilon > 0$ there is $\delta > 0$ depending on ε and x such that $\sigma(f(x), f(x')) < \varepsilon$ in Y whenever $\rho(x, x') < \delta$ in X, where ρ and σ denote the ***metrics*** on X and Y, respectively.

There are various generalizations of continuity. In the following are two of them.

Let $f: X \to \mathbb{R}$ be a real-valued function defined on X. If for every $x \in X$ and every $a > f(x)$ ($a < f(x)$), there is a neighbourhood U of x such that $a > f(x')$ ($a < f(x')$) for all $x' \in U$, then f is called an **upper semi-continuous function** (**lower semi-continuous function**). $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ for $x < 0$ ($x \leqslant 0$) and $f(x) = x + 1$ for $x \geqslant 0$ ($x > 0$) is an example of an upper semi- (lower semi-) continuous function. See the article on "Generalized continuities" in this Encyclopedia for more generalizations of continuous maps.

2. Topological mappings

Let $f: X \to Y$ be a continuous map from X onto Y. If X is a ***compact*** space, then so is Y. If X is ***connected***, then so is Y. Namely compactness and connectedness are **preserved** by continuous maps. But, generally speaking, not so many properties are preserved by continuous maps. The image of a closed (open) set in X by the continuous map f is not necessarily closed (open) in Y. Suppose $X = \prod_{\alpha \in A} X_\alpha$, i.e., X is the product of spaces X_α, $\alpha \in A$. Then the map $p: X \to X_\alpha$ defined by $p(x) = x_\alpha$ for $x = (x_\alpha)_{\alpha \in A} \in X$ is called a ***projection***. The projection is continuous on X and maps open sets to open sets but does not necessarily map a closed set to a closed set. To see it, consider the particular projection $p: \mathbb{R}^2 \to \mathbb{R}$; then the closed subset $F = \{(x, y): xy = 1\}$ of

[1] In the present Encyclopedia $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{R}^n$, S^n and I denote the set of natural numbers, the real line, the set of all rationals in $\mathbb{R}$, the complex plane, the n-dimensional Euclidean space, the n-dimensional sphere and the closed segment [0, 1], respectively.

$\mathbb{R}^2$ is mapped by p onto $\mathbb{R} - \{0\}$, which is not closed in $\mathbb{R}$. A more trivial example is the bijection $i : \mathbb{R}_d \to \mathbb{R}$, where $\mathbb{R}_d$ is the discrete space consisting of all real numbers, and $\mathbb{R}$ is the real line while $i(x) = x$ is the identity map. Then every subset of $\mathbb{R}_d$ is closed and open but not necessarily mapped to a closed (open) subset of $\mathbb{R}$. Continuous maps which map closed (open) sets to closed (open) sets are especially important, and discussions on those maps can be found in the articles "Open maps" and "Closed maps" in this Encyclopedia.

Perhaps the most important type of continuous map is "topological map", whose definition is given below.

Let $f : X \to Y$ be a continuous bijection (= one-to-one onto map) from X onto Y. If the inverse map f^{-1} is also continuous, then f is called a **topological map** or **homeomorphism**, and X and Y are said to be **homeomorphic spaces**. The homeomorphic relation is sometimes denoted by $X \cong Y$.

Note that if f is a topological map, then so is f^{-1}. If $f : X \to Y$ is a topological map and $X' \subset X$, then so is $f|_{X'} : X' \to f(X')$. The composite of two (and accordingly of finitely many) topological maps is a topological map. A continuous bijection from X onto Y is a topological map iff it maps every closed (open) set in X to a closed (open) set in Y.

The simplest example of a topological map is the identity map $i : X \to X$ defined by $i(x) = x$. In the following are two more well known examples. The map

$$f : (-1, 1) = \{x \in \mathbb{R}: |x| < 1\} \to \mathbb{R}$$

defined by $f(x) = \tan \frac{\pi}{2} x$ is a topological map between $(-1, 1)$ and $\mathbb{R}$. The map g illustrated by Figure 1 is a topological map from $S^1 - \{p_0\}$ onto $\mathbb{R}$. Thus

$$(-1, 1) \cong \mathbb{R} \cong S^1 - \{p_0\}.$$

In general $\mathbb{R}^n \cong S^n - \{p_0\}$ is known for each $n \in \mathbb{N}$.

A property of spaces is called a **topological property** (or **topological invariant**) if it is preserved by every topological map. Since a topological map maps topology to topology, every property which can be defined in terms of topology is a topological property, and vice versa. ***Compactness***,

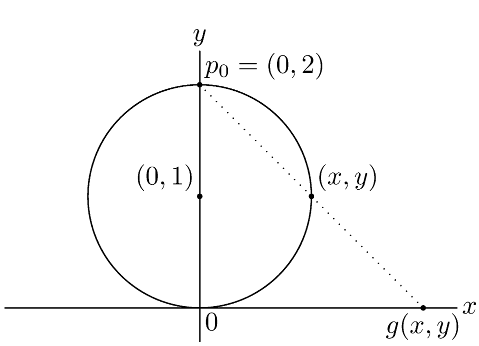

Fig. 1. $\mathbb{R} \cong S^1 - \{p_0\}$.

first-countability, ***second-countability***, ***regularity***, ***normality*** and ***paracompactness***, they are all topological properties. The term, topological invariant may be more suitable for topological properties like ***character***, ***weight*** and ***dimension***. "Metric" is not a topological property, but "***metrizability***" is a topological property. "***Completeness***" of a metric space is not a topological property, but "***Čech-completeness***" is a topological property. Generally two homeomorphic topological spaces may be regarded as the same space since they have the same topological properties.

In 1872 F. Klein defined **topology**[2] (then called **Analysis Situs**) as the geometry whose subject is the study of topological properties. Since then the field of study has been greatly expanded, and thus the definition of topology should be also extended to include studies of various aspects relevant to topological properties. But still continuous map and topological map are among the most significant and most basic concepts in topology.

Topology today is divided into many subfields depending on the methods and objects of study, e.g., set-theoretic topology, algebraic topology, differential topology, geometric topology and so on, among which **general topology** occupies the most basic position. Because general topology aims at the study of topological spaces themselves (and relevant matters) while all topological theories are developed essentially on topological spaces, and the concepts and languages of general topology are used in most areas of modern mathematics.

Let f be a topological map from X onto a subspace Y' of Y; then f is called a **topological embedding** (**imbedding**) of X into Y, and X is said to be **topologically embedded** in Y. In this case, since X and $Y' = f(X) \subset Y$ may be regarded as the same space, we can regard X as a subspace of Y. For example, $\mathbb{R}$ is topologically embedded in S^1, because $\mathbb{R} \cong S^1 - \{p_0\}$. See the article "Topological embeddings" for the embedding theory.

A continuous bijection need not be a topological map as the identity map $i : \mathbb{R}_d \to \mathbb{R}$ in the previous section shows. However, under certain conditions a continuous bijection will be a topological map. Every continuous bijection from $\mathbb{R}$ onto $\mathbb{R}$ is a topological map. Every continuous bijection from a compact space onto a ***Hausdorff space*** is a topological map.

A localization of the concept of homeomorphism is in the following. A continuous map $f : X \to Y$ is called **locally homeomorphic at** $x \in X$ if x has an open neighbourhood U such that $f(U)$ is an open set in Y and the restriction $f|_U$ of f to U is a topological map from U onto $f(U)$. If f is locally homeomorphic at every point of X, then it is called a **local homeomorphism**. A local homeomorphism is not necessarily a homeomorphism. Let $\mathbb{R}_0$ and $\mathbb{R}_1$ be two copies of $\mathbb{R}$ and

$$f : \mathbb{R}_0 \oplus \mathbb{R}_1 = \mathbb{R} \times \{0, 1\} \to \mathbb{R}$$

[2] Note that this term can be used in two different senses. It is used here to mean a subfield of mathematics, and it can be also used to mean the collection of all open sets in a topological space.

be the projection. Then f is a local homeomorphism but not a homeomorphism. A local homeomorphism from X onto Y is a homeomorphism if it is a bijection.

It is an interesting fact that under certain circumstances one can prove that *every* real-valued function is continuous on some ***dense subspace***. A space X is said to have **Blumberg property** if for every real-valued function f on X, there is a dense subspace D of X such that the restriction of f to D is continuous. H. Blumberg proved that every ***separable complete metric space*** has this property. It is also known that a metric space has the Blumberg property iff it is a ***Baire space*** and that there is a ***compact Hausdorff space*** which does not have the Blumberg property. See W.A.R. Weiss [1].

One can find something more about continuous and topological maps in [E] and [N].

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c-2 Open Maps

It seems that the study of open (interior) maps began with papers [13, 14] by S. Stoïlow. Clearly, openness of maps was first studied extensively by G.T. Whyburn [19, 20]. No doubt, the study of open maps and, in particular, light open maps, was motivated by the fact that a non-constant function $w = f(z)$, analytic in a region R of the z-plane which takes R into the w-plane, is strongly open. That is, if U is open in R, then $f(U)$ is open in the w-plane. Furthermore, f is a **light map** (for each x in R, $f^{-1}f(x)$ is ***totally disconnected***). In fact, $f^{-1}f(x)$ is a **discrete set** (has no limit point). The study of light open maps is the result of abstracting these concepts from such analytic functions $w = f(z)$.

For the purpose of this article, all spaces are metric except when otherwise specified. Clearly, generalizations to T_1-topological spaces exist and are useful.

A map f of X into Y is an **open map** if and only if for each U open in X, $f(U)$ is open relative to $f(X)$ in Y; f is a **strongly open map** if and only if for each U open in X, $f(U)$ is open in Y.

If f is an open map of X onto Y where X is compact, then f can be factored uniquely: $f = lm$ where m is a ***monotone*** map ($m^{-1}m(x)$ is connected) of X onto Z and l is a light open map of Z onto Y. This is a special case of the monotone-light factorization theorem of S. Eilenberg which also motivates the study of light open maps.

G.T. Whyburn characterized openness and closedness of maps in terms of quasi-compactness: A map f of X onto Y is open (closed) if and only if it is quasi-compact and generates a ***lower semi-continuous decomposition*** (***upper semi-continuous decomposition***) G of X into point inverses ($G = \{f^{-1}f(x) \mid x \in X\}$). A map f of X onto Y is **quasi-compact** if and only if the image of each closed (open) inverse set ($U = f^{-1}f(U)$) is closed (open). The property is due to P. Alexandroff and H. Hopf and called **strong continuity**.

A property of all open or closed maps and all retractions is that ***local connectedness*** is invariant.

All **continuous decompositions** (lower and upper semi-continuous) generate open and closed maps.

THEOREM 1 (G.T. Whyburn [19]). *Suppose that f is an open map of X onto Y where X is compact. If R is any connected open set in Y, then each quasi-component Q of $f^{-1}(R)$ maps onto R under f.*

If $p \in f^{-1}(R)$, then the **quasi-component** of $f^{-1}(R)$ which contains p is the set consisting of p together with all points x of $f^{-1}(R)$ such that $f^{-1}(R) \neq A \cup B$, two separated sets where $p \in A$ and $x \in B$ (p is not separated from x in $f^{-1}(R)$).

THEOREM 2 (G.T. Whyburn [19]). *If f is an open map of X onto Y where X is compact and C is any continuum in Y, then each component K of $f^{-1}(C)$ maps onto C under f ($f(K) = C$).*

THEOREM 3 (G.T. Whyburn [19]). *Suppose that f is an open map of X onto Y. If A is a closed non empty subset of Y and $f^{-1}(A)$ does not locally separate X at a point $x \in f^{-1}(A)$, then A does not locally separate Y at $f(x)$.*

THEOREM 4 (G.T. Whyburn [19]). *Suppose that f is an open map of X onto Y where X is compact and locally connected. If A is a closed subset of Y and R is a component of $Y - A$, then $f^{-1}(R)$ has at most a finite number of components and each maps onto R under f. Furthermore, if f is light and A is a locally connected continuum in Y (A is a **Peano continuum** in Y) whose interior is dense in A, then $f^{-1}(A)$ is locally connected.*

THEOREM 5 (F. Raymond – using Whyburn's work [11]). *Suppose that M is a locally compact, locally connected, Hausdorff space. Let f be a proper open map onto the Hausdorff space M^*. If U^* is an open connected set in M^*, then the components of $U = f^{-1}(U^*)$ are finite in number and each is mapped by f onto U^*. Furthermore, if G is a compact group which acts on M and $f: M \to M/G$ is the orbit map (f is open and closed), then for each component U_i of $U = f^{-1}(U^*)$, there exists an open and closed subgroup G_i of G which is the largest subgroup of G leaving U_i invariant and the maps induced by f map U_i/G_i homeomorphically onto U^*. Moreover, if one of the G_i is a normal subgroup of G, then all G_i are identical and the order of G/G_i is the same as the number of components of U.*

QUESTION 1. Suppose that f is an open map of X onto Y where X is compact. What can be said about the relationships between the ***homology***, ***cohomology***, and ***homotopy*** groups of X and Y?

The first result is due to S. Eilenberg [6]. Soon afterwards, G.T. Whyburn [18] proved: If f is open and $H_1(X)$ is finitely generated (rational coefficients), then the induced homomorphism $f_*: H_1(X) \to H_1(Y)$ is onto. He also proved: Suppose that N is a nodal set (N is a closed subset of X and $\overline{X - N} \cap N$ is a point). Then $f_*: H_1(N) \to H_1(f(N))$ is onto.

THEOREM 6 (S. Smale [12]). *Suppose that p is an open map of X onto Y, X is a locally arcwise connected Hausdorff space, and Y is semi-locally simply connected met-*

ric space. Then (X, p, Y) *has the* ***covering homotopy property*** *for a point up to a homotopy (fixed at both ends of* $[0,1]$*). It follows that the induced homomorphism* $f_* : \pi_1(X, f^{-1}(y)) \to \pi_1(Y, y)$ *is onto.*

G.E. Bredon gave an example [2] of a light open map of a ***contractible*** non-collapsible 2-complex K^2 onto S^2 (a 2-sphere). Thus, $H_2(K^2) = 0$ and $H_2(S^2) \neq 0$. P. Roy gave a much simpler example of a light open map of the dunce's hat onto a 2-sphere S^2 (see L.F. McAuley [8]). J. Baildon [1] gave an example of a light open map of Bing's house B with two rooms onto a 2-sphere S^2. The space B is a contractible non-collapsible two complex.

1. Conditions for openness

In addition to characterizations of open maps, there are some conditions which imply openness.

THEOREM 7 (C.J. Titus and G.S. Young [15]). *Let* D *be an open set in* E^n *and* $f : D \to E^n$ *be light, of class* C^1*, and have a non-negative (non-positive) Jacobian in* D*. Then* f *is open.*

There are results which relate openness to various types of path-lifting.

THEOREM 8 (G.T. Whyburn and S. Stoïlow [19, 13, 14]). *Suppose that* f *is a light open map of* X *onto* Y *where each of* X *and* Y *is a compact metric space. Then each simple arc in* Y *(homeomorphism of* $[0,1]$ *into* Y*) can be lifted to* X*. That is,* (X, f, Y) *has the* ***covering isotopy property*** *for a point.*

E.E. Floyd generalized this theorem as follows:

THEOREM 9 (E.E. Floyd [7]). *A light map of a Peano continuum* E *onto a Peano continuum* B *is open if and only if it has the covering homotopy property for points.*

In considering the question of "which fibrations are open?", P.T. McAuley defined the notion of path-lifting continuous at constants (PL-cc). For a map this notion lies between that of having the covering homotopy property for points and being a ***Hurewicz fibration***.

THEOREM 10 (P.T. McAuley [10]). *If* $p : E \to B$ *has Pl-cc and* B *is locally* 0*-connected, then* p *is open.*

THEOREM 11 (P.T. McAuley [10]). *If* $p : E \to B$ *has the covering homotopy property for points and* B *is a* 1*st Axiom space, then* p *is open.*

THEOREM 12 (P.T. McAuley [10]). *If* $p : E \to B$ *is a light map of a Peano continuum* E *onto* B*, then* p *is open if and only if* p *has PL-cc.*

2. Some properties of open maps on simple spaces

THEOREM 13 (G.T. Whyburn [19]). *If* X *is a simple closed curve and* f *is an open map of* X *onto* Y*, then* Y *is either a simple closed curve or a simple arc. If* Y *is a scc, there exists an integer* k *such that* f *is topologically equivalent to the map* $w = z^k$ *on the circle* $|z| = 1$*. If* Y *is a simple arc, then there is an integer* k *such that* f *is topologically equivalent to the map* $f(1,\theta) = \sin\left(\frac{h\theta}{2}\right)$ *of the circle* $\rho = 1$ *into the interval* $(-1, 1)$*. If* X *is a simple arc and* f *is an open map of* X *onto* Y*, then* Y *is a simple arc.*

3. Some important theorems on light open and closed maps

THEOREM 14 (A.V. Cernavskii [3]). *Suppose that* f *is a finite-to-one open and closed map on a connected (metric)* n*-manifold* M^n *onto a Hausdorff space* Y*. Then*

(1) *There is a natural number* k *so that for each* $x \in M^n$*,* $\mathrm{card}(f^{-1}f(x)) \leqslant k$ *and*
(2) *the elements of maximal multiplicity, those* x *with* $\mathrm{card}(f^{-1}f(x)) = k$*, form a dense open set in* M^n*. Furthermore,*
(3) *for each open set* U *of* M^n*, there is* $\varepsilon > 0$ *such that if* f *is any finite-to-one open and closed map of* M^n *onto a metric space* Y *and if* f *is not a homeomorphism, then for some* $x \in U$*,* $\mathrm{diam}(f^{-1}f(x)) \geqslant \varepsilon$*.*

The important work of D.C. Wilson should be mentioned (an answer to a question raised by S. Eilenberg).

THEOREM 15 (D.C. Wilson [21]). *If* M^3 *is any compact connected triangulated* 3*-manifold and* $m \geqslant 3$*, then there is a light open map* f *of* M^3 *onto* I^m *such that for each* $x \in M^3$*,* $f^{-1}f(x)$ *is homeomorphic to the Cantor Set.*

4. Approximating maps with open maps

THEOREM 16 (J.J. Walsh [16]). *For* $n \geqslant 3$*, a map* $f : M^n \to Y$ *can be approximated by an open map* $f : M^n \to Y$ *iff* f *is quasi-monotone.*

THEOREM 17 (J.J. Walsh [17]). *A map from a compact* PL *manifold* M^m $(m \geqslant 3)$ *to a compact connected polyhedron* Q *(or connected ANR) is homotopic to an open map iff the index of* $f_*(\pi_1(M^m))$ *in* $\pi_1(Q)$ *is finite.*

5. Differential open maps

The important work of P.T. Church on differentiable open maps must be mentioned (only a few examples are cited here).

THEOREM 18 (P.T. Church [4]). *Let* $f: E^n \to E^n$ *be open and* C^1. *If the rank of the Jacobian matrix of* f *at* $\bar{x}$ *is at least* $n-1$, *then* f *is locally a homeomorphism at* $\bar{x}$.

THEOREM 19 (P.T. Church [4]). *Let* $f: M^n \to N^n$ *be* C^n *and open* ($n \geqslant 2$); *let* M *be compact or let* f *be light. Then there exists a closed set* E, $\dim E \leqslant n-3$, *such that for each* x *in* $M^n - E$ *there exists a neighbourhood* U *of* x *on which* f *is topologically equivalent to one of the canonical maps* $F_{n,d}$ ($d = 1, 2, \ldots$). *Moreover,* E *is nowhere dense in* B_f *unless* f *is a local homeomorphism.*

THEOREM 20 (P.T. Church and J.G. Timourian [5]). *If* $f: M^{p+1} \to N^p$ *is real analytic and open with* $p \geqslant 1$, *then there is a closed subspace* $X \subset M^{p+1}$ *such that* $\dim f(X) \leqslant p-2$ *and, for each* $x \in M^{p+1} - X$, *there is a natural number* $d(x)$ *with* f *at* x *locally topologically equivalent to the map* $\chi_{d(x)}: C \times R^{p-1} \to R \times R^{p-1}$ *defined by* $\chi_{d(x)}(z, t_1, \ldots, t_{p-1}) = (\mathrm{Re}(z^{d(x)}), t_1, \ldots, t_{p-1})$.

There are a number of very useful survey articles some of which are contained in "The Proceedings of Conference on Monotone Mappings and Open Mappings", dedicated to the memory of G.T. Whyburn. A survey article on monotone mappings (including monotone open maps and light open maps) [9] is included in "General Topology and Modern Analysis" along with other related articles on kinds of open mappings.

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c-3 Closed Maps

A ***continuous map*** $f: X \to Y$ is called a **closed map** if for every closed set $A \subseteq X$, the image $f(A)$ is closed in Y. The notion of a closed map was introduced by W. Hurewicz and by P.S. Alexandroff in 1925 (see [1], and [E]).

The classes of ***T_1-spaces***, ***normal*** spaces, ***perfectly normal*** spaces and ***collectionwise normal*** spaces are preserved under closed maps. That is, if X has one of these properties, and $f: X \to Y$ is a closed map onto Y, then Y also has the corresponding property. The separation properties of ***Hausdorff***, ***regular*** and ***Tychonoff*** are not generally preserved to closed images. Simple examples will be given later. Some other properties preserved by closed maps are ***paracompactness*** (E. Michael, [E, 5.1.33]), ***metacompactness*** onto a Hausdorff range (Worrell, [E, 5.3.7]), ***subparacompactness***, ***submetacompactness***, ***σ-spaces***, $\Sigma^{\#}$-spaces, and ***stratifiable*** spaces (***M_3-spaces***), see, e.g., the survey article [4]. We will mention some other properties.

The Tychonoff property is preserved under an ***open***-and-***closed*** map, as can be seen by the following result due to Ponomarev: If $f: X \to Y$ is an open-and-closed map and $u: X \to [0,1]$ is continuous, then the map $v: Y \to [0,1]$, defined by $v(y) = \sup\{u(x)\colon x \in f^{-1}(y)\}$, is continuous [E, 1.5.L]. This can also be useful in showing that if $f: X \to Y$ is an open-and-closed map of a normal space X onto a space Y, then the continuous extension βf over the ***Stone–Čech compactification***s βX and βY is also open-and-closed (A.V. Arhangel'skiĭ and A.D. Taimanov).

If $f: X \to Y$ is a continuous map of a normal space X onto a Tychonoff space Y and $\beta f: \beta X \to \beta Y$ is its continuous extension, then the map f is closed if and only if $(\beta f)^{-1}(y) = \mathrm{cl}_{\beta X} f^{-1}(y)$ for every $y \in Y$.

Let $f: X \to Y$ be a closed map of a Hausdorff space X onto a space Y, $\mathcal{P}$ be a topological property, Y be a space with the property $\mathcal{P}$ and all fibers $f^{-1}(y)$ have the property $\mathcal{P}$. Then X has the property $\mathcal{P}$ for each of the following cases:

(i) $\mathcal{P}$ is the property of ***compactness***;
(ii) $\mathcal{P}$ is the property of having ***Lindelöf degree*** $\leqslant \tau$, where τ is a given infinite cardinal;
(iii) $\mathcal{P}$ is the property of ***$[a,b]$-compactness***, where a, b are infinite cardinals and $a \leqslant b$.

The Hanai–Morita–Stone Theorem [E, Theorem 4.4.17] affirms that for every closed map $f: X \to Y$ of a ***metrizable space*** X onto a space Y the following conditions are equivalent:

(i) Y is metrizable;
(ii) the space Y is ***first-countable***;
(iii) for every $y \in Y$ the set $\partial f^{-1}(y)$ is compact (Vaĭnšteĭn).

Furthermore, if $f: X \to Y$ is a closed map between metric spaces and X is ***completely metrizable*** then Y is completely metrizable (Vaĭnšteĭn [E, 4.5.13(e)]). An open-and-closed image of a metrizable space is metrizable.

A continuous map $f: X \to Y$ is called a **perfect map** if f is a closed map and all fibers $f^{-1}(y)$ are ***compact*** subsets of X (some authors also require that X be a Hausdorff space). The Encyclopedia article by Burke on "Perfect Maps" is devoted to this topic.

Closed maps and perfect maps are closely related. For example, if X is compact and $f: X \to Y$ is a closed map onto a T_1-space Y, then f is perfect. If X is compact and Y arbitrary, the projection map $\pi_Y: X \times Y \to Y$ is a closed (hence perfect) map. Conversely, S. Mrówka proved that if X is arbitrary and for every Y the projection map $\pi_Y: X \times Y \to Y$ is closed, then X is compact [4, p. 3]. However, if X is ***countably compact*** and Y is ***sequential*** then $\pi_Y: X \times Y \to Y$ is a closed map [E, 3.10.7].

Closed maps arise as projections from certain topological sums: Let $\mathcal{P}$ be a topological property that is preserved under closed (respectively perfect) maps. If $\{A_\alpha\colon \alpha \in \Lambda\}$ is a ***hereditarily closure-preserving*** (respectively ***locally finite***) closed cover of X with each A_α having property $\mathcal{P}$, and if the ***topological sum*** of the A_α satisfies $\mathcal{P}$ then X satisfies $\mathcal{P}$ [4, 2.7].

A map $f: X \to Y$ is called **inductively perfect** if there exists a subspace Z of X such that $f(Z) = Y$ and the restriction $f|_Z: Z \to Y$ is perfect. If X is a Hausdorff space, then Z must be closed in X. If $f: X \to Y$ is a closed map of a Hausdorff space X onto a space Y and for every $y \in Y$ the set $\partial f^{-1}(y)$ is compact (such maps are called **peripherally compact**), then the map f is inductively perfect.

A space X is called a **q-space** if for every point $x \in X$ there exists a sequence $\{U_n\colon n \in \mathbb{N}\}$ of ***open neighbourhoods*** of x in X such that for every choice $x_n \in U_n$, the sequence $\{x_n\colon n \in \mathbb{N}\}$ has a cluster point. We mention some important relations between closed maps and peripherally compact maps found by E. Michael [9] (and Vaĭnšteĭn [E] in the class of metric spaces). Let $f: X \to Y$ be a closed onto map of a Tychonoff space X. Michael [9] proved the following assertions: If Y is a q-space, then for every $y \in Y$, every continuous real-valued function on X is bounded on $\partial f^{-1}(y)$. If X is paracompact and Y is a q-space, then f is peripherally compact and hence also a **compact-covering map**, i.e., for every compact subset F of Y there exists a compact set $\Phi \subseteq X$ such that $f(\Phi) = F$. If $y \in Y$ and $\{y_n\colon n \in \mathbb{N}\}$ is a non-trivial sequence converging to y, then $f^{-1}(y) \cap \mathrm{cl}(\bigcup\{f^{-1}(y_n)\colon n \in \mathbb{N}\})$ is a bounded subset of X. From this it follows that for every onto closed map $f: X \to Y$ of a paracompact ***Čech-complete*** space (paracompact ***p-space***) X, the space Y is a Čech-complete space (p-space) if and only if Y is a q-space. N. Lašnev [E], using Michael's result, proved that for a closed map $f: X \to Y$ of a paracompact space X onto a sequential space Y there exists a closed subset Z of X such that $f(Z) = Y$ and the map $f|_Z : Z \to Y$ is ***irreducible***.

Let $f: X \to Y$ be continuous and $C(f) = \{y \in Y : f^{-1}(y)$ is compact$\}$. A.V. Arhangel'skiĭ [2] posed the problem of describing properties of the set $C(f)$. In particular, he asked when is the set $C(f)$ non-empty? When is the set $C(f)$ σ-discrete? V.V. Filippov gave the following simple example. Fix a space X in which every G_δ-subset is open and $\mathbb{N}$ is the discrete space of natural numbers. Then the projection $\pi_X : X \times \mathbb{N} \to X$ is an open-and-closed map and $C(\pi_X) = \emptyset$. If τ is an uncountable cardinal number and $\tau = \sum\{2^m : m < \tau\}$, then there exists a paracompact space with a ***base of rank*** 1 such that the weight $w(X) = \tau$, $|X| = \exp(\tau)$ and every G_δ-subset of X is open.

N. Lašnev [8] proved that the complement of the set $C(f)$ is σ-discrete for every closed map $f: X \to Y$ of a metrizable space X onto Y. A complete characterization of such images has been given by Lašnev [8]. Closed images of metric spaces are called **Lašnev space**s. F. Slaughter, Jr. proved that every Lašnev space is an M_1***-space*** (see [KV, Chapter 10, §5]). For more on Lašnev spaces see [HvM, Chapter 7, §9] and [4].

Let X be a space and E be an equivalence relation on X. Denote by X/E the set of equivalence classes of E and by $q: X \to X/E$ the map, where $q^{-1}(q(x))$ is the equivalence class of $x \in X$. The topology $\{U \subseteq X/E$: $q^{-1}(U)$ is open in $X\}$ is called the **quotient topology** on X/E and q is the natural quotient map. The decomposition E is an **upper semi-continuous decomposition** if the natural map q is closed. Quotient spaces appeared in the R.L. Moore's paper [12] and in P.S. Alexandroff's paper [1].

By identifying a closed subset A of a space X to a point we obtain the quotient space X/A and the closed map $q: X \to X/A$. This construction provides examples showing that many properties are not preserved by closed maps: local compactness, metrizability, first-countability. For example, treating $\mathbb{N}$ as a subspace of the reals $\mathbb{R}$, then $\mathbb{R}/\mathbb{N}$ is not first-countable, not ***locally compact*** and not Čech-complete. One can show that the separation properties of Tychonoff or regular are not preserved under a closed map by starting with a non-normal Tychonoff space X. If A, B are two disjoint closed subsets of X, which cannot be separated, then the quotient space $Z = X/A$ is not Tychonoff (and not regular). The space Z/B is a closed image of X which is not Hausdorff. See [4] for a construction of this type, by P. Zenor, who shows that ***countable paracompactness*** is not preserved under a closed map.

A map $f: X \to Y$ of X onto Y is said to be **hereditarily quotient** if for every $B \subseteq Y$ the restriction $f|_{f^{-1}(B)}$: $f^{-1}(B) \to B$ is a quotient map. A map $f: X \to Y$ is hereditarily quotient if and only if for every $y \in Y$ and any open set $U \subseteq X$, with $f^{-1}(y) \subseteq U$, we have $y \in \operatorname{Int} f(U)$. Every closed map is hereditarily quotient.

E. Michael has introduced bi-quotient maps [10] (also studied by O. Hajek) and tri-quotient maps [11] for studying mapping properties "between" the properties of perfect maps and hereditarily quotient maps. An onto map $f: X \to Y$ is a **bi-quotient map** if for every cover γ of any fiber $f^{-1}(y)$ there exists a finite subfamily $\mathcal{U} \subseteq \gamma$ such that $y \in \operatorname{Int} \bigcup \mathcal{U}$. The map f is a **tri-quotient map** if one can assign to every open $U \subseteq X$ an open $U^* \subseteq Y$ such that $U^* \subseteq f(U)$; $X^* = Y$; $U_1 \subseteq U_2$ implies $U_1^* \subseteq U_2^*$; and if $y \in U^*$ and $\mathcal{W}$ is an open cover of $f^{-1}(y) \cap U$ in X, then there is a finite $\mathcal{V} \subseteq \mathcal{W}$ such that $y \in (\bigcup \mathcal{V})^*$. These are common generalizations of perfect maps and of open maps. In fact, every perfect (or open) map $f: X \to Y$ is tri-quotient. Closed maps are tri-quotient if the domain is paracompact and the range is first-countable [11]. Every tri-quotient map is bi-quotient and every bi-quotient map is hereditarily quotient. Every hereditarily quotient map with compact fibers is a bi-quotient map. In an early application of these notions, V.V. Filippov [7] proved that if $f: X \to Y$ is a quotient map of a space X with a ***point-countable base*** onto a space of ***point-countable type*** and all fibers $f^{-1}(y)$ are separable, then Y has a point-countable base. This follows because his results show that this map is actually a bi-quotient map and bi-quotient maps with separable fibers preserve the point-countable base property [7] – improving his own result that perfect maps preserve the point-countable base property. If $f: X \to Y$ is a quotient map onto a first-countable space Y and all fibers $f^{-1}(y)$ are Lindelöf, then f is bi-quotient [10].

Every inductively perfect map is tri-quotient. E. Michael [11] proved that every tri-quotient map $f: X \to Y$ from a complete metric space X onto a paracompact space Y is inductively perfect. Michael asked whether a tri-quotient map $f: X \to Y$ of a metric space X onto paracompact Y is inductively perfect if the assumption, that X is complete, is weakened to only assuming that each $f^{-1}(y)$ is complete. This problem has generated much work (see [5, 11, 13]) and was recently solved by G. Debs and J. Saint Raymond with an example of a tri-quotient (and compact-covering) map $f: X \to Y$ where X, Y are separable metric, every fiber of f is compact and f is not inductively perfect [6].

Closed maps arise in dimension theory. For example, if $f: X \to Y$ is a closed map between ***metric spaces*** such that $\dim f^{-1}(y) \leqslant k$ for every $y \in Y$ then $\dim X \leqslant \dim Y + k$ [E, 7.4.20(b)]. If X is metrizable and $f: X \to Y$ is a closed onto map such that $|f^{-1}(y)| \leqslant n + 1$ $(n \geqslant 0)$, for all $y \in Y$, then $\dim Y \leqslant \dim X + n$ [E, 7.4.19(b)]. (Recall that for a metric space X, ***covering dimension*** $\dim X$ is equal to the ***large inductive dimension*** $\operatorname{Ind} X$ [E].) If X is a ***topological space*** with $\operatorname{Ind} X = 0$ and $f: X \to Y$ is a closed onto map such that $|f^{-1}(y)| \leqslant n + 1$, for all $y \in Y$, then Y is a ***normal*** space and $\operatorname{Ind} Y \leqslant n$.

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c-4 Perfect Maps

A **perfect map** $f: X \to Y$ is a closed, continuous and onto map with $f^{-1}(y)$ ***compact*** in X for every $y \in Y$. In this article the term map (or mapping) will always mean a continuous map. For convenience, we assume all topological spaces considered here are ***Hausdorff spaces*** (even though for emphasis, Hausdorff may be indicated). By a **closed map** we mean that $f(A)$ is closed in Y for every closed set $A \subseteq X$. The point inverses $f^{-1}(y)$, for $y \in Y$, are sometimes called the **fibers** of the map f. It is sometimes interesting to consider a continuous closed map with a condition on the fibers which is weaker than compact (such as ***Lindelöf***) or with no condition at all. A closed map with ***countably compact*** fibers is called a **quasi-perfect map**. Many proofs of results quoted here may be found in [E] and a survey on closed maps and perfect maps may be found in [3].

A closed map $f: X \to Y$ is always a ***quotient*** map [E, Section 2.4]. If f is a quotient map it is possible to check conditions on the fibers of f to see if f is closed. If $W \subseteq X$ let the **saturated part** of W be given by the union of all fibers which are fully contained in W. This is the set $\mathrm{S}(W) = f^{-1}(Y \setminus f(X \setminus W))$. A set $A \subseteq X$ would be called **saturated** (wrt f) if $A = \mathrm{S}(A)$. We see that a quotient map $f: X \to Y$ is closed if and only if, for every open set $W \subseteq X$, the saturated part $\mathrm{S}(W)$ is open. In the language of ***decomposition spaces*** (or ***quotient spaces***) this says that a decomposition space is ***upper semi-continuous*** if and only if the associated quotient map is closed. To construct a perfect map via the upper semi-continuous decomposition process it would be necessary to start with a partition $\mathcal{P}$ of X into compact subsets.

If $f: X \to Y$ is a closed map and $A \subseteq X$ is a saturated subset of X then $f|_A : A \to f(A)$ is a closed map and hence a quotient map – so closed maps are ***hereditarily quotient*** [E, 2.4.F]. However, the restriction of f to an arbitrary subset of X need not give a closed map or even a quotient map. If f is restricted to a closed subset $Z \subseteq X$ then $f|_Z : Z \to f(Z)$ is a closed map. Certainly, if f was actually a perfect map then $f|_Z$ would be a perfect map. The composition of two closed maps clearly yields a closed map. The composition of two perfect maps also gives a perfect map. This follows because, under any perfect map $f: X \to Y$, if $C \subseteq Y$ is a compact subset of Y then the preimage $f^{-1}(C)$ is a compact subset of X [E, 3.7.2]. This property of the inverse preservation of compactness under perfect maps is close to a characterization of perfect maps and is so when the range of the map is a Hausdorff ***k-space***. That is, if $f: X \to Y$ is a continuous map, with the preimage $f^{-1}(C)$ compact in X for every compact $C \subseteq Y$ (sometimes called a **proper map**), and Y is a Hausdorff k-space then f is a perfect map.

Since compactness is preserved under a continuous map and compact subspaces of a Hausdorff space are closed, it is clear that any continuous map from a compact space X onto a Hausdorff space Y is a perfect map. If X is compact Hausdorff and Y is any space then the projection map $\pi_2 : X \times Y \to Y$ is a closed map [E, 3.1.16] with fibers all homeomorphic to X. So π_2 is a perfect map. If X is an arbitrary Hausdorff space and, for all spaces Y, the projections $\pi_2 : X \times Y \to Y$, are closed maps then X, in fact, must be compact. If Z is a closed subspace of $X \times Y$ and X is compact then the restricted map $\pi_2|_Z$ is a perfect map onto its range $\pi_2|_Z(Z)$. In a sense, all perfect maps can be considered to arise in this fashion since it is known that a continuous onto map $g: Z \to Y$ (for ***Tychonoff space*** Z) is perfect if and only if for any ***compactification*** X of Z, the map $\varphi: Z \to X \times Y : \varphi(z) = (z, g(z))$ is an embedding map onto a closed subspace Z' of $X \times Y$. This fact is also the basis for the following useful observation: If P is a topological property which is closed hereditary then P is inversely preserved under perfect maps (with Tychonoff domain) if and only if $X \times Y$ has property P whenever X is compact and Y has property P.

When maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are given it is natural to consider the **product map** $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ defined in the obvious manner. If $f_1 \times f_2$ is a closed map then both f_1 and f_2 are closed. However, if f_1 is perfect and f_2 is closed the product map $f_1 \times f_2$ may not be closed [E, 2.3.28]. To illustrate by an example, let $\mathrm{id}_{\mathbb{R}}$ be the identity map of the real line $\mathbb{R}$ onto itself (a nice perfect map) and let f_2 be the map of $\mathbb{R}$ onto the one-point discrete space $\{0\}$. The product map $\mathrm{id}_{\mathbb{R}} \times f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \{0\}$ is not closed. The graph $G = \{(x, y) \colon y = \frac{1}{x}\}$ is a closed subset of $\mathbb{R} \times \mathbb{R}$, but the image of G under $\mathrm{id}_{\mathbb{R}} \times f_2$ is not closed in $\mathbb{R} \times \{0\}$. If f_1 and f_2 are both perfect then $f_1 \times f_2$ is a perfect map (and conversely). A continuous map $f: X \to Y$ (with X Hausdorff) is perfect if and only if, for every Hausdorff space Z, the product map $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is closed [E, 3.7.14].

The above result about the product of two perfect maps can be upgraded to an infinite family $\{f_\alpha \colon \alpha \in \Lambda\}$ of (nonempty) maps. That is, the product map $\prod_{\alpha \in \Lambda} f_\alpha$ is perfect if and only if, for every $\alpha \in \Lambda$, f_α is perfect. When all of the maps $f_\alpha : X \to Y_\alpha$ have a common domain X then we can talk about the **diagonal map** $\Delta_{\alpha \in \Lambda} f_\alpha : X \to \prod_{\alpha \in \Lambda} Y_\alpha$, defined by $\Delta_{\alpha \in \Lambda} f_\alpha(x) = (f_\alpha(x))_{\alpha \in \Lambda}$. Since the ***diagonal*** Δ of $\prod_{\alpha \in \Lambda} X$ is homeomorphic to X this carries the same information as the product map $\prod_{\alpha \in \Lambda} f_\alpha$ restricted to the diagonal of $\prod_{\alpha \in \Lambda} X$. If at least one f_β is perfect and all Y_α are Hausdorff then this is enough to say that the diagonal map $\Delta_{\alpha \in \Lambda} f_\alpha$ is perfect [E, 3.7.11].

Product maps are naturally involved in the study of ***inverse limit*** systems. Suppose $\mathbf{S} = \langle X_n, f_n \rangle$ is an inverse limit sequence with $\mathbf{X} = \varprojlim X_n$. If all ***bonding map***s f_n are perfect then all projection maps $\pi_k|_{\mathbf{X}} : \mathbf{X} \to X_k$ are perfect. (Here, π_k represents the usual projection map from $\prod_{n\in\omega} X_n$ to X_k.) If $\mathbf{T} = \langle Y_n, g_n\rangle$ is another inverse limit sequence with $\mathbf{Y} = \varprojlim Y_n$ and $\Phi = \langle \varphi_n\rangle$ is a ***system map*** from $\mathbf{S}$ to $\mathbf{T}$, with all φ_n perfect, then the induced map $\varphi = \varprojlim \varphi_n = \prod_{n\in\omega} \varphi_n|_{\mathbf{X}}$ is a perfect map from $\mathbf{X}$ to $\mathbf{Y}$. Similar results hold for more general inverse systems [E].

A perfect map $f : X \to Y$ has the curious property that if f is extended continuously over a Hausdorff space $Z \supseteq X$ (with the same range Y) then X must be closed in Z [E, 3.7.4]. That is, X cannot be a dense proper subspace of Z. For Tychonoff spaces X, Y this gives the frequently useful result: A continuous map $f : X \to Y$ is perfect if and only if for any compactification bY of Y, the continuous extension $F : \beta X \to bY$ of f satisfies the condition that $F(\beta X \setminus X) \subseteq bY \setminus Y$. We also see that if X, Y are ***locally compact*** Hausdorff spaces and ωX, ωY denote the respective ***one-point compactification***s then a map $f : X \to Y$ is perfect if and only if the extension of f obtained by assigning the point at infinity of ωX to the point at infinity of ωY gives a continuous extension.

Much of the utility of closed or perfect maps comes from results about the preservation of topological properties in the image or preimage direction. Many of the common "base axiom" properties are preserved in the image direction under perfect maps. This includes the properties of being ***metrizable***, ***second-countable*** (more generally, with ***weight*** $\leqslant \kappa$), ***developable***, ***quasi-developable***, ***quasi-metrizable***, a ***γ-space***, ***stratifiable*** (for closed maps) or ***semistratifiable*** (closed maps), or having a ***σ-locally countable*** base, ***uniform base***, ***σ-point-finite*** base, ***point-countable*** base, ***$\delta\theta$-base***, ***base of countable order*** or ***primitive base***. Notable exceptions to this would include spaces with a ***σ-disjoint*** base, ***first-countable*** spaces, ***semi-metrizable*** spaces, ***symmetrizable*** spaces and ***Nagata spaces*** – these classes are not generally preserved under perfect maps. It is interesting that the class of stratifiable spaces (same as ***M_3-spaces***) is even preserved under closed maps but it is not known whether the class of ***M_1-spaces*** is preserved under perfect maps. This question of whether M_1-spaces are preserved under perfect maps is actually equivalent to the well-known "$M_3 \Longrightarrow M_1$" question [KV, Chapter 10]. However, M. Ito [HvM, Chapter 7] has shown that the class of hereditarily M_1-spaces is preserved under closed maps, and C. Borges and D.J. Lutzer have shown that M_1-spaces are preserved under perfect irreducible maps.

The preservation result for metrizable spaces is worthy of special note. The **Hanai–Morita–Stone Theorem** [E, 4.4.17] says that if $f : X \to Y$ is a closed continuous map from a ***metric*** space X onto a space Y then Y is metrizable $\Longleftrightarrow$ Y is first-countable $\Longleftrightarrow$ for every $y \in Y$, $\partial f^{-1}(y)$ is compact. Not only do we see that perfect maps preserve metrizability (and complete metrizability) but if a closed continuous map f does preserve metrizability then f must have fibers with compact boundaries. In this sense f is close to being a perfect map. Indeed, for this f, there is always a closed subspace Z of X such that $f|_Z$ is a perfect map from Z onto Y. Such a map is said to be **inductively perfect**. In case X is connected (so every fiber has a non-empty boundary) the closed set Z can simply be $Z = X \setminus \bigcup\{\operatorname{Int} f^{-1}(y) : y \in Y\}$. A stronger result in this direction is given by E.A. Michael [10]. He shows that if X is ***paracompact***, Y is first-countable or ***locally compact*** and $f : X \to Y$ is a closed continuous map then $\partial f^{-1}(y)$ is compact for every $y \in Y$; hence f is inductively perfect. An interesting corollary to this says that any closed map f from a paracompact space X onto any space Y is compact-covering. (A map $f : X \to Y$ is said to be **compact-covering** if every compact subset of Y is the image of some compact subset of X under f.) A classic result by Michael says that an open map f from metric X onto paracompact Y such that all fibers are complete wrt the metric on X, is inductively perfect. There are several variations and improvements on this result – see [vMR, Chapter 17] for a discussion and a beginning on references.

Since perfect maps are compact-covering it is clear that inductively perfect maps are also compact-covering maps. It is natural to ask when compact-covering maps are inductively perfect [vMR, Chapter 17]. Answering a question by Michael, G. Debs and J. Saint Raymond have given an example of a compact-covering map $f : X \to Y$ between ***separable*** metric spaces X, Y such that every fiber of f is compact but f is not inductively perfect. In [5], Debs and Saint Raymond have continued a descriptive set-theoretic approach to the validity of the statement $\mathbb{A}(X, Y)$, for separable metric spaces X and Y. $\mathbb{A}(X, Y)$: "Any compact-covering map $f : X \to Y$ is inductively perfect". It is known (J.P.R. Christensen; J. Saint Raymond) that $\mathbb{A}(X, Y)$ holds in **ZFC** if X is a **Polish space** (= separable complete metrizable). A.V. Ostrovskiĭ has shown that $\mathbb{A}(X, Y)$ holds in ZFC if Y is ***σ-compact***. Debs and Saint Raymond have shown that in the universe L, there exists an F_σ-subset X of ω^ω and a compact-covering map $f : X \to \omega^\omega$ where f is not inductively perfect. See [5] for references.

Closed maps preserve several important covering properties in the image direction. The classes of paracompact spaces, ***metacompact*** spaces, ***subparacompact*** spaces, and ***submetacompact*** spaces are all preserved under closed map images. The classes of ***para-Lindelöf*** and σ-para-Lindelöf spaces are preserved under perfect images. See [KV, Chapter 9] for a more complete list with corresponding proofs. The classes of ***screenable*** spaces and ***orthocompact*** spaces are not generally preserved under closed maps or even perfect maps, and ***countably paracompact*** spaces are not preserved under closed maps. It is not known whether the class of ***meta-Lindelöf*** spaces is preserved under perfect maps. All of the covering properties mentioned in this paragraph, except orthocompactness, are inversely preserved under closed maps with Lindelöf fibers (and ***regular*** domain).

When showing that topological properties are preserved under a closed or perfect map it is often useful to rely on certain structures of collections of sets being preserved in the image direction. If $f: X \to Y$ is closed and $\mathcal{A}$ is a ***closure-preserving*** collection in X then $\{f(A): A \in \mathcal{A}\}$ is closure-preserving in Y. If f is perfect and $\mathcal{A}$ is ***locally finite*** in X then $\{f(A): A \in \mathcal{A}\}$ is locally finite in Y. If $\mathcal{A}$ is a ***σ-relatively-discrete*** cover of X (and f is perfect) then Y has a σ-relatively-discrete cover $\mathcal{E}$ which is a refinement of $\mathcal{A}$ [4]. Moreover, if $\mathcal{A}$ is a ***network*** for X then $\mathcal{E}$ can be chosen to be a network for Y.

The phrase **perfect class** of spaces [E] is used to describe a class $\mathcal{C}$ of Hausdorff topological spaces which is closed under the formation of perfect images and perfect preimages of spaces from $\mathcal{C}$. The class of compact spaces is certainly a well-known perfect class. Other examples of perfect classes of spaces include the classes of regular spaces, Lindelöf spaces, paracompact spaces, σ-compact spaces, countably compact spaces, locally compact spaces, ***Čech-complete*** spaces (restricted to Tychonoff spaces), spaces of ***countable type***, ***strict*** ***p-space***s, ***Σ-space***s and k-spaces. The class of ***normal*** spaces is not perfect – normality is certainly preserved under a continuous closed map but for the failure of the inverse image direction, look at the space $(\omega_1 + 1) \times \omega_1$ [E, 3.12.20], where $(\omega_1 + 1)$ and ω_1 have the ***order topology***. This is a well-known non-normal countably compact space and the projection $\pi_2: (\omega_1 + 1) \times \omega_1 \to \omega_1$ is a perfect map onto the normal space ω_1. The Tychonoff property is not generally preserved in either the image or preimage direction.

Other classes known to be invariant under closed images include ***sequential*** (quotient maps), normal, ***realcompact***, ***Fréchet space***s, ***monotonically normal*** spaces, ***collectionwise normal*** spaces, ***σ-space***s and spaces with all closed sets G_δ. The classes of ***scattered*** spaces, ***pseudocompact*** spaces, ***Čech-analytic*** [HvM, Chapter 8] (with Tychonoff range) [8], and countably paracompact spaces are preserved under perfect images. Regular spaces with ***property D*** are preserved under finite-to-one perfect maps [15]. Spaces of ***point-countable type***, ***p-spaces*** [6], ***q-spaces***, spaces with a ***G_δ-diagonal***, ***Baire spaces***, ***submetrizable*** spaces, realcompact spaces, ***elastic spaces***, ***zero-dimensional*** spaces, ***extremally disconnected*** spaces and spaces with ***property wD*** [15] are not generally preserved to the image under a perfect map. The property of being a G_δ-set is not preserved under a perfect map. It is not known if the class of ***$w\Delta$-space***s is preserved under perfect maps.

A topological property will not be inversely preserved under a perfect map unless compact spaces always have that property. Properties which are inversely preserved under perfect maps (and not yet mentioned above) include point-countable type, q-spaces, realcompact, regular property D (quasi-perfect) and property wD (quasi-perfect).

There is a general situation when certain topological properties can be "pulled back" under a perfect map [E, 3.7.27]: Suppose P is a closed hereditary topological property which is also finitely productive. If there exists a perfect map $f: X \to Y$ where Y has property P and there exists a **condensation** (one-to-one continuous map) $g: X \to Z$ onto a space Z with property P then X also has property P. The map g tends to give X and fibers of f certain properties which may otherwise not be present. For example, if Z has a G_δ-diagonal then so does X and all fibers of f would be metrizable. This result, due to A. Arhangel'skiĭ has been strengthened (in some cases) by replacing the condensation space Z with a cleavability condition. If $\mathcal{P}$ is a class of spaces, a space X is said to be **cleavable** over $\mathcal{P}$ if for every $A \subseteq X$ there is some $Z \in \mathcal{P}$ and a map $g: X \to Z$ with $A = f^{-1}(f(A))$. Clearly, if X admits a condensation onto some $Z \in \mathcal{P}$ then X is cleavable over $\mathcal{P}$. See [HvM, Chapter 7] for references to the following: Let $\mathcal{P}$ be one of the classes of metric spaces, ***Moore spaces***, σ-spaces, ***semi-metric*** spaces, (semi)-stratifiable spaces or spaces with a G_δ-diagonal. If X admits a perfect map onto an element of $\mathcal{P}$ and is cleavable over $\mathcal{P}$ then $X \in \mathcal{P}$.

The class of Baire spaces deserves special attention. As mentioned above, Baire spaces are not generally preserved under perfect maps. However, Baire spaces are preserved and inversely preserved under closed irreducible maps [1]. A map $f: X \to Y$ is said to be **irreducible** if the only closed subset Z of X, with $f(Z) = Y$, is $Z = X$. Equivalently, if W is any non-empty open subset of X then the saturated part $\mathrm{S}(W) \neq \emptyset$. Observe that any continuous map, for which the set of one-point fibers is dense in the domain, is an irreducible map. If $f: X \to Y$ is any map with compact fibers, a Zorn's Lemma argument shows there exists a closed subset $Z \subseteq X$ such that $f|_Z: Z \to X$ is irreducible. N. Lašnev [9] showed that if $f: X \to Y$ is a closed surjection with X paracompact and Y Fréchet, then there exists closed $Z \subseteq X$ such that $f(Z) = Y$ and $f|_Z$ is irreducible. A closed surjection $f: X \to Y$ is said to be **inductively irreducible** if f has an irreducible restriction to some closed subspace of X. See [7] for results on when closed maps with a paracompact or Lindelöf domain are inductively irreducible.

In [12], J. van Mill and R.G. Woods discuss the perfect and perfect irreducible images of many of the classic zero-dimensional separable metric spaces. They characterize the perfect and perfect irreducible, zero-dimensional images of the Baire spaces $\mathbb{C}$ (***Cantor set***), $\mathbb{C} \setminus \{p\}$, and $\mathbb{P}$ (space of irrational numbers). One sample result: If $f: X \to Y$ is a perfect irreducible map from one of $\mathbb{C}$, $\mathbb{C} \setminus \{p\}$, or $\mathbb{P}$ there is a dense subset $S \subseteq Y$ such that $f|_{f^{-1}(S)}: f^{-1}(S) \to S$ is a homeomorphism, and, if Y is zero-dimensional, then Y is homeomorphic (respectively) to $\mathbb{C}$, $\mathbb{C} \setminus \{p\}$ or $\mathbb{P}$. They also prove that if Y is any σ-compact, nowhere locally compact metric space then there is a perfect irreducible map $f: \mathbb{Q} \times \mathbb{C} \to Y$ such that every fiber of f is homeomorphic to $\mathbb{C}$.

There are several classes of topological spaces which can be characterized as the class of images or preimages of another class under a closed or perfect map. One such class is the class of **Lašnev space**s [9] which is defined as the class of closed images of metric spaces. Any Lašnev space Y, given by the image of a closed $f: X \to Y$ with X metrizable, can be "decomposed" as $Y = Y_0 \cup (\bigcup_{i=1}^{\infty} Y_i)$, where $f^{-1}(y)$ is compact for every $y \in Y_0$ and each Y_i, for $i \geqslant 1$, is closed discrete. Prior to Lašnev's result K. Morita proved a similar decomposition theorem for closed images of paracompact locally compact spaces and since then the decomposition result has been extended to closed images of regular σ-spaces by J. Chaber and metacompact p-spaces by N.V. Veličko. See [HvM, Chapter 7] for a discussion of other extensions of Lašnev's theorem. L. Foged (see [HvM, Chapter 7]) obtained the following internal characterization of Lašnev spaces: A (Hausdorff) space X is a Lašnev space if and only if X is a Fréchet space with a ***σ-hereditarily closure-preserving*** closed ***k-network***.

The class of paracompact p-spaces (equivalently, paracompact ***M-spaces***) can be characterized as the class of all perfect preimages of metric spaces [KV, Chapter 10]. Combining this with an earlier remark in this section about perfect maps we see that the class of paracompact p-spaces is exactly the class of all closed subsets of the product of a compact space with a metric space. The class of M-spaces is characterized as the class of quasi-perfect preimages of metric spaces. The class of paracompact Čech-complete spaces is characterized as the class of all perfect preimages of complete metric spaces [E, 5.5.9].

Every metric space of weight $\kappa \geqslant \aleph_0$ can be expressed as a perfect image of a subspace of the ***Baire space of weight*** κ, $B(\kappa)$ [E, 4.4.J], and every complete metric space of weight κ can be expressed as a closed image of $B(\kappa)$.

Any perfect image of the ordinal space ω_1 is just homeomorphic to ω_1 [14]. See [14] for a characterization of the quotients (perfect images) of compact ordinals. The perfect preimages of ω_1 can be much different than ω_1 and do not have to contain a copy of ω_1 as a subspace. For a quick example, let D be a discrete space of cardinality $\aleph_1$ and let $g : D \to \omega_1$ be a bijection. If $G : \beta D \to \omega_1 + 1$ is the Stone–Čech continuous extension of g then $Z = G^{-1}(\omega_1)$ is a perfect preimage of ω_1. Z cannot contain a copy of ω_1 since Z does not contain any nontrivial convergent sequences. D. Burke and G. Gruenhage (see [3] for reference) have independently shown that any noncompact space X, in which all countable sets have compact closure, contains a perfect preimage of ω_1. This can be useful in questions of how countably compactness relates to compactness. Assuming $\mathsf{MA} + \neg\mathsf{CH}$, this can be used to show that if X is a regular ***L-space*** and bX is any compactification of X then the remainder b$X \setminus X$ contains a perfect preimage of ω_1.

A continuous surjection $f : X \to Y$ is a **light map** (**zero-dimensional map**) if all fibers of f are ***hereditarily disconnected*** (respectively, zero-dimensional). A **monotone map** has all of its fibers connected. Zero-dimensional maps and monotone maps seem to be "opposite" notions. See [E, 6.2.22] for the result: If $f : X \to Y$ is a perfect map then f can be expressed as a composition $f = h \circ g$ where $g : X \to Z$ is a monotone perfect map and $h : Z \to Y$ is a zero-dimensional perfect map.

If X and Y are metric spaces, a perfect map $f : X \to Y$ is said to be a **cell-like map** if the fibers $f^{-1}(y)$ have ***trivial shape*** for every $y \in Y$. Cell-like maps play a prominent role in geometric and infinite dimensional topology, particularly in the study of the ***Hilbert cube*** and Hilbert cube manifolds. With a "nice" domain and range, cell-like maps often turn out to be ***near-homeomorphisms***. See [11] for an introduction to cell-like maps.

It is useful to have a few examples of perfect or closed maps in order to see the types of possible constructions. In [2] an example is given of a perfect map from a locally compact space with a G_δ-diagonal onto a space with no G_δ-diagonal. The fibers in this map each have cardinality $\leqslant 2$. In [3] an example, due to Lutzer, is described of a Lindelöf semi-metrizable space with a countable ***network*** and a perfect map onto a space Y which is not of point-countable type and not symmetrizable, hence not first-countable or semi-metrizable. The perfect map in this case has just one fiber which is not a singleton. J. Chaber [6] has given an example of a perfect map from a p-space with a σ-locally countable base and a σ-disjoint base onto a space which is not a p-space. The construction of the domain space and resulting perfect map is general enough that it may have applications in other situations. In [3] there is an example, due to P. Zenor, of a locally compact, countably paracompact space and a closed map onto a space which is not countably paracompact. See [KV, Chapter 9] for an example by Burke of a perfect map from an orthocompact space onto a space which is not orthocompact. The fibers in this map are infinite – it is not known if orthocompactness is preserved under finite-to-one closed maps. See [15] for references to three examples which destroy submetrizability under finite-to-one perfect maps. These examples actually kill one of the properties of realcompactness, G_δ-diagonal or property wD. In [13] there is an open perfect map $f : X \to Y$ where is X is hereditarily disconnected (no nontrivial connected subsets) and Y is connected with a ***dispersion point***. Both X and Y can be embedded into $\mathbb{R}^3$.

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c-5 Cell-Like Maps

A compact subset C of a space X is a **cell-like set** in X if for each neighbourhood U of C the inclusion $i_U : C \to U$ is ***homotopic*** to a constant. Within ANRs cell-likeness is an embedding invariant, not merely a positional feature relative to the superspace, because if C is cell-like in X, and $e : C \to Y$ is an embedding into an ANR Y, then $e(C)$ is cell-like in Y. In addition, each neighbourhood W of $e(C)$ then contains a smaller neighbourhood W' of $e(C)$ such that the inclusion $W' \to W$ is null-homotopic. Thus, one speaks of a compact metric space simply as being **cell-like**, provided under some (hence, under any) embedding in an ANR Y, its image is cell-like in Y.

Cell-likeness clearly generalizes the concept of ***contractibility***; the ***topologists' sine curve*** is a familiar example of a cell-like, non-contractible space

A compact, connected subset C of $\mathbb{R}^2$, the Euclidean plane, is cell-like if and only if $\mathbb{R}^2 \setminus C$ is connected; consequently, the famous Fixed Point Problem for planar continua amounts to asking whether cell-like subsets of the plane must have the ***fixed-point property***. For 1-dimensional metric compacta, being cell-like is equivalent to being ***tree-like***. Generally, a compact metric space C is cell-like if and only if (1) it is acyclic with respect to Čech homology and (2) it has the following **1-UV Property**: for some embedding $e : C \to Q$, the ***Hilbert cube***, each neighbourhood U of $e(C)$ contains a smaller neighbourhood V of $e(C)$ such that every map $\partial B^2 \to V$ extends to a map $B^2 \to U$. Also, cell-likeness is equivalent to having the ***shape*** of a point.

A map $f : X \to Y$ is **cell-like** if each point preimage $f^{-1}(y)$, $y \in Y$, is cell-like in X. (This definition can be, and will be, interpreted as requiring such maps to be surjective.) Due to the simplicity of their point preimages, cell-like maps preserve most standard algebraic invariants used to distinguish spaces. Every cell-like map $f : X \to Y$ between ANRs is a ***homotopy equivalence***; in fact, f is cell-like if and only if it is a **fine homotopy equivalence**, meaning that for each open subset V of Y, f restricts to a homotopy equivalence $f^{-1}(V) \to V$.

Cell-like maps $f : X_1 \to X_2$ between finite-dimensional metric compacta exhibit a significant amount of regularity. First of all, f preserves ***shape***; in particular, whenever one of the X_i is cell-like, so is the other. Furthermore, f cannot raise ***dimension***, essentially because the dimensions of X_1, X_2, when finite, equal their respective ***cohomological dimensions***, and cohomological invariants are preserved by cell-like maps. Pathology arises, however, in the infinite-dimensional realm. J.L. Taylor [15] produced a cell-like map $X \to Q$, where Q is the Hilbert cube and X is compact but not cell-like; others observed that Taylor's example could be modified to produce a cell-like map for which only the domain, not the image, is cell-like. A.N. Dranishnikov [8] solved a foundational problem of Alexandroff by constructing an infinite-dimensional compactum having finite cohomological dimension, thereby establishing the existence of a cell-like map $f : X_1 \to X_2$ from a finite dimensional X_1 onto an infinite-dimensional X_2.

Cell-like images of ANRs, if finite-dimensional, are ANRs. However, cell-like images of ***manifolds*** need not be manifolds, not even when finite-dimensional. Instead, such cell-like images provide concrete examples of a related class called **generalized n-manifolds**, consisting of the finite-dimensional ANRs X with $H_*(X, X \setminus \{x\}; \mathbb{Z}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{\text{origin}\}; \mathbb{Z})$ for all $x \in X$.

In the space of all continuous functions of a compact ANR Y onto itself, any limit of a set of homeomorphisms is cell-like. Better yet, in many categories cell-like maps can be uniformly approximated by homeomorphisms; this is a recurring theme in what follows. Say that a map $f : X \to Y$ onto a compact metric space (Y, ρ) is a **near-homeomorphism** if for each $\varepsilon > 0$ there exists a homeomorphism $h : X \to Y$ such that $\rho(h(x), f(x)) < \varepsilon$, for all $x \in X$. A classical theorem of R.L. Moore [11] establishes that for any cell-like map $f : M \to Y$ defined on a 2-manifold M, the image space Y is a 2-manifold. Although Moore's methods did not address the matter, in case M is metrizable, a presumption on all spaces arising throughout this unit, f must be a near-homeomorphism (see [7, Section 25]).

Cell-likeness is closely related to cellularity. A subset K of an n-manifold M is a **cellular set** provided there exists a sequence of n-cells $C_1, C_2, \dots$ in M such that $\operatorname{Int} C_i \supset C_{i+1}$ and $K = \bigcap_i C_i$. It should be clear that cellular sets are cell-like, but in manifolds of dimension at least three the converse does not hold. However, the two concepts are intimately related: a compact subset K of $M \times 0 \subset M \times \mathbb{R}$ is cell-like if and only if K is cellular in $M \times \mathbb{R}$. As usual, a map $f : M \to X$ defined on an n-manifold M is a **cellular map** if each point preimage is. Combined work of S. Armentrout [1] ($n = 3$), F. Quinn [12] ($n = 4$) and L.C. Siebenmann [14] ($n \geqslant 5$) established that cellular maps between compact n-manifolds are near-homeomorphisms. Indeed, with an exception only for $n = 3$, to allow for the possible failure of the 3-dimensional Poincaré Conjecture, cell-like maps between compact n-manifolds. $n \neq 3$, are near-homeomorphisms, because the cell-like maps and the cellular maps between n-manifolds coincide ($n \neq 3$), despite the fact that the class of cell-like subsets of n-manifolds, $n > 2$, is strictly larger than that of the cellular subsets.

Cell-like maps play a fundamental role in two important manifold recognition criteria, for ordinary topological manifolds and for infinite-dimensional manifolds modeled on the

Hilbert cube (***Hilbert cube manifolds***). The first depends on a basic general position property possessed by all n-manifolds, $n > 4$. A metric space Y satisfies the **disjoint discs property** if any two maps $\mu_1, \mu_2 : B^2 \to Y$ can be approximated, arbitrarily closely, by maps $\mu_1', \mu_2' : B^2 \to Y$ such that $\mu_1'(B^2) \cap \mu_2'(B^2) = \emptyset$. R.D. Edwards [9] (for a detailed proof, see [7]) proved a pivotal Cell-like Approximation Theorem: a cell-like, surjective map $f : M^n \to Y$ defined on an n-manifold M^n, $n \geqslant 5$, is a near-homeomorphism if and only if Y is an ANR satisfying the disjoint discs property.

Any finite-dimensional, cell-like image of an n-manifold is a generalized n-manifold; the generalized n-manifolds that arise in this way are said to be **resolvable**. Quinn [13] showed that corresponding to any connected generalized n-manifold X, $n > 3$, is an integer-valued index, $i(X) \in 1 + 8\mathbb{Z}$, locally determined and locally constant, such that $i(X) = 1$ if and only if X is resolvable. Quinn's index satisfies $i(X_1 \times X_2) = i(X_1) \cdot i(X_2)$, so every resolvable Cartesian product must be composed of resolvable factors. Edwards's theorem can be viewed as assuring that a generalized n-manifold X, $n \geqslant 5$, is a genuine n-manifold if and only if $i(X) = 1$ and X has the disjoint discs property. It turns out that for every generalized n-manifold X with $i(X) = 1$, $X \times \mathbb{R}^2$ is an $(n+2)$-manifold; still unsettled is whether then $X \times \mathbb{R}$ is a manifold.

Non-resolvable generalized manifolds do exist. For $n > 5$ and every integer $\xi \in 1 + 8\mathbb{Z}$ J. Bryant, S. Ferry, W. Mio and S. Weinberger [2] produced a generalized n-manifold X homotopy equivalent to S^n with $i(X) = \xi$. Hence, Quinn's obstruction is essential to the characterization of topological manifolds.

Resolvability and cell-like approximation are topics interwoven through the solution of the long-standing Double Suspension Problem, which asked whether the double suspension of a non-simply connected homology n-sphere could yield the $(n+2)$-sphere. The **suspension** of a space Z, written $\mathrm{Susp}(Z)$, is the quotient space obtained from $Z \times [-1, 1]$ by identifying the sets $Z \times \{-1\}$ and $Z \times \{1\}$ to points, and the **double suspension** of Z then is the suspension of $\mathrm{Susp}(Z)$. If Σ^n denotes a compact n-manifold homologically but not homotopically the same as S^n, could its double suspension be a manifold? If so, the manifold would necessarily be S^{n+2}. Edwards [9] and J.W. Cannon [3] proved that the double suspension of any such Σ^n is S^{n+2}. One can always build resolutions locally, since Σ^n bounds a compact, contractible C^{n+1} (but this was not completely established at the time of [3, 9]) embedded in some $(n+1)$-manifold W^{n+1} in such a way that $(W^{n+1}/C^{n+1}) \times \mathbb{R}$, a cell-like image of $W^{n+1} \times \mathbb{R}$, is equivalent to a neighbourhood of any point on the suspension circle (the suspension of the two suspension points) in $\mathrm{Susp}(\mathrm{Susp}(\Sigma^n))$. It is not difficult to see that $(W^{n+1}/C^{n+1}) \times \mathbb{R}$ satisfies the disjoint discs property and, hence, is an $(n+2)$-manifold.

While Edwards was developing his cell-like approximation theorem for finite-dimensional manifolds, H. Toruńczyk [16] produced one for Hilbert cube manifolds. A metric space M is called a **Hilbert cube manifold**, abbreviated simply as **Q-manifold**, if each point has a closed neighbourhood homeomorphic to Q, the Hilbert cube. Say that a metric space Y has the **disjoint cells property** if any two maps $\mu_1, \mu_2 : B^k \to Y$, $k \geqslant 0$, can be approximated, arbitrarily closely, by maps $\mu_1', \mu_2' : B^k \to Y$ such that $\mu_1'(B^k) \cap \mu_2'(B^k) = \emptyset$. Toruńczyk's result assures that a cell-like map $f : M^Q \to Y$ defined on a Q-manifold M^Q is a near-homeomorphism if and only if Y is an ANR satisfying the disjoint cells property. One readily obtains an approximation theorem originally due to T.A. Chapman [4] as a corollary: cell-like maps between (compact) Q-manifolds are near-homeomorphisms.

Improving slightly upon previous work of R.T. Miller and J.E. West, R.D. Edwards [5, Chapter XIV] proved that, for any compact ANR Y, $Y \times Q$ is a Q-manifold. Hence, Y is the cell-like image, even the contractible image, of some Q-manifold. This yields a striking characterization: a compact ANR Y is a Q-manifold if and only if it possesses the disjoint cells property. Moreover, every compact AR with the disjoint cells property is homeomorphic to Q.

Given a map $g : X \to Z$ its **mapping cylinder**, $\mathrm{Map}(g)$, is the **adjunction space** obtained from (the disjoint union) $X \times [0, 1] \cup Z$ by identifying each $z \in Z$ with $g^{-1}(z) \times 0 \subset X \times [0, 1]$. Mapping cylinders often appear in algebraic topology as a means for analyzing algebraic properties of a map g by studying those of the space pair $(\mathrm{Map}(g), X')$, where X' denote the implicit image of $X \times 1$ in $\mathrm{Map}(g)$. It turns out that for any cell-like map $f : M^Q \to Y$ from a Q-manifold M^Q onto an ANR Y, $\mathrm{Map}(f)$ is also a Q-manifold. An analogous finite-dimensional result is available for open double mapping cylinders: if $f : M^n \to Y$ is a cell-like map from an n-manifold M^n, $n > 3$, onto an ANR Y, then the adjunction space obtained from (the disjoint union) $(M^n \times (-1, 1)) \cup Y$ by identifying each $y \in Y$ with $f^{-1}(y) \times 0 \subset M^n \times (-1, 1)$ is an $(n+1)$-manifold.

Finally, cell-like maps play a role in analysis of geometric homotopy equivalences. Let $K \supset K'$ be ***simplicial complexes***; K admits an **elementary collapse** to K', written $K \searrow^e K'$, if K contains some k-simplex σ with a $(k-1)$-dimensional face τ such that $K \setminus \{\sigma, \tau\} = K'$; this occurs provided τ is a proper face of no other simplex in K except σ. When there is such an elementary collapse, K' is a ***strong deformation retract*** of K. The equivalence relation generated by elementary collapses is more discriminating than that of homotopy equivalence. Two complexes K and K' are said to be **simple-homotopy equivalent** (and so to have the same **simple-homotopy type**) if there exists a finite sequence $K = K_0, K_1, \ldots, K_m = K'$ of complexes such that, for $i \in \{0, 1, \ldots, m-1\}$, either $K_i \searrow^e K_{i+1}$ or $K_{i+1} \searrow^e K_i$. Moreover, given a finite complex L, using all possible pairs of finite complexes (K, L), K variable with subcomplex L fixed and inclusion $|L| \to |K|$ of underlying point sets a homotopy equivalence, one can define a geometric **Whitehead group**, $\mathrm{Wh}(L)$; to do so one passes to equivalence classes under those finite sequences of elementary collapses and their inverses that preserve L throughout and employs abstract adjunction along

L as addition operator. In a related vein, to any group G, there also corresponds an algebraically defined **Whitehead group**, $\mathrm{Wh}(G)$ [10]; a crucial point, detailed by M.M. Cohen [6], is that the geometric $\mathrm{Wh}(L)$ is isomorphic to the algebraic $\mathrm{Wh}(\pi_1(L))$. Given a simplicial homotopy equivalence $p: L \to K$ between finite simplicial complexes, one can define the **Whitehead torsion** $\tau(p) \in \mathrm{Wh}(L)$ as the class of the pair $(\mathrm{Map}(p), L)$, where L is taken to be the implicit image of $L \times 1$ in $\mathrm{Map}(p)$; more generally, for any map $f: |L| \to |K|$, the torsion $\tau(f)$ of f is defined as $\tau(p)$, where p is any ***simplicial approximation*** to f. The triviality of some such $\tau(f)$ implies the simple-homotopy equivalence of L and K. It is known, for instance, that the Whitehead group $\mathrm{Wh}(A) = 0$ for any free Abelian group A; thus, homotopy equivalent finite complexes K and L are simple-homotopy equivalent whenever $\pi_1(|L|)$ is free Abelian. J. Milnor [10] raised the natural question of whether homeomorphisms between simplicial complexes must have trivial torsion, which T.A. Chapman [4] eventually answered in the affirmative, showing that even cell-like maps $f: |L| \to |K|$ satisfy $\tau(f) = 0 \in \mathrm{Wh}(L)$. In work which exploited the theory of Q-manifolds and of cell-like maps, Chapman showed that $f: |L| \to |K|$ is a simple-homotopy equivalence if and only if $f \times$ identity: $|L| \times Q \to |K| \times Q$ is homotopic to a homeomorphism. This means that two finite complexes K and L are simple-homotopy equivalent precisely when $|K| \times Q$ and $|L| \times Q$ are homeomorphic. Chapman's work also afforded extension of the notion of simple-homotopy equivalence from simplicial and CW-complexes to compact ANRs.

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c-6 Extensions of Maps

Let $C(X, Y)$ denote the collection of all continuous functions from X into Y. If $A \subset X$, and $f : A \to Y$ is a continuous map from A to Y, then a continuous map $ef : X \to Y$ such that $ef(x) = f(x)$ for all $x \in A$ is called an **extension of** f, and the operator $e : C(A, Y) \to C(X, Y)$ is called an **extender**. A subset A is said to be C**-embedded** in X if every real valued continuous map $f : A \to \mathbb{R}$ can be extended to a real valued continuous map $ef : X \to \mathbb{R}$; and A is C^***-embedded** in X if every bounded real-valued continuous map $f : A \to \mathbb{R}$ can be extended to a real valued continuous map $ef : X \to \mathbb{R}$. A remarkable theorem is the following:

TIETZE–URYSOHN EXTENSION THEOREM 1 [E, Chapter 2]. *If A is a closed subset of a* ***normal*** *space X, then every real-valued continuous map $f : A \to \mathbb{R}$ can be extended to a real-valued continuous map $ef : X \to \mathbb{R}$. Thus, every closed subset of X is C-embedded in X. Conversely, if every closed subset of X is C-embedded in X, then X is normal.*

The Tietze–Urysohn Theorem implies that, for any pair of disjoint closed sets H and K in a normal space X, one can assign to each point of X a single real coordinate, with all points of H having coordinate 1 and all points of K having coordinate 0. From that one easily derives a pseudometric on X that separates H and K. A metrization theorem follows directly:

URYSOHN METRIZATION THEOREM 1. *In a* ***regular, Hausdorff*** *space with a countable neighbourhood base, one can impose a countable dimensional rectangular coordinate system, that is, the space can be* ***topologically embedded*** *in the* ***Hilbert cube****.*

In fact, a consequence of the Tietze–Urysohn Theorem is that every normal space is a subspace of a real vector space, since it can be embedded in a product of real lines. The Nagata–Smirnov–Bing Metrization Theorem [E, Chapter 4], that a regular Hausdorff space with a σ-locally finite base is metrizable, follows, too.

The Hahn–Banach Theorem can be regarded as an analogue to the Tietze–Urysohn Theorem for ***normed linear spaces***. A **linear functional** p on a real linear space E is a map $p : E \to \mathbb{R}$ such that $p(x + y) = p(x) + p(y)$ and $p(kx) = kp(x)$ for all $x, y \in E$ and all $k \in \mathbb{R}$. A **sublinear functional** on E is a map $p : E \to \mathbb{R}$ such that $p(x + y) \leqslant p(x) + p(y)$ and $p(kx) = kp(x)$ for all $x, y \in E$ and all $k \geqslant 0$.

HAHN–BANACH THEOREM 1. *Let E be a real linear space and F a subspace of E and p a sublinear functional on E. If $f : F \to \mathbb{R}$ is a linear mapping dominated by p on F, then there is a linear functional $ef : E \to \mathbb{R}$ that extends f and such that ef is dominated by p on E.*

The vector space of all continuous real-valued functions on A is denoted by $C(A)$, while $C^*(A)$ denotes the vector space of all bounded continuous real-valued functions on A. Many topologists and analysts, starting with K. Borsuk, S. Kakutani, R. Arens, J. Dugundji, E.A. Michael and E. van Douwen have studied situations in which the extension process from f to ef could be carried out in a natural way, so that, for each closed $A \subset X$, the extender e would be

(1) a linear transformation from $C(A)$ to $C(X)$, or, at least, monotone (whenever g dominates f on A, eg would dominate ef on X;
(2) continuous in the sense that, for each $f \in C(A)$, the range of ef would be a subset of the convex hull of the range of f.

The key result is:

BORSUK EXTENSION THEOREM 1 [3]. *Let A be a closed subset of a separable, metrizable space X. Then there is a linear transformation $e : C(A) \to C(X)$, satisfying*:

(1) *for each $f \in C(A)$, $ef : X \to \mathbb{R}$ extends $f : A \to \mathbb{R}$,*
(2) *for each $f \in C(A)$, the range of ef is a subset of the convex hull of the range of f.*

The **Dugundji Extension Theorem** generalized Borsuk's theorem to all metric spaces [1, 12, 6, 13].

THEOREM 1. *If A is a closed subset of a metric space X, there is a linear transformation $e : C(A) \to C(X)$ such that if $C(A)$ and $C(X)$ are both equipped with the* ***sup-norm topology****, or with the* ***compact-open topology*** *or with the* ***topology of pointwise convergence****, then $e : C(A) \to C(X)$ is continuous.*

The continuity of the extender e is crucial. By considering a Hamel basis for $C(A)$, it may be deduced from the Tietze–Urysohn Theorem that, in any normal space X, there always exists a linear extender $e : C(A) \to C(X)$ [9].

Subsequent generalizations of the Dugundji theorem have relaxed the requirement that X be metrizable and have considered functions having values in a locally convex topological linear space. A space X is said to have the **Dugundji extension property** if, for every closed $A \subset X$, there is a continuous linear extender $e : C(A) \to C(X)$. The space X has the **bounded Dugundji extension property** if, for every closed $A \subset X$, there is a continuous linear extender $e : C^*(A) \to C^*(X)$. C. Borges [2] showed that all ***stratifiable*** spaces have the Dugundji extension property. Another large class of spaces in which some version of the bounded Dugundji Extension Theorem is known to hold is the class of generalized ordered spaces. A **linearly ordered topological space** is a linearly ordered set with the open interval topology. A **generalized ordered space** is a space that can be embedded in a linearly ordered topological space.

THEOREM 2 [10]. *If A is a closed subset of a generalized ordered space X, there is a linear transformation $e : C^*(A) \to C^*(X)$ such that*

(1) *e is an extender*;
(2) *if $C^*(A)$ and $C^*(X)$ are both equipped with the **sup-norm topology**, then e is an **isometry**.*

An example shows that this is the strongest version of the Dugundji extension theorem that holds in all generalized ordered spaces. The **Michael line** is the set $\mathbb{R}$ of real numbers with topology obtained by starting with the usual topology and declaring all irrational points to be isolated. The Michael line M is a paracompact, generalized ordered space which has a closed subset, the set Q of all rational numbers, for which there is no continuous linear extender $e : C(Q) \to C(M)$. Furthermore, there is no extender $e : C^*(Q) \to C^*(M)$, linear or otherwise, that is continuous when both $C^*(Q)$ and $C^*(M)$ are equipped with the compact-open topology or with the topology of pointwise convergence [9]. The Michael line is not perfectly normal, but G. Gruenhage, Y. Hattori and H. Ohta have shown that, even in perfectly normal, linearly ordered topological spaces the Dugundji Extension Theorem may fail [7]. On the other hand D. Lutzer and H. Martin showed that if A is a closed metrizable G_δ subset of any collectionwise normal space, then there exists a continuous linear extender $e : C(A) \to C(X)$.

By the Tietze–Urysohn Theorem, a space X is normal if and only if every closed subset of X is C-embedded in X, and, also, if and only if every closed subset of X is C^*-embedded in X. A natural question is: is there some form of normality on a space X that is equivalent to X having the Dugundji extension property? R. Heath and D. Lutzer showed that, if X has the bounded Dugundji extension property then X is ***collectionwise normal***. But neither collectionwise normality, nor even ***paracompactness***, implies the Dugundji extension property (or the bounded Dugundji extension property). Two examples of Erik van Douwen [15] have the ***Lindelöf*** property and hence are paracompact, but do not have the Dugundji extension property: the first is a "cosmic space" (a continuous image of a separable metric space); the second (van Douwen and Roman Pol), is a *countable*, scattered, regular space that is first-countable at all but one of its points. A. Pełczyński [14] cites several examples of compact Hausdorff spaces which do not have the Dugundji extension property, including in particular βN.

Strengthening collectionwise normality in a different direction, a topological space X is **monotonically normal** provided there is a function G that assigns to each ordered pair H, K of disjoint closed sets an open neighbourhood $G(H, K)$ of H such that

(1) $G(H, K)$ and $G(K, H)$ are disjoint,
(2) whenever $A \subset B$ and $D \subset C$, then $G(A, C) \subset G(B, D)$.

Generalized ordered spaces and stratifiable spaces, the two large classes of generalized metric spaces that have the bounded Dugundji extension property, are monotonically normal, but not all monotonically normal spaces do satisfy the Dugundji extension property nor does that extension property imply monotonically normality. However, P. Zenor has shown that monotonically normal spaces do satisfy a monotone extension theorem.

THEOREM 3 [11]. *Let A be a closed subset of a monotonically normal space X and let I denote the closed interval $[0, 1]$. There is a function $e : C(A, I) \to C(X, I)$, satisfying*:

(1) *for each $f \in C(A)$, $ef : X \to \mathbb{R}$ extends $f : A \to \mathbb{R}$,*
(2) *whenever g dominates f on A, eg dominates ef on X.*

There is one extension property that characterizes collectionwise normality. A subset A is said to be **P-embedded** in X if every continuous map $f : A \to Y$ from A into an arbitrary ***Čech-complete*** AR, Y, can be extended to continuous map $ef : X \to Y$. It is noted by Takao Hoshina [HvM, 405–416] that a space X is collectionwise normal if and only if every closed subset of X is P-embedded in X. T. Przymusiński showed that a subset A of a ***completely regular*** space X is P-embedded in X if and only if $A \times Z$ is P-embedded in $X \times Z$ for every compact Z, and also if and only if $A \times \beta X$ is C^*-embedded in $X \times \beta X$. It remains open whether some version of normality is equivalent to the Dugundji extension property.

One class of spaces, retractifiable spaces, does have all extension properties [15]. A **retractifiable space** is a space in which every closed subset is a retract of the space. For $A \subset X$, a **retraction** from X onto A is a continuous map $f : X \to A$ such that $f(x) = x$ for all $x \in A$, and A is then called a retract of X.

A space Y is called an AE **for metric spaces** (respectively an ANE **for metric spaces** if, whenever X is a metric space and A is a closed subspace of X, then any continuous function $f : A \to Y$ can be extended to a continuous function from X (respectively, some neighbourhood of A in X) into Y.

A space Y is called an AR **for metric spaces** (respectively an ANR **for metric spaces**) if, whenever Y is a closed subspace of a metric space X, there is a continuous retraction from X (respectively, some neighbourhood of Y in X) onto Y. Further clarifying the relation between extensions and retractions are the following:

THEOREM 4 [4]. *A subset A of a space X is a retract of X if and only if, for every subset Y, every continuous map $f : A \to Y$ has a continuous extension $f : X \to Y$.*

KURATOWSKI–WOJDYSŁAWSKI THEOREM 1 [4]. *For each metrizable space X there is a normed linear space Z and a homeomorphism* $h : X \to Z$ *such that* $h(X)$ *is closed in its convex hull.*

THEOREM 5 [4, 13]. *A metrizable space Y is an* AR (*respectively an* ANR) *for metric spaces if and only if Y is an* AE (*respectively an* ANE) *for metric spaces.*

The above theorem also holds if one replaces "metric" with "stratifiable" [5].

A useful notation for an "AR (respectively an ANR) for metric spaces" is $\mathrm{AR}(\mathcal{M})$ (respectively $\mathrm{ANR}(\mathcal{M})$). Similarly, an AE (respectively an ANE) for metric spaces is denoted by $\mathrm{AE}(\mathcal{M})$ (respectively $\mathrm{ANE}(\mathcal{M})$). One of the most important properties of $\mathrm{ANR}(\mathcal{M})$-*spaces* is in the following theorem:

BORSUK HOMOTOPY EXTENSION THEOREM 1 [4]. *Let A be a closed subset of a metrizable space X, and let Y be an* $\mathrm{ANR}(\mathcal{M})$. *Let* $\{f_t\colon t \in [0,1]\}$ *be a continuous family of maps* $f_t : A \to Y$ *such that* f_0 *has a continuous extension* $ef_0 : X \to Y$. *Then there is a continuous family* $\{ef_t : t \in [0,1]\}$ *of maps* $ef_t : X \to Y$ *such that* ef_t *extends* f_t *for each* $t \in [0,1]$.

By a **continuous** family $\{f_t\colon t \in [0,1]\}$ of maps $f_t : X \to Y$ one means a continuous function $F : X \times [0,1] \to Y$ such that $F(s,t) = f_t(s)$ for all $s \in X$ and $t \in [0,1]$. Following Hoshina [MN]: if (X, A) is an ordered pair with A a subspace of a space X and if Y is another space, (X, A) is said to have the **Homotopy Extension Property** (abbreviated **HEP**) with respect to Y if every continuous map $f : (X \times \{0\}) \cup (A \times [0,1]) \to Y$ can be continuously extended over $X \times [0,1]$. The Borsuk extension theorem was generalized by C.H. Dowker: every pair (X, A) with A a closed subset of a ***countably paracompact*** normal space X has the HEP with respect Čech-complete separable ANR-spaces. A more general theorem by K. Morita and T. Hoshina is the following.

THEOREM 6 [MN, p. 77]. *For a subspace A of a space X the following are equivalent*:

(1) *A is C-embedded in X*;
(2) $(X \times B) \cup (A \times Z)$ *is C-embedded in* $X \times Z$ *for every compact metric space Z and its closed subset B*;
(3) $(X \times \{0\}) \cup (A \times [0,1])$ *is C-embedded in* $X \times [0,1]$;
(4) (X, A) *has the HEP with respect to every separable Čech-complete* ANR.

In general, if the subspace X of Y is not closed in Y, there may exist no extender $e : C(X) \to C(Y)$. However by theorems of Stone and Čech:

THEOREM 7 [8]. *Every completely regular space X is a dense subspace of a compact space* βX *with following properties*:

(1) *every continuous map* $f \in C^*(X)$ *has an extension* $ef \in C(\beta X)$;
(2) *every continuous map f from X into any compact space Y has a continuous extension ef from* βX *into Y*.

The space βX is unique and is called the Stone–Čech compactification of X. Among other things, the Stone–Čech compactification is a major source of examples in the study of abstract spaces [8].

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c-7 Topological Embeddings (Universal Spaces)

Since the beginning of the 20th century universal spaces have been playing an important role in the classical dimension theory as well as in some other areas of general and geometric topology. The existence of a universal object in a certain class of spaces allows us to reduce the study of spaces from this class to the study of subspaces of a fixed topological space.

Let $\mathcal{P}$ be a class of ***completely regular*** and ***Hausdorff*** (also known as ***Tychonoff***) spaces. A space X is said to be **universal in the class** $\mathcal{P}$ (shortly $\mathcal{P}$**-universal**) if $X \in \mathcal{P}$ and if every $Y \in \mathcal{P}$ admits a topological embedding into X.

It is remarkable that ***powers*** of simple topological spaces serve as examples of universal spaces in the corresponding classes of spaces. Here are some of the examples:

(1) The three-point set $\{0, 1, 2\}$ topologized so that only $\emptyset$, $\{0\}$ and $\{0, 1, 2\}$ are open. The space $\{0, 1, 2\}^\tau$ is universal in the class of all topological spaces of ***weight*** $\leqslant \tau$, $\tau \geqslant \omega$ (cf. [E, 2.3.I]).

(2) The two-point set $\{0, 1\}$ with the topology consisting of $\emptyset$, $\{0\}$ and $\{0, 1\}$. The space $\{0, 1\}^\tau$, also known as the **Alexandroff cube** of weight τ, is universal in the class of all T_0-spaces of weight $\leqslant \tau$, $\tau \geqslant \omega$ (P. Alexandroff [E, 2.3.26]).

(3) The two-point set $\{0,1\}$ with the ***discrete topology***. The space $\{0, 1\}^\tau$, also known as the **Cantor cube** of weight τ, is universal in the class of all ***0-dimensional*** Tychonoff spaces of weight $\leqslant \tau$ (N. Vedenisoff for $\tau > \omega$, [E, 6.2.16]; W. Sierpiński for $\tau = \omega$, [8, 1.3.15]). The countable power $\{0, 1\}^\omega$ is topologically equivalent to the **Cantor discontinuum** described below.

(4) The τth power $\mathbb{I}^\tau$ of the segment $\mathbb{I} = [0, 1]$ with the standard topology, also known as the **Tychonoff cube** of weight τ, $\tau \geqslant \omega$, is universal in the class of all Tychonoff spaces of weight $\leqslant \tau$ (A. Tychonoff for $\tau > \omega$, [E, 2.3.23]; P. Urysohn for $\tau = \omega$, [1, Footnote 4 on p. 100]). The countable power $\mathbb{I}^\omega$ is called the **Hilbert cube** and is universal in the class of all ***separable metrizable spaces*** [E, 4.2.10].

(5) Let $D(\tau)$ denote the discrete space of cardinality τ, $\tau \geqslant \omega$. The countable power $B(\tau) = D(\tau)^\omega$, also known as the **Baire space** of weight τ is universal in the class of all metrizable spaces of dimension $\dim \leqslant 0$ and weight $\leqslant \tau$ (W. Sierpiński [E, 7.3.15]). It is known that the space $B(\omega)$ is topologically equivalent to the space of irrational numbers (with the topology inherited from the real line).

(6) Let $J(\tau)$ denote the **hedgehog** with τ spines, i.e., the set of equivalence classes of the equivalence relation $\sim$ defined on the set $\bigcup\{\mathbb{I} \times \{t\}\colon t \in T\}$, where $|T| = \tau$, by letting $(x, t_1) \sim (y, t_2)$ if and only if $x = 0 = y$ or $x = y$ and $t_1 = t_2$. A topology on this set is generated by the following metric

$$\rho\big([(x, t_1)], [(y, t_2)]\big) = \begin{cases} |x - y|, & \text{if } t_1 = t_2, \\ x + y, & \text{if } t_1 \neq t_2. \end{cases}$$

The countable power $J(\tau)^\omega$ is universal in the class of all metrizable spaces of weight $\leqslant \tau$, $\tau \geqslant \omega$ (H.J. Kowalsky [E, 4.4.9]).

In light of Example 5 it is interesting to note (M. Fréchet [E, 4.3.H(b)]) that the space of all rational numbers (with the topology inherited from the real line) is universal in the class of all countable metrizable spaces.

The following statement is of a somewhat different nature since instead of a class of topological spaces it deals with the class of metric (i.e., the class of metrizable spaces each furnished with a given metric) spaces. Consider the set $C(\mathbb{I})$ of all continuous real-valued functions defined on the interval $\mathbb{I} = [0, 1]$ endowed with the standard sup metric defined by letting

$$\sigma(f, g) = \sup\big\{\big|f(t) - g(t)\big|\colon t \in \mathbb{I}\big\}.$$

The metric space $(C(\mathbb{I}), \sigma)$ contains an isometric copy of every separable metric space (S. Banach and S. Mazur [2, Proposition II.1.5]).

The next group of examples has been formally obtained within the classical dimension theory, but undoubtedly is of significant importance in various areas of general and geometric topology.

We begin by outlining the construction of the Menger compacta. Partition the standard unit cube $\mathbb{I}^{2k+1}$, lying in $(2k + 1)$-dimensional ***Euclidean space*** $\mathbb{R}^{2k+1}$, into 3^{2k+1} congruent cubes of the "first rank" by hyperplanes drawn perpendicularly to the edges of the cube $\mathbb{I}^{2k+1}$ at points dividing the edges into three equal parts, and we choose from these 3^{2k+1} cubes those that intersect the ***k-dimensional skeleton*** of the cube $\mathbb{I}^{2k+1}$. The union of the selected cubes of the "first rank" is denoted by $\mathbb{I}(k, 1)$. In an analogous way, divide every cube entering as a term in $\mathbb{I}(k, 1)$ into 3^{2k+1} congruent cubes of the "second rank" and the union of all analogously selected cubes of the "second rank" is denoted by $\mathbb{I}(k, 2)$. If we continue the process we get a decreasing sequence of compacta

$$\mathbb{I}(k, 1) \supseteq \mathbb{I}(k, 2) \supseteq \cdots.$$

The compactum $M_k^{2k+1} = \bigcap\{\mathbb{I}(k, i)\colon i \in N\}$ is called the k-dimensional **Menger Universal Space**. This construction, due to K. Menger [10], can be extended in order to produce

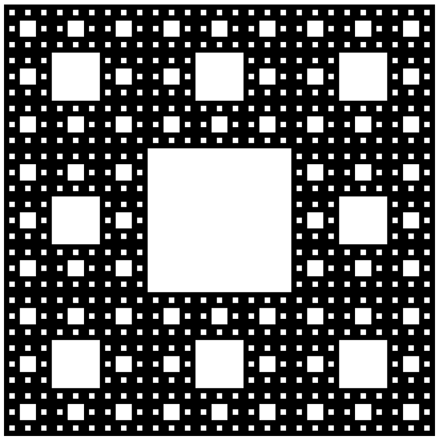

Fig. 1. 4th step in the construction of the Sierpiński Carpet M_1^2.

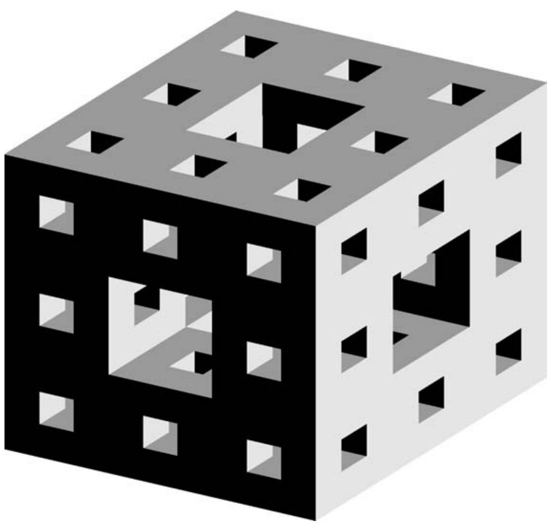

Fig. 2. 2nd step in the construction of the Menger Curve M_1^3.

***k*-dimensional** compacta M_k^n lying in $\mathbb{R}^n$ for all pairs of non-negative integers (k, n) with $k \leqslant n$, $n \geqslant 1$. Somewhat different constructions of the compacta M_k^{2k+1} have been given by S. Lefschetz [9] and M. Bestvina [3].

Compactum M_0^1 is precisely the above-mentioned Cantor discontinuum. We have already mentioned (Example 3) that M_0^1 is universal in the class of 0-dimensional metrizable compact spaces. Compactum M_1^2, also known as the **Sierpiński universal curve** or the **Sierpiński Carpet**, is universal in the class of all at most 1-dimensional planar compacta (W. Sierpiński [8, Chapter 1, §11]).

Further, it was shown by K. Menger [10] that the 1-dimensional Menger compactum M_1^3, also known as the **Menger Universal Curve** or the **Menger Sponge**, is universal in the class of all at most 1-dimensional metrizable compacta.

In general, it was conjectured [10] that the compactum M_k^n is universal in the class of all at most k-dimensional compact spaces embeddable in $\mathbb{R}^n$ (**Menger's problem**). As it was already mentioned, this problem was known to have a positive solution for the cases $(k, n) = (1, 2)$ and $(k, n) = (1, 3)$. Positive solution in the case $k = n - 1$ was also given by K. Menger [10]. Results of S. Lefschetz [9], H. Bothe [4] and M. Bestvina [3] show that the compacta M_k^{2k+1} are universal in the class of at most k-dimensional metrizable compact spaces. The ultimate affirmative solution of Menger's problem was obtained by M. Štanko [8, Chapter 1, §11].

Closely associated with the Menger compacta M_k^n are the **Nöbeling universal spaces** N_k^n defined for every (k, n) with $k \leqslant n$, $n \geqslant 1$. The space N_k^{2k+1}, defined as the subspace of $\mathbb{R}^{2k+1}$ consisting of points with at most k rational coordinates, is universal in the class of all at most k-dimensional separable metrizable spaces (G. Nöbeling [8, 1.11.5]. Obviously, N_0^1 is precisely the space of irrational numbers (Example 5).

Universal spaces for non-separable metrizable spaces of a given ***covering dimension*** and given weight have been explicitly constructed by J.-I. Nagata [8, 4.2.6]. Later, B. Pasynkov obtained the same result by using a completely different approach based on his factorization theorem [8, 4.2.5]. Very similar approach, based on Mardešic's factorization theorem [8, 3.3.2], was used by B. Pasynkov [8, 3.3.4] in order to obtain universal compact spaces of given covering dimension dim and given weight (independently, simultaneously, and by a completely different method, the same result was established by A. Zarelua [8, 3.3.4]). Mardešic's factorization theorem also allows us to prove the so-called compactification theorem (originally proved, using a different method, by E. Skljarenko [8, 3.3.3]) stating that for every Tychonoff space there exists a compactification preserving both the dimension dim and weight.

It is important to note that the approach based on factorization theorems does not produce explicit constructions of universal spaces. On the other hand, this approach proves significantly more than just the existence of universal spaces. Let us say that a map $f : X \to Y$ is a **$\mathcal{P}$-invertible map** if for any map $g : Z \to Y$ of a space $Z \in \mathcal{P}$, there exists a map $h : Z \to X$ such that $g = f \circ h$. In certain situations, an approach based on factorization theorems makes it possible to prove that there exists a $\mathcal{P}$-invertible map $f : X \to \mathbb{I}^\tau$ of a compact space $X \in \mathcal{P}$ of weight τ onto the Tychonoff cube. Since $\mathbb{I}^\tau$ contains a topological copy of every Tychonoff space Z of weight $\leqslant \tau$ and since f is $\mathcal{P}$-invertible, it follows immediately that X contains a topological copy of Z. Also the closure of this copy of Z in X provides a compactification of Z that is an element of the class $\mathcal{P}$ and has weight τ. This illustrates how the existence of universal compacta of given dimension dim and given weight as well as the existence of dimension and weight preserving compactifications follow from the existence of invertible maps. A deeper connection between the existence of invertible maps, universal spaces and compactifications was discovered in [5].

The same approach works for some classes of ***infinite-dimensional*** spaces (we consider only **covering dimension** dim). For instance J. Nagata (see [1, Chapter 10, §2]) proved that the class of **countable-dimensional** metrizable spaces (a space is countable-dimensional if it can be represented as the union of countably many 0-dimensional subspaces) of weight $\leqslant \tau$ contains a universal space. B. Pasynkov

[1, Chapter 10, §3] proved that the class of **weakly countable-dimensional** (these are spaces representable as the unions of countably many finite dimensional closed subspaces) ***normal*** spaces of weight $\leqslant \tau$ also contains a universal space. Similar result for metrizable spaces was obtained by A. Arkhangel'skiĭ [1, Chapter 10, §3].

Recent developments in infinite-dimensional topology and theory of ***Absolute Extensor***s brought a completely new light to the role of universal spaces. Somewhat stronger versions of the standard concept of universality, coupled with certain extension properties, provide complete topological characterizations of topological spaces such as Cantor, Hilbert and Tychonoff cubes, infinite powers of the real line, Menger compacta, and Nöbeling spaces.

Historically, the first result (K. Menger, G. Nöbeling, L. Pontrjagin and G. Tolstowa, S. Lefschetz, W. Hurewicz [8, 1.11.4]), displaying a prototype of the strong universality property in a particular situation, states that not only every at most k-dimensional compactum X admits a topological embedding into $\mathbb{R}^{2k+1}$, but that every continuous map $f : X \to \mathbb{R}^{2k+1}$ can be arbitrarily closely approximated by topological embeddings.

In order to properly measure "closeness" of maps between continuous maps of (generally speaking) non-metrizable and non-compact spaces first introduce the corresponding topology on spaces of maps. Let X and Y be arbitrary Tychonoff spaces and τ be an arbitrary infinite cardinal number. For each map $f \in C(Y, X)$ declare sets of the form

$$\bigl\{g \in C_\tau(Y, X)\colon g \text{ is } \mathcal{U}_t\text{-close to } f \text{ for each } t \in T\bigr\},$$

where $|T| < \tau$ and $\mathcal{U}_t \in \operatorname{cov}(X)$ for each $t \in T$, to be open basic neighbourhoods of the point f in $C_\tau(Y, X)$. Here $\operatorname{cov}(X)$ denotes the collection of all countable ***functionally open*** covers of the space X. Also two maps $f, g : Y \to X$ are said to be $\mathcal{U}$-close, where $\mathcal{U} \in \operatorname{cov}(X)$, if for each point $y \in Y$, there exists an element $U \in \mathcal{U}$ such that $f(y), g(y) \in U$. The corresponding space is denoted by $C_\tau(Y, X)$.

If both spaces Y and X have countable bases and Y is compact, then the topology of $C_\omega(Y, X)$ coincides with the standard ***compact-open topology***.

It follows from Tychonoff's theorem (Example 4) that the τth power $\mathbb{R}^\tau$ of the real line contains a topological copy of every Tychonoff space of weight $\leqslant \tau$. As we will shortly see the following ***cardinal invariant*** allows us to detect a much finer universality property of $\mathbb{R}^\tau$. Every Tychonoff space admits not only an embedding but even a ***C-embedding*** into an infinite power of the real line. Indeed, consider $i : X \to \mathbb{R}^{C(X)}$ defined as $i(x) = \{f(x)\colon f \in C(X)\} \in \mathbb{R}^{C(X)}$ for each $x \in X$. The $\mathbb{R}$-weight (cf. [6, Definition 1.3.14]) of a Tychonoff space X is defined as the minimal infinite cardinal number τ such that X admits a C-embedding into $\mathbb{R}^\tau$ (notation: $\mathbb{R}\text{-}w(X) = \tau$). Note that Tychonoff spaces of countable $\mathbb{R}$-weight are precisely the ***Polish spaces***. Also, for compact spaces, weight and $\mathbb{R}$-weight coincide.

Let $\tau \geqslant \omega$ and $k \in \omega$. For a class $\mathcal{P}$ of Tychonoff spaces let

$$\mathcal{P}_{k,\tau} = \bigl\{X \in \mathcal{P}\colon \dim X \leqslant n \text{ and } \mathbb{R}\text{-}w(X) \leqslant \tau\bigr\}.$$

A space $X \in \mathcal{P}_{k,\tau}$ is said to be **strongly $\mathcal{P}_{k,\tau}$-universal** [6, §6.5] if for each $Y \in \mathcal{P}_{k,\tau}$, the set of C-embeddings is dense in the space $C_\tau(Y, X)$.

Below we consider two classes of spaces only: $\mathcal{C}$ – the class of all compact Hausdorff spaces and $\mathcal{T}$ – the class of all Tychonoff spaces.

A space X is said to be an **Absolute Extensor in dimension** k (notation $AE(k)$), $k = 0, 1, 2, \dots, \infty$, if any continuous map $f : A \to X$ defined on a closed subspace of an at most k-dimensional space B can be continuously extended over the whole space B. If $k = \infty$, then such spaces are called Absolute Extensors (notation: $AE(\infty) = AE$). This definition works well if all spaces under consideration are normal, but requires some modifications [6, §6.1] for Tychonoff spaces.

(7) Let $\tau \geqslant \omega$. The following conditions are equivalent for any space X (L. Brouwer [6, 1.1.10] for $\tau = \omega$; E. Ščepin [6, 8.1.6] for $\tau > \omega$):
  - (a) X is topologically equivalent to the Cantor cube $\{0, 1\}^\tau$;
  - (b) X is a strongly $\mathcal{C}_{0,\tau}$-universal $AE(0)$-space.

(8) Let $\tau \geqslant \omega$. The following conditions are equivalent for any space X (H. Toruńczyk [12] for $\tau = \omega$; E. Ščepin [6, 7.2.9] for $\tau > \omega$):
  - (a) X is topologically equivalent to the Tychonoff cube $\mathbb{I}^\tau$;
  - (b) X is a strongly $\mathcal{C}_{\infty,\tau}$-universal AE-space.

(9) Let $k = 1, 2, \dots$. The following conditions are equivalent for any space X (M. Bestvina [3]):
  - (a) X is topologically equivalent to the Menger cube M_k^{2k+1};
  - (b) X is a strongly $\mathcal{C}_{k,\omega}$-universal $AE(k)$-space.

(10) Let $\tau \geqslant \omega$. The following conditions are equivalent for any space X (P. Alexandroff, P. Urysohn [6, 1.1.5] for $\tau = \omega$; A. Chigogidze [6, 8.1.4] for $\tau > \omega$):
  - (a) X is topologically equivalent to the τth power $D(\omega)^\tau$;
  - (b) X is a strongly $\mathcal{T}_{0,\tau}$-universal $AE(0)$-space.

(11) Let $\tau \geqslant \omega$. The following conditions are equivalent for any space X (H. Toruńczyk [13] for $\tau = \omega$; A. Chigogidze [6, 7.3.3] for $\tau > \omega$):
  - (a) X is topologically equivalent to the τth power $\mathbb{R}^\tau$ of the real line;
  - (b) X is a strongly $\mathcal{T}_{\infty,\tau}$-universal AE-space.

(12) Let $k = 1, 2, \dots$. Then the following conditions are equivalent for any space X (S. Ageev):

(a) X is topologically equivalent to the Nöbeling space N_k^{2k+1};
(b) X is a strongly $\mathcal{T}_{k,\omega}$-universal $AE(k)$-space.

Finally, let us discuss non-separable metrizable spaces. In light of Examples 5, 6 and the existence of Nagata's universal metrizable spaces of given weight and dimension, it is natural to ask whether in this case also strongly universal spaces exist. We consider the class $\mathcal{CM}$ of ***completely metrizable*** spaces. By $\mathcal{CM}_{k,\tau}$ we denote the class of all at most k-dimensional (dim) completely metrizable spaces of weight $\leqslant \tau$. Also, we say that a space $X \in \mathcal{CM}_{k,\tau}$ is strongly $\mathcal{CM}_{k,\tau}$-universal if for each $Y \in \mathcal{CM}_{k,\tau}$ the set of closed embeddings is dense in the space $C_\omega(Y, X)$. The Baire space $B(\tau)$ of weight τ and the countable power $J(\tau)^\omega$ of the hedgehog of spinning τ not only are strongly universal in the corresponding classes of spaces but even are characterized by this property.

(13) Let $\tau > \omega$. The following conditions are equivalent for any space X (A. Stone [11]; in the present form [7]):

(a) X is topologically equivalent to the Baire space $B(\tau)$;
(b) X is a strongly $\mathcal{CM}_{0,\tau}$-universal space.

(14) Let $\tau \geqslant \omega$. The following conditions are equivalent for any space X (H. Toruńczyk [13]):

(a) X is topologically equivalent to $J(\tau)^\omega$;
(b) X is a strongly $\mathcal{CM}_{\infty,\tau}$-universal AE-space.

The existence of a strongly $\mathcal{CM}_{k,\tau}$-universal $AE(k)$-space with $0 < k < \infty$ and $\tau > \omega$ was proved by A. Chigogidze and V. Valov in [7, Corollary 2.8]. The problem of topological uniqueness of a strongly $\mathcal{CM}_{k,\tau}$-universal $AE(k)$-space with $0 < k < \infty$ and $\tau > \omega$ remains open [7] (the case $\tau = \omega$ is covered in Example 12).

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c-8 Continuous Selections

1. Introduction

This article is primarily concerned with the following question: Given a map $\varphi : X \to 2^Y$ (where 2^Y denotes $\{E \subset Y\colon E \neq \emptyset\}$), under what conditions does φ have a **selection**, that is, a ***continuous*** $f : X \to Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$?[1] More generally, given $\varphi : X \to 2^Y$ and a ***closed*** $A \subset X$, when does every selection for $\varphi|A$ extend to a selection for φ?

Most of our results assume that X is ***paracompact***, that Y is **completely metrizable**,[2] that every set $\varphi(x)$ is ***closed*** in Y, and that φ is a **lower semi-continuous map** (= **l.s.c.**) (which means that $\{x \in X\colon \varphi(x) \cap V \neq \emptyset\}$ is ***open*** in X for every open V in Y). These assumptions, supplemented by others in particular theorems, yield interesting and quite general results under conditions which, in many respects, are not only sufficient but also necessary. For example, if one wants φ not merely to have a selection but to have the somewhat stronger property that, whenever $x_0 \in X$ and $y_0 \in \varphi(x_0)$, then φ has a selection f such that $f(x_0) = y_0$, then φ *must* be l.s.c.

Only a few highlights of this subject can be covered in this article. A more comprehensive treatment, including proofs, can be found in the recent monograph by D. Repovš and P.V. Semenov [15], where additional references are provided in the excellent bibliography.

2. Examples of lower semi-continuous maps

The following simple examples may convey some insight into the properties of lower semi-continuous maps.

2.1. If $u : Y \to X$ maps Y onto X, then u is an **open map** (i.e., $u(V)$ is open in X for every open V in Y) if and only if the map $u^{-1} : X \to 2^Y$ is l.s.c.

2.2. Let $\varphi : X \to 2^Y$ be l.s.c., let $A \subset X$ be closed, and let g be a selection for $\varphi|A$. Then the map $\varphi_g : X \to 2^Y$, defined by $\varphi_g(x) = \{g(x)\}$ when $x \in A$ and $\varphi_g(x) = \varphi(x)$ when $x \in X \setminus A$, is also l.s.c.

2.3. If $\varphi : X \to 2^Y$ is l.s.c., then so is the map $\bar{\varphi} : X \to 2^Y$ defined by $\bar{\varphi}(x) = (\varphi(x))^-$.

2.4. If Y is a topological linear space, and if $\varphi : X \to 2^Y$ is l.s.c., then so is the map $\psi : X \to 2^Y$ defined by $\psi(x) = \operatorname{conv}(\varphi(x))$.

[1] For many authors, continuity is not part of the definition of a selection.

[2] I.e., ***metrizable*** with a ***complete metric***.

2.5. If $\varphi : X \to 2^Y$ is l.s.c., if $V \subset Y$ is open, and if $\varphi(x) \cap V \neq \emptyset$ for every $x \in X$, then the map $\psi : X \to 2^Y$, defined by $\psi(x) = \varphi(x) \cap V$, is also l.s.c.

3. Selection theorems

In the sequel, $\mathcal{F}(Y)$ denotes $\{E \in 2^Y\colon E \text{ is closed in } Y\}$.

Perhaps the simplest and most generally useful selection theorem is the following result. (See [5, Theorem 1] for (a), and note that (b) follows from (a) and 2.2 above.)

THEOREM 1. *Let X be paracompact, Y a **Banach space**, and let $\varphi : X \to \mathcal{F}(Y)$ be l.s.c. with $\varphi(x)$ **convex** for all $x \in X$. Then*:

(a) *φ has a selection.*
(b) *More generally, if $A \subset X$ is closed, then every selection for $\varphi \restriction A$ extends to a selection for φ.*

It might be noted that Theorem 1(a) is valid for a T_1***-space*** X if *and only if* X is paracompact (see [6, Theorem 3.2″]).

Theorem 1 remains true, with essentially the same proof, if Y is only a completely metrizable, ***locally convex topological linear space***. More generally, and with rather more effort, one can show that it suffices if Y has a suitably defined "convex structure" (see [8], and an improvement in [3]). However, these and similar results all involve conditions on the $\varphi(x)$ which are variants of convexity, and which, while sufficient, are far from necessary.

When X is ***finite-dimensional*** (in the sense of the ***covering dimension*** dim (see [E, 7.1.7])), the situation is entirely different, and we have a result (see Theorem 3) with purely topological conditions on the sets $\varphi(x)$ which are, in a sense, necessary as well as sufficient. We first consider the ***zero-dimensional*** case, where matters are particularly simple and where there are no conditions at all on the map $\varphi : X \to \mathcal{F}(Y)$ beyond being l.s.c. (see [5, Theorem 2] for (a) and [7, Theorem 1.2 with $n = -1$] for (b)).

THEOREM 2. *Let X be paracompact, Y completely metrizable, and let $\varphi : X \to \mathcal{F}(Y)$ be l.s.c. Then*:

(a) *If* $\dim X = 0$, *then φ has a selection.*
(b) *More generally, if $A \subset X$ is closed, and if* $\dim X = 0$ *or* $\dim X \setminus A = 0$, *then every selection for $\varphi \restriction A$ extends to a selection for φ.*

It might be noted that Theorem 2(a) is valid for a metrizable space Y if *and only if* Y is completely metrizable (see [12]). Similarly for Theorems 5 and 7 below.

Theorems 1 and 2 are easy to state and relatively easy to prove. By contrast, Theorem 3 below, which reduces to Theorem 2 when $n = -1$, is much more difficult, and also requires some additional definitions: A set $E \subset Y$ is called C^n if, for all $k \leqslant n$, every continuous $\alpha : S^k \to E$ (where S^k is the k-sphere) extends to a continuous $\beta : B^{k+1} \to E$ (where B^{k+1} is the $(k+1)$-ball). A collection $\mathcal{E} \subset 2^Y$ is called **equi-**LC^n if, for all $y \in \bigcup \mathcal{E}$, every neighbourhood V of y in Y contains a neighbourhood W of y in Y such that, for all $E \in \mathcal{E}$ and all $k \leqslant n$, every continuous $\alpha : S^k \to W \cap E$ extends to a continuous $\beta : B^{k+1} \to V \cap E$. (For $n = -1$, these requirements are vacuous.)

The following theorem was originally proved in [7, Theorem 1.2]. Using a different approach, an interesting generalization was recently obtained by E.V. Shchepin and N.B. Brodsky in [16] (see also [15, Theorem 7.2]).

THEOREM 3. *Let X be paracompact, Y completely metrizable, and let $\varphi : X \to \mathcal{F}$ be l.s.c. with $\{\varphi(x): x \in X\}$ equi-LC^n. Then*:

(a) *If* $\dim X \leqslant n + 1$, *and if every $\varphi(x)$ is C^n, then φ has a selection.*
(b) *More generally, if $A \subset X$ is closed, and if* $\dim X \leqslant n+1$ *or* $\dim X \setminus A \leqslant n + 1$, *then every selection for $\varphi \restriction A$ extends to a selection for $\varphi \restriction U$ for some open $U \supset A$ in X; if every $\varphi(x)$ is C^n, then one can take $U = X$.*

It is plausible to conjecture that one could sharpen Theorem 1 by replacing the convexity conditions on the sets $\varphi(x)$ by suitable infinite-dimensional analogs of the purely topological conditions in Theorem 3. Unfortunately, an example of C. Pixley [14] shows that a very natural candidate for such conditions will not work. A simpler example is given in [11, Example 10.3], but there the sets $\varphi(x)$, unlike those in [14], are not ***compact***.

Our next result, a modification of Theorem 2, is the only one in this article which does not assume that Y is completely metrizable or that the sets $\varphi(x)$ are closed in Y. See [10, Theorems 1.1 and 1.2].

THEOREM 4. *Let X be paracompact, Y metrizable,*[3] *and let $\varphi : X \to 2^Y$ be l.s.c. Then*:

(a) *If X is countable, then φ has a selection.*
(b) *More generally, if $A \subset X$ is closed, and if $X \setminus A$ is countable, then every selection for $\varphi \restriction A$ extends to a selection for φ.*

Observe that, when comparing Theorems 1, 2 and 4, the hypotheses on X and $A \subset X$ become progressively stronger while those on Y and the sets $\varphi(x) \subset Y$ become weaker.

The following result, which should be compared to Theorem 2(a), may be regarded as a set-valued selection theorem. It was first proved in [9], but a better proof, using Theorem 2(a), is given in [2, Theorem 3.2]. A map $\varphi : X \to \mathcal{F}(Y)$ is an **upper semi-continuous map** (= **u.s.c.**) if $\{x \in X: \varphi(x) \subset V\}$ is open in X for every open V in Y.

[3] For (a), it suffices if Y is only ***first-countable***.

THEOREM 5. *Let X be paracompact, Y completely metrizable, and let $\varphi : X \to \mathcal{F}(Y)$ be l.s.c. Then there exists a u.s.c. $\theta : X \to \mathcal{F}(Y)$ and an l.s.c. $\psi : X \to \mathcal{F}(Y)$ with $\theta(x)$ and $\psi(x)$ compact and with $\psi(x) \subset \theta(x) \subset \varphi(x)$ for every $x \in X$.*

Now call a map $\psi : X \to \mathcal{F}(Y)$ **continuous** (or **Vietoris-continuous**[4]) if it is both l.s.c. and u.s.c. In the following result (see [13, Theorem 1.1]), both the hypothesis and the conclusion are stronger than in Theorem 5 but weaker than in Theorem 3(a) with $n = 0$.

THEOREM 6. *Let X be paracompact, Y completely metrizable, and let $\varphi : X \to \mathcal{F}(Y)$ be l.s.c. with each $\varphi(x)$ connected and with $\{\varphi(x): x \in X\}$ equi-LC^0. Then there exists a continuous $\psi : X \to \mathcal{F}(Y)$ with $\psi(x)$ compact and connected and with $\psi(x) \subset \varphi(x)$ for every $x \in X$.*

We now come to our last theorem, which rephrases a result in [1] and [4].

THEOREM 7. *Let X be any space, and let Y be completely metrizable with* $\dim Y = 0$. *Then every continuous $\varphi : X \to \mathcal{F}(Y)$ has a selection.*

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c-9 Multivalued Functions

For any topological space (X, τ), we let $\mathcal{P}(X) = \{A\colon A$ is a nonempty subset of $X\}$, $2^X = \{A \in \mathcal{P}(X)\colon A$ is closed$\}$, $\mathcal{C}(X) = \{A \in 2^X\colon A$ is compact$\}$, $\mathcal{F}_n(X) = \{A \in 2^X\colon A$ has at most n elements$\}$, and $\mathcal{F}(X) = \{A \in 2^X\colon A$ is finite$\}$. For a ***locally convex*** linear space Y, we let $\mathcal{K}(Y) = \{S \in \mathcal{P}(Y)\colon S$ is ***convex***$\}$, $\overline{\mathcal{K}}(Y) = \{S \in \mathcal{K}(Y)\colon S$ is closed$\}$, and $c\overline{\mathcal{K}}(Y) = \{S \in \overline{\mathcal{K}}(Y)\colon S$ is ***compact*** or $S = Y\}$. For $U_1, \ldots, U_n \in \tau$, we let $\langle U_1, \ldots, U_n\rangle = \{A \in 2^X\colon A \subset \bigcup_i U_i$ and each $U_i \cap A \neq \emptyset\}$, $\lceil U_1, \ldots, U_n\rceil = \{A \in 2^X\colon A \subset \bigcup_i U_i\}$, $\lfloor U_1, \ldots, U_n\rfloor = \{A \in 2^X\colon$ each $U_i \cap A \neq \emptyset\}$. $\mathbb{N}$ denotes the natural numbers and I the closed unit interval.

The **finite topology** or the **Vietoris topology**, 2^τ, is the topology generated by $\mathcal{B}_f = \{\langle U_1, \ldots, U_n\rangle\colon U_i \in \tau,\ n \in \mathbb{N}\}$. The **upper semi-finite topology**, L^τ, is the topology generated by $\mathcal{B}_u = \{\lceil U_1, \ldots, U_n\rceil\colon U_i \in \tau,\ n \in \mathbb{N}\}$. The **lower semi-finite topology**, L^τ, is the topology generated by $\mathcal{B}_l = \{\lfloor U_1, \ldots, U_n\rfloor\colon U_i \in \tau,\ n \in \mathbb{N}\}$. Obviously, these topologies can be defined for $\mathcal{P}(X)$. Clearly, 2^τ is the largest topology which is contained in U^τ and L^τ.

All the preceding spaces of subsets of a topological space are generally called **hyperspace**s and [9] has a lot of information about them.

For any sets X and Y, $\mu\colon X \to Y$ is a **multivalued function** or **multifunction** (or **carrier**) provided that, for each $x \in X$, $\mu(x) \in 2^Y$. (The more general requirement that $\mu(x) \in \mathcal{P}(Y)$ is generally too weak.) For $A \subset X$ and $B \subset Y$, $\mu(A) = \bigcup\{\mu(x)\colon x \in A\}$, $\mu^{-1}(B) = \{x \in X\colon \mu(x) \cap B \neq \emptyset\}$.

Clearly, $\mu^{-1}\colon Y \to X$ (defined by $\mu^{-1}(y) = \mu^{-1}(\{y\})$) is a multifunction and any multifunction μ can be viewed as a (single-valued) function $\mu\colon X \to 2^Y$. Throughout, we identify $\mu\colon X \to Y$ with $\mu\colon X \to 2^Y$, because it is obvious how to translate from one notation to another.

Given a multifunction $\mu\colon X \to Y$, we let $gr\,\mu = \{(x, y) \in X \times Y\colon y \in \mathcal{F}(x), x \in X\}$, and $p_X\colon gr\,\mu \to X$ and $p_Y\colon gr\,\mu \to Y$ be the functions defined by $p_X(x, y) = x$ and $p_Y(x, y) = y$.

Let $X$and Y be topological spaces and $\gamma\colon X \to Y$ a multifunction (or, equivalently, $\gamma\colon X \to 2^Y$ a function).

(a) γ is a **usc-function** (an **upper semi-continuous map**) provided that $\gamma^{-1}(B)$ is closed for each closed $B \subset Y$ (equivalently, $\gamma\colon X \to 2^Y$ is U^τ-continuous),
(b) γ is a **lsc-function** (i.e., **lower semi-continuous**) provided that $\gamma^{-1}(U)$ is open for each open $U \subset Y$ (equivalently, $\gamma\colon X \to 2^Y$ is L^τ-continuous),
(c) γ is a **continuous multifunction** provided that γ is usc and lsc,
(d) γ is a **closed multifunction** (an **open multifunction**) provided that $\gamma(A)$ is closed (open) for each closed (open) $A \subset X$,
(e) γ is Y**-compact** (Y**-separable**) (X**-compact** or X**-separable**) provided that $\gamma(x)$ is compact (***separable***) for each $x \in X$ ($\gamma^{-1}(y)$ is compact or separable for each $y \in Y$),
(f) γ is Y**-perfect** (X-perfect) provided that γ is closed, Y-compact and usc (γ is closed, X-compact and usc),
(g) γ is **perfect multifunction** provided that γ is X-perfect and Y-perfect,
(h) a function $g\colon X \to Y$ is a **selection for** γ provided that g is continuous and each $g(x) \in \gamma(x)$,
(i) for a topological space X and $G \subset 2^X$, a function $f\colon G \to X$ is a **selection** provided that f is continuous and each $f(E) \in E$.

It is noteworthy, that if $f\colon X \to Y$ is a single-valued, continuous and closed (open) function then $f^{-1}\colon Y \to X$ is a usc-function (lsc-function). These are very special functions which send distinct points to disjoint sets.

Even though the concepts of Definitions (h) and (i) appear very different, they are closely related as the following result asserts: A space Y has a selection $f\colon 2^Y \to Y$ if and only if every continuous multifunction $\mu\colon X \to 2^Y$ has a selection (see [1]).

Selection theorems

Since selection theorems are closely related to the ***extension*** of continuous functions (indeed, for a closed $A \subset X$ and continuous (single-valued) function $f\colon A \to Y$, a continuous selection for a related lsc-function $\nu\colon X \to 2^Y$ leads to a continuous extension $\bar{f}\colon X \to Y$ of f (see Corollary 1.5 of [10])) or measurable functions, we will first state a few theorems of E.A. Michael [10] which demonstrate their usefulness:

(i) A T_1***-space*** is ***normal*** if and only if, for any separable ***Banach space*** Y, every lsc-function $\mu\colon X \to c\overline{\mathcal{K}}(Y)$ has a selection.
(ii) A T_1-space is ***collectionwise normal*** if and only if, for any Banach space Y, every lsc-function $\mu\colon X \to c\overline{\mathcal{K}}(Y)$ has a selection.
(iii) A T_1-space is ***paracompact*** if and only if, for any Banach space Y, every lsc-function $\mu\colon X \to \mathcal{F}(Y)$ has a selection.
(iv) A T_1-space is ***perfectly normal*** if and only if, for any separable Banach space Y, every lsc-function $\mu\colon X \to \mathcal{D}(Y)$ has a selection. ($\mathcal{D}(Y)$ is a rather complicated subspace of $\mathcal{K}(Y)$.)

Michael [11] contains a variety of highly specialized selection theorems with applications to fiber spaces and ***deformation retracts***. It seems appropriate to mention the following results from [1] and [8], respectively: If Y is a connected,

locally pathwise connected space which is not an AR ***for metric spaces***, then there exist a metric space X and one-to-finite continuous multifunction $\mu : X \to Y$ which does not have a selection. If X is ***extremally disconnected***, then every usc-function $\gamma : X \to \mathcal{C}(Y)$, Y ***Hausdorff***, has a selection.

[12] has a rather simple proof of an Abstract Selection Theorem which easily implies several continuous selection theorems of E.A. Michael and others, and also implies several measurable selection theorems. Several of K. Prikry's publications deal with measurable selections. [14] has a rather complete study of usc-functions, including a very general measurable selection theorem for these functions. We conclude the subject of selections by noting that liftings and cross-sections are special cases of selections.

Preservation of topological properties

A rather successful way of studying the preservation of topological properties by a multifunction $\mu : X \to Y$ involves the maps p_X and p_Y, because, as proved in [2], μ is usc and Y-compact iff p_Y is perfect, p_X is open if μ is lsc and p_Y is open if μ is open.

Let us mention the following two theorems from [2]: Let $\mu : X \to Y$ be a perfect multifunction, where X and Y are ***T_1-spaces*** with ***G_δ-diagonals***. Then X is ***metrizable*** (***stratifiable***) iff Y is metrizable (stratifiable). Let X and Y be regular T_1-spaces and $\mu : X \to Y$ be an onto perfect multifunction. Then X is paracompact (***locally compact***; ***countably paracompact***; ***strongly paracompact***) iff Y is.

As explained in [2], the concept of a multivalued ***quotient map*** is necessarily unusual. Let $\mu : X \to Y$ be an onto multifunction. μ is a **us-quotient** (**ls-quotient**) function provided that $U \subset Y$ is closed (open) iff $\mu^{-1}(U)$ is closed (open) in X. μ is said to be a **quotient function** if it is both a us- and ls-quotient function) but it does lead to useful results: Let $\mu : X \to Y$ be a multivalued Y-compact quotient map from the separable metrizable space X onto a ***regular first-countable*** T_1-space Y with a G_δ-diagonal. Then Y is separable metrizable.

Extensions of multivalued functions

There are interesting results on the extension of lsc-functions or usc-functions or continuous multifunctions, but it appears that much remains to be done. The following result appears in [2]: Let X be a stratifiable space, A a closed subset of X and E any topological space. If $f : A \to 2^E$ is a lsc-function (usc-function; continuous multifunction and $\dim(X - A) = 0$) then there exists a lsc-function (usc-function; continuous multifunction) $\bar{f} : X \to 2^E$ such that $\bar{f}|A = f$ (i.e., $\bar{f}$ extends f) and range $\bar{f} =$ range f. If $f : A \to \mathcal{F}(E)$ or $f : A \to \mathcal{C}(E)$ then $\bar{f} : X \to \mathcal{F}(E)$ or $\bar{f} : X \to \mathcal{C}(E)$.

The proof of the preceding theorem shows that if $f : A \to E$ is a continuous multifunction and $\dim(X - A) \neq 0$, then the lsc-extension $f_\ell : X \to E$ and the usc-extension $f_u : X \to E$ of f are different and that $f_\ell(x) \subset f_u(x)$ for each $x \in X$. This raises the obvious question: Is there a continuous extension $f_c : X \to Y$ such that $f_\ell(x) \subset f_c(x) \subset f_u(x)$ for each $x \in X$? Theorem 4.3 of [3] provides a negative answer to this question, as follows: Let Y be any compact metric, connected space which is not ***locally connected***. Then there exist compact metric spaces X and continuous multifunctions $f : A \to Y$ (A closed in X) which have no continuous extensions to all of X.

For non-metric spaces, the addition of local connectedness does not help. (See Example 3.5 of [3].) For metric spaces, the addition of local connectedness does help! For the sake of brevity, let us first recall some terminology: For a class $\mathcal{Q}$ of topological spaces, Y is said to be an AE($\mathcal{Q}$) provided that every continuous function $f : A \to 2^Y$, where A is a closed subset of some $X \in \mathcal{Q}$, has a continuous extension $\bar{f} : X \to 2^Y$. It follows that for a metrizable space Y, 2^Y is an AE (normal) iff Y is compact, connected and locally connected (see [4]) and for a metrizable space Y, 2^Y is an AE (metrizable) iff Y is ***arcwise connected*** and *locally* arcwise connected. (If Y is a complete metric space then "arcwise" is implicit.) (See [6].) The following result follows from Lemma 3.6 of [5]. For a metrizable space Y, $\mathcal{F}(Y)$ is an AE (metrizable) if and only if Y is connected and locally arcwise connected.

Fixed points

If $\mu : X \to X$ is a multifunction, $z \in X$ is a **fixed point** of μ provided that $z \in \mu(z)$. Many fixed point theorems for multifunctions are like the ***Banach Contraction Principle***; others are like Brouwer's Fixed-point Theorem. The papers [13], [7] and [15] give a good account of what has been done.

Other topologies

There are several very useful topologies for hyperspaces which naturally yield different continuity conditions for multifunctions. [9] carefully examines the relationships between the finite topology, the Hausdorff metric topology on 2^X or $\mathcal{C}(X)$ whenever X is a (bounded) metric space and the ***uniform topology*** (generated by a ***uniformity*** on X).

Recently, other topologies, such as the ***locally finite topology*** and the ***Mosco topology*** have proved useful. The work of G. Beer on this subject gives an excellent account of what has been accomplished and what remains to be done.

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c-10 Applications of the Baire Category Theorem to Real Analysis

The **Baire Category Theorem** (BCT) has many applications in various parts of mathematics. Some of the important ones are discussed in the article "Baire spaces" by J. Chaber and R. Pol in this volume. We shall be concerned with applications related to real-valued functions. The spaces under consideration will be ***metric spaces***. The **BCT** states that a ***complete metric space*** cannot be expressed as a countable union of ***nowhere dense*** sets. An equivalent formulation is that a countable intersection of ***dense*** open sets in a complete metric space is dense. The BCT is valid also for a space that is **topologically complete**, i.e., one that is ***homeomorphic*** to a complete metric space.

The BCT has numerous applications in the setting of real-valued functions of a real variable. Some uses involve showing that a condition that is satisfied pointwise, is actually satisfied globally. The ***Banach–Steinhaus Theorem***, mentioned in the article "Baire spaces", is an example. Here is another example of this type of use of the BCT. If a function $f:\mathbb{R}\to\mathbb{R}$ has the property that for each $x\in\mathbb{R}$ there exists $n=n(x)\in\mathbb{N}$ such that $f^{(n)}(x)=0$, then there exists $N\in\mathbb{N}$ such that $f^{(N)}(x)=0$ for every $x\in\mathbb{R}$, so f is a polynomial. (See [3, Section 10].)

Another important type of application of the BCT involves its use in establishing the existence of certain objects that might be difficult to visualize or construct. A classical example is the use of the BCT to show the existence of continuous **nowhere differentiable** functions (i.e., functions that at no point have a finite derivative). Such theorems can often be viewed as "structure" theorems – they provide information on the structure of ***residually*** many members of a complete metric space (or of any space in which the BCT is valid).

We shall focus our attention on such structure theorems for several familiar metric spaces that arise in the theory of real-valued functions. It has become the custom of many authors to use the terms 'most', **generic**, or 'typical' to describe that a certain property is shared by all members of a residual subset of a complete space. Thus the statement that the subset of $C[a,b]$ that consists of nowhere differentiable functions is residual in $C[a,b]$ can reduce to the statement that the typical (or generic) continuous function is nowhere differentiable or that most continuous functions are nowhere differentiable.

Nondifferentiability behavior

Among the early applications of the Baire Category Theorem to spaces of real functions were theorems obtained by members of the Polish school concerning the behavior of the typical continuous function with respect to differentiation. In this connection the space under consideration is the space $C[a,b]$ of continuous functions on an interval $[a,b]$ furnished with the **sup-norm**, $\|f\|=\max_{t\in[a,b]}|f(t)|$. In separate papers in 1931, S. Banach [2] and S. Mazurkiewicz [10] showed that the typical continuous function is nowhere differentiable. The same result holds if we allow infinite derivatives or unilateral finite derivatives. Interestingly enough, S. Saks [13] showed that the result fails if we allow infinite unilateral derivatives. These results together with variants can be summarized as follows:

THEOREM 1. *Let $A=\{f\in C[a,b]$: f satisfies conditions* (i), (ii) *and* (iii) *below$\}$*:

(i) *For every $x\in[a,b]$, f does not have a finite or infinite derivative at x.*
(ii) *For every $x\in[a,b]$, f does not have a finite unilateral derivative at x.*
(iii) *There exists a set $S\subset[a,b]$ with cardinality of the continuum in every subinterval of $[a,b]$ such that f has an infinite unilateral derivative at every $x\in S$.*

Then A is a residual subset of $C[a,b]$.

Part (iii) suggests that the typical function in $C[a,b]$ does have some differentiability structure – one sided infinite derivatives at some points. Known constructions of continuous functions that do not have finite or infinite unilateral derivatives at any point (so called **Besicovitch functions**) were extremely complicated, so a proof using the BCT seemed desirable. Because of Saks' result no such proof is available in $C[a,b]$. J. Malý [8], however found a closed subspace of $C[a,b]$ in which the class of Besicovitch functions is residual. (See [4, pp. 144 and 167] for further discussion and references.)

The previous results do not give a full picture of just how the typical $f\in C[a,b]$ fails to be differentiable. More information, involving derived numbers, was obtained by V. Jarnik in 1933 and J. Marcinkiewicz in 1935. Recall that α is a **derived number** of the function f at the point x if there exists a sequence $\{h_n\}\to 0$ such that $(f(x+h_n)-f(x))/h_n\to\alpha$ as $n\to\infty$.

Jarnik showed that for the typical $f\in C[a,b]$, every extended real number is a derived number at every point. (See [4, p. 144].) Here is Marcinkiewicz's result.

THEOREM 2 (Marcinkiewicz). *Let $\{h_n\}$ be a sequence of nonzero numbers such that $h_n\to 0$. Let S be the set of* *functions $f \in C[a,b]$ satisfying the following condition: For every measurable function g that is finite on $[a,b]$ except for a set of measure zero, there exists a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ for which*

$$\lim_{k\to\infty} \frac{f(x+h_{n_k}) - f(x)}{h_{n_k}} = g(x)$$

on $[a,b]$ except on a set of measure zero.

Then S is residual in $C[a,b]$.

Marcinkiewicz's remarkable theorem can be interpreted as saying that the typical $f \in C[a,b]$ is a "universal generalized anti-derivative". (See [4, p. 82] for a proof and further discussion.)

Intersection patterns

Theorems concerning the intersection pattern of the graphs of typical continuous functions with the family of straight lines give further information on the differentiability structure of such functions. The result that follows, obtained by A. Bruckner and K.M. Garg in 1977, shows also that while typical continuous functions behave pathologically in various senses, they are virtually alike with respect to these intersection patterns. One such function differs from another only in the exact locations of certain exceptional lines. (See [4, Chapter 13].) Specifically:

THEOREM 3 (Bruckner–Garg). *For $\alpha \in \mathbb{R}$ let L_α be the family of lines with slope α, and for $f \in C[a,b]$, let $L(f,\alpha)$ consist of those lines in L_α that intersect the graph of f. For all f in a residual subset of $C[a,b]$ the following are true: There exists a countable dense subset $S \subset \mathbb{R}$ such that if $\alpha \notin S$ then*

- (i) *Each of the extreme lines in $L(f,\alpha)$ intersects the graph of f in a single point.*
- (ii) *There is a countable set of lines in $L(f,\alpha)$ each of which intersects the graph of f in the union of a* ***Cantor set*** *with a singleton not in the Cantor set. The union of these lines is dense in the strip determined by the two extreme lines in $L(f,\alpha)$.*
- (iii) *Every other line in $L(f,\alpha)$ intersects the graph of f in a Cantor set.*

For $\alpha \in S$, the pattern above holds with one exception: There exists a single line in $L(f,\alpha)$ that intersects the graph of f in the union of a Cantor set with a two-point set not in the Cantor set.

While the functions described in the above theorem form a residual subset of $C[a,b]$, no function exhibiting the properties has been constructed explicitly.

Actually more is known. A. Bruckner and J. Hausermann showed in 1985 that the intersections of lines with the graph of a typical continuous function are very "thin" (technically, bilaterally strongly porous). This implies that the typical f is nowhere differentiable with respect to various forms of generalized differentiability. (See [4, pp. 165–166] for references, relevant definitions and further discussion.)

Derivatives

We have discussed continuous functions with no derivative at any point. When a function f has a finite derivative at every point (i.e., is differentiable), its derivative f' might exhibit unusual properties. For example, f' might be discontinuous on a set that is large in the sense of measure (though it must be small in the sense of category, that is, first category). This situation is actually typical.

Let $M\Delta'[a,b]$ be the metric space of bounded derivatives on $[a,b]$ furnished with the ***uniform metric***. Thus, $f \in M\Delta'[a,b]$ if there exists a differentiable function F such that $F' = f$ on $[a,b]$ and f is bounded. This space is complete since a uniform limit of derivatives is a derivative. For $f \in M\Delta'[a,b]$ let $C_f = \{x\colon f \text{ is continuous at } x\}$. Then, for each f in a residual subset of $M\Delta'[a,b]$, C_f has Lebesgue measure zero [5] and f maps C_f onto a set having cardinality of the continuum whose closure has Lebesgue measure zero [7]. In particular, the residual set C_f maps onto a ***nowhere dense set***.

In the late nineteenth century P. du Bois-Reymond and U. Dini had different opinions on whether or not there could exist a function F, with a bounded derivative, such that F was not monotone in any interval (i.e., there is no interval on which F is either nondecreasing or nonincreasing). Du Bois-Reymond thought no such functions could exist – Dini believed the existence of such a function highly probable. Observe that if F were such a function, then F' would have to be positive on a dense set and negative on another dense set! A number of mathematicians in the late nineteenth and early twentieth centuries provided examples (usually with errors) until A. Denjoy finally provided an exhaustive study of such functions in 1915. Later, other mathematicians provided "simpler" examples, but none of these examples was really simple. In 1976, C. Weil [16] provided a proof using the BCT. The proof was short, direct and simple. (See [4, pp. 24 and 25] for a further discussion.) Here is Weil's result.

THEOREM 4 (Weil). *Let $M\Delta_0'[a,b]$ denote the members of $M\Delta'[a,b]$ that vanish on a dense subset of $[a,b]$. Then $M\Delta_0'[a,b]$ is a complete metric space. Each member of some residual subset W of $M\Delta_0'[a,b]$ is positive on one dense subset of $[a,b]$ and negative on another dense subset of $[a,b]$.*

Thus, for $F' \in W$, F is a differentiable function that fails to be monotone on any interval.

Iteration

During the last third of the twentieth century considerable attention was paid to the study of attractive properties of continuous functions mapping an interval I into itself. For $f: I \to I$, let f^0 be the identity function, and for $n \in \mathbb{N}$, define $f^n = f \circ f^{n-1}$ and $f^{-n} = (f^n)^{-1}$, the inverse of f^n. For $x \in I$, the **ω-limit set** of f at x, denoted by $\omega(f, x)$, is the **cluster set** of the sequence $\{f^n(x)\}_1^\infty$, this is the intersection $\bigcap_n \overline{\{f^m(x)\colon m \geqslant n\}}$. The set $\omega(f, x)$ is always closed, and f maps this set onto itself. For members of large classes of smooth functions that have been studied, there is just one set that serves as the ω-limit set for all points in the interval, except those in a set of measure zero. That ω-limit set can be a Cantor set, or a finite union of disjoint intervals $J_1, \ldots, J_n$, with $f(J_i) = J_{i+1}$ for $i < n$, $f(J_n) = J_1$. It is possible that all of these intervals are single points. (See J. Milnor's article [11].)

For the typical $f \in C(I, I)$, the space of continuous functions mapping the closed interval I into itself, the attractive behavior is very different. The following was proved in [1].

THEOREM 5. *Let $C(I, I)$ be furnished with the metric of uniform convergence, and let K be a **first category** subset of I whose complement in I has measure zero. For $f \in C(I, I)$ let $K^*(f) = \bigcup_{n=-\infty}^{\infty} f^n(K)$, $A(f) = \bigcup_{x \in I} \omega(f, x)$ and $L(f) = I \setminus K^*(f)$. Then the set of functions in $C(I, I)$ that satisfy the conditions below forms a residual subset of $C(I, I)$.*

(i) *$A(f) \subset L(f)$. (Hence $A(f)$ has measure zero.)*
(ii) *For every $x \in K^*(f)$, $\omega(f, x)$ is a Cantor set, $H(x)$.*
(iii) *If x and y are distinct points of K, then $H(x) \cap H(y) = \emptyset$.*
(iv) *For every $x \in K$, f maps $H(x)$ **homeomorphically** onto itself, and for $y \in H(x)$, $\omega(f, y) = H(x)$.*
(v) *The set $\{y\colon \omega(f, y) \subset H(x)\}$ is nowhere dense in I.*
(vi) *f is one-to-one on $\bigcup_{n=1}^{\infty} f^n(K)$.*

Thus, the typical f in $C(I, I)$ is one-to-one on a set K whose complement in I has zero measure, and each point of that set has a Cantor set as its ω-limit set. All ω-limit sets for f are contained in a zero measure set in the complement of K. Each ω-limit set attracts only a zero measure set.

For functions in $C(I, I)$, all ω-limit sets are either finite unions of intervals or are nowhere dense, nonempty closed sets. Conversely, if A is either a finite union of intervals or is a nonempty, nowhere dense closed subset of I, there exists an $f \in C(I, I)$ such that $\omega(f, x) = A$.

If one requires more smoothness of f, then "most" closed nowhere dense sets are not ω-limit sets for f. More precisely, let $\mathcal{K}$ denote the nonempty closed subsets of I furnished with the ***Hausdorff metric***. Then $\mathcal{K}$ is a complete metric space. A. Bruckner and T. Steele proved [6] that the typical member E of $\mathcal{K}$ is a Cantor set with the property that if E is an ω-limit set for $f \in C(I, I)$ then f maps its set of points of differentiability onto a first category subset of E, and if f is a **Lipschitz map** on a subset A of E (that is, if there exists $M \geqslant 0$ such that $|f(x) - f(y)| \leqslant M|x - y|$ for all $x, y \in A$), then f maps A onto a nowhere dense subset of E.

Infinitely differentiable functions

Let C^∞ denote the class of functions that have derivatives of all orders on $\mathbb{R}$. If the Taylor series about a point $x = a$ for a function $f \in C^\infty$ converges to f in a neighbourhood of a, we say that f is **analytic** at a. Functions in $\mathcal{C}^\infty$ can fail to be analytic in various ways – a Taylor series might converge to the wrong function, or it might diverge at every point except at its center.

In 1940 H. Cartan provided a complicated construction of a C^∞ function whose Taylor series diverges at every $x \neq a$, no matter which point a is taken as center. In 1954 A. Morgenstern [12] provided a simple proof of the existence of such functions using the BCT. Under an appropriate metric for which convergence of f_k to f means that for all $n \in \mathbb{N}$, $f_k^{(n)}$ converges uniformly to $f^{(n)}$, C^∞ becomes a complete metric space. Each member of a residual subset of this space has the property that the Taylor series diverges everywhere except at its center, no matter which point is taken as center.

The BCT can also be used to obtain positive results in this area – if, for each point a of an interval, the Taylor series of a function about a has a positive radius of convergence, then the function is analytic except on a nowhere dense set. (See [3, Section 24] for further discussion and references.)

Fourier series

In 1876 du Bois-Reymond produced an example of a continuous 2π-periodic function whose Fourier series diverged at a point in the interval $(-\pi, \pi)$. By use of the Banach–Steinhaus theorem (which depends on the BCT), one can obtain such a function that diverges on a residual subset of $\mathbb{R}$. Such functions are, in fact, typical.

Let $C(T)$ denote the ***Banach space*** of continuous 2π-periodic functions equipped with the sup-norm. The set of functions in $C(T)$ that diverge on a residual subset of $\mathbb{R}$ is residual in $C(T)$.

While the set of divergence for such functions is large in the sense of category, it is small in the sense of measure. This was proved in 1965 by L. Carleson, who proved that for every $f \in L_2(T)$ the Fourier series of f does converge except on a set of measure zero, thereby solving the problem from 1915 known as Lusin's problem. For $f \in L_1(T)$, this result is no longer valid. The Fourier series of the typical function in $L_1(T)$ diverges except on a set of measure zero.

Representation of functions

In a famous speech delivered before the International Congress of Mathematicians in Paris in 1900, D. Hilbert identified twenty-three problems whose study became central parts of twentieth century mathematical research. (See [9] for Hilbert's speech and discussion of the problems.) In 1957 A. Kolmogorov solved Hilbert's thirteenth problem by showing that every continuous function of several variables can be expressed in terms of functions of one variable. A somewhat improved version of Kolmogorov's theorem can be stated as follows:

THEOREM 6. *There exist fixed continuous functions* $\varphi_k : [0,1] \to \mathbb{R}$, $k = 1, 2, \dots, 2n+1$, *and constants* λ_i, $i = 1, \dots, n$, *with the following properties*: *for every continuous real-valued function* f *on* $[0,1]^n$, *there exists a continuous function* $g : \mathbb{R} \to \mathbb{R}$ *such that* $f(x_1, \dots, x_n) = \sum_{k=1}^{2n+1} g(\lambda_1 \varphi_k(x_1) + \cdots + \lambda_n \varphi_k(x_n))$.

The functions φ_k exhibit special properties and were difficult to construct. Shorter proofs for the existence of the functions φ_k were advanced later. These used the BCT. For example, J. Kahane showed in 1975 that most $(2n+1)$-tuples of increasing functions will have the desired properties. (See [9, p. 420].) For further discussion of Hilbert's thirteenth problem, Kolmogorov's theorem and variants, see D. Sprecher [15] and G. Lorentz [9, pp. 419–430]. See also the article "Baire spaces".

Transitive homeomorphisms

For some of the applications of the Baire Category Theorem that have been mentioned, the BCT served to provide a simpler proof of the existence of objects whose existence was already known (e.g., continuous nowhere differentiable functions, differentiable nowhere monotone functions, Cantor sets that are not ω-limit sets for Lipschitz functions). In other applications an existence proof via the BCT preceded any constructive example (e.g., the intersection patterns of the graph of a continuous function with the family of straight lines). Another example of this type involves the existence of homeomorphisms H of the closed unit square S onto itself with the following properties: (i) H preserves Lebesgue measure: $\lambda(H(E)) = \lambda(E)$ for every measurable subset E of S; (ii) H is **transitive**, i.e., H has a point x with **dense orbit** (the sequence of iterates $\{H^n(x)\}$ is dense in S). J. Oxtoby proved the existence of such homeomorphisms using the BCT. Let M denote the set of all homeomorphism of S onto itself for which (i) is satisfied. Furnish M with the metric $\rho(H_1, H_2) = \sup_{x \in S} |H_1(x) - H_2(x)|$. Then (M, ρ) is a metric space that is ***topologically complete***, hence the BCT applies. Oxtoby showed that all members of a residual subset of M satisfy condition (ii). Thus the typical homeomorphism mapping S onto itself and preserving measure has a point with a dense orbit. For further discussion see Oxtoby [14, Chapter 18].

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c-11 Absolute Retracts

A **retraction** is a continuous map $r: X \to X$ of a topological space into itself that restricts to the identity on its image. The image of a retraction is called a **retract** of X. Retractions are the topological analog of projection operators in analysis and of idempotents in algebra. The concept of retraction is an important special case of that of extension (being an extension to its domain of the identity map of its image). The extension of maps is perhaps the most fundamental topic of topology and is discussed in depth in the article "Extensors" in this volume, which should be consulted in conjunction with this one, as should "Cell-Like Maps".

Throughout this article, $\mathcal{C}$ will denote a class of ***normal*** spaces containing each closed subset of each of its members. A topological space X is an **Absolute Retract** (AR) for $\mathcal{C}$ provided that it is in $\mathcal{C}$ and is a retract of each space Y in $\mathcal{C}$ of which it is embedded as a closed subset. An **Absolute Neighborhood Retract** (ANR) for a class $\mathcal{C}$ of spaces is a space in $\mathcal{C}$ that is a retract of some neighbourhood of itself in each space Y in $\mathcal{C}$ of which it is embedded as a closed subset, i.e., it is a **neighbourhood retract** whenever it is embedded as a closed subset of a space in $\mathcal{C}$. The acronyms AR($\mathcal{C}$) and ANR($\mathcal{C}$) are used.

Similarly, X is termed an **Absolute Extensor** (AE) for a class, $\mathcal{C}$, of spaces provided that each map $f: A \to X$, into X from a closed subset A of a space Y in $\mathcal{C}$ has an extension to Y. X is an **Absolute Neighborhood Extensor** (ANE) for $\mathcal{C}$, provided that each such f has an extension to some neighbourhood of A in Y. The terminology AE($\mathcal{C}$), ANE($\mathcal{C}$) is used. Frequently encountered classes include the normal spaces $\mathcal{N}$, the metrizable spaces $\mathcal{M}$, the compact Hausdorff spaces $\mathcal{CH}$, the compact metric spaces $\mathcal{CM}$, the finite-dimensional subclasses of these, here denoted ANR($\mathcal{C}, n$), etc, and the generalized ***homology manifolds*** (or ANR-***homology manifolds***), $\mathcal{GM}$ (see below).

Absolute Retracts were introduced in 1931 by Karol Borsuk in $\mathcal{CM}$ to axiomatize the universally important Tietze Extension Theorem, which states that (in the present terminology) the closed unit interval is in AE($\mathcal{N}$), and were immediately recognized as being of central importance, with early development by the Polish School, especially Borsuk, K. Kuratowski, and M. Wojdysławski, but with notable contributions also by P.S. Alexandroff, S. Eilenberg, R. Fox, and S. Lefschetz before or during the Second World War. In the next fifteen years, the theory was brought to maturity by C.H. Dowker, J. Dugundji, O. Hanner, S. Hu, Y. Kodama, S. Liao, and many others. Subsequently, it has become a fundamentally important part of topology, of metric spaces in particular.

There are many variants of the ANR concept. For example, equivariant ANRs, in which one restricts to the category of spaces with a particular topological group G acting by homeomorphisms with equivariant maps as morphisms. (See below.) Allied concepts include the Shape Theory version Absolute Neighborhood Shape Retracts [11], and related topics such as ***cell-like maps*** [9, 11, 17], ***soft maps*** [8], and invertible maps [8], and several ***homotopy*** notions, e.g., the ***Homotopy Extension Property***, the ***homotopy lifting property***, and ***fibration***s. These are too extensive to be discussed in this article. Many more generalizations are mentioned in "Extensors".

It is immediate from the definitions that $A(N)E(\mathcal{C}) \cap \mathcal{C} \subset A(N)R(\mathcal{C})$, since the identity map of X will extend to a map of (a neighbourhood of X in) any space Y in $\mathcal{C}$ of which X is a closed subset. $A(N)R(\mathcal{C}) \subset A(N)E(\mathcal{C})$, for $\mathcal{C} = \mathcal{CH}$, $\mathcal{CM}$, or $\mathcal{M}$, since given an extension problem $f: A \to X$ with A closed in Y, the adjunction space $Z = Y \cup_f X$ with the ***quotient topology*** (or an appropriate metric topology in the case of $\mathcal{M}$) is again in $\mathcal{C}$ with X as a closed subset, and a retraction $r: Z \to X$ defines an extension of f to Y. For this reason, the acronyms $A(N)R$ and $A(N)E$ are often used interchangeably in the literature.

Open subsets of ANR($\mathcal{M}$)s are in ANR($\mathcal{M}$) and in general, open subsets of ANE($\mathcal{C}$) spaces are in ANE($\mathcal{C}$).

For $\mathcal{C} = \mathcal{CH}$, $\mathcal{CM}$, or $\mathcal{M}$, the existence of (appropriately topologized) cones $C(X) = X \times I/(X \times \{0\})$, and reduced cones $C(X)/(\{x_0\} \times I)$ ensures that ANR($\mathcal{C}$)s are locally contractible and that contractible ANR($\mathcal{C}$)s and ANE($\mathcal{C}$)s are AR($\mathcal{C}$)s and AE($\mathcal{C}$)s, respectively.

In the metric case the Kuratowski–Wojdysławski Embedding Theorem that any metric space may be embedded as a relatively closed subset of a convex set in a ***Banach space*** is extremely useful ([3, p. 79], [14]). For example, it immediately shows that if $X \in$ ANR($\mathcal{M}$), then for each open cover α of X there is an open cover β of X such that any two maps $f, g: Y \to X$ that are β-close are α-homotopic. The maps f and g are β**-close** if for each $y \in Y$ there is a $B \in \beta$ containing both $f(y)$ and $g(y)$; f and g are α**-homotopic** provided that there is a homotopy $F: Y \times I \to X$ such that each $F(\{y\} \times I) \subset A$ for some $A \in \alpha$ depending on y.

The $A(N)R(\mathcal{M})$ spaces occupy a central role in that they are $A(N)R(\mathcal{C})$s for many $\mathcal{C}$. The following is taken from [14], which collects results of Dowker, Hanner, E. Michael, and K. Iseki. An $A(N)R(\mathcal{M})$ is an $A(N)R$ for the spaces simultaneously ***fully normal*** and ***perfectly normal***, it is an $A(N)R$ for the fully normal spaces if and only if it is ***topologically complete***, it is an $A(N)R$ for the perfectly normal spaces if and only if it is ***separable***, it is an $A(N)R$ for normal spaces if and only if separable and topologically complete, and it is an $A(N)R$ for the ***Tychonoff space***s if and only if it is (separable and locally) compact.

Examples of ANRs and ANEs

Arbitrary Cartesian products are ANE($\mathcal{C}$)s provided that all factors are ANE($\mathcal{C}$)s and only finitely many factors are not AE($\mathcal{C}$)s. Dugundji's Extension Theorem [14, 3] states that convex subsets of ***locally convex topological vector space***s are AE($\mathcal{M}$)s , and convex subsets of locally convex metrizable topological vector spaces are ARs. On the other hand the Lebesgue spaces ℓ^p and L^p, $0 < p < 1$, which are not locally convex, are AR($\mathcal{M}$)s.

In general, ***simplicial complexes*** with the metric topology (induced from a linear embedding in a ***Hilbert space*** with vertices on members of an orthonormal basis) are $A(N)R(\mathcal{M})$s and with the Whitehead ***weak topology*** (the ***direct limit*** of the finite subcomplexes under inclusion) are $A(N)E(\mathcal{M})$s [14]. Manifolds modeled on AR($\mathcal{M}$)s (e.g., n-manifolds, ***Hilbert manifold***s, and ***Hilbert cube manifold***s) are $A(N)R(\mathcal{M})$s.

Many function spaces are in ANR($\mathcal{M}$), including $(Y, A)^{(X,B)}$ if $Y, A \in$ ANR($\mathcal{M}$), X is compact metric and B is closed in X (using the ***compact-open topology*** on the function space [14, 3]). Many spaces of differentiable maps (see R. Geoghegan's article [vMR, Chapter 30] for suggestions of compact ANRs). R. Luke and W. Mason [vMR, Chapter 30] proved that the spaces of all homeomorphisms of compact surfaces are ANR($\mathcal{M}$)s, and S. Ferry and H. Toruńczyk [vMR, Chapter 30] showed the same for the homeomorphism groups of compact Hilbert cube manifolds, but this is not known for the homeomorphism group of any compact n-manifold for $n > 2$. (These are locally contractible (Cernavskiĭ, Edwards–Kirby [9]) and hence ANEs for the finite-dimensional metric spaces by the Borsuk–Kuratowski Theorem below.) However, Geoghegan and W. Haver established that the groups of all piecewise linear homeomorphisms of compact n-manifolds are ANR($\mathcal{M}$)s [vMR, Chapter 30]. (Establishing that these function spaces are ANR($\mathcal{M}$)s is often the most difficult part of showing that they are, in fact, manifolds modeled on infinite-dimensional linear spaces.)

Many spaces of subsets are also ANR($\mathcal{M}$)s. Wojdysławski showed that the space 2^X, also termed $\exp(X)$, of all nonvoid closed subsets of any ***Peano continuum*** X, equipped with the **Hausdorff metric** ($d(A, B) < \varepsilon$ if $A \in N_\varepsilon(B)$ and $B \in N_\varepsilon(A)$) is an ANR($\mathcal{M}$) [17], [vMR, Chapter 30] and conjectured that they are homeomorphic with the **Hilbert cube**, the countably infinite product of closed intervals. (This was proved by D. Curtis, R. Schori and J. West [17], [vMR, Chapter 30].)

The universal spaces μ^n of K. Menger and N_n^{2n+1} of G. Nöbeling are in AE($\mathcal{M}, n$). The theory of these spaces and of manifolds modeled on them is now a fairly mature subject in itself and was developed by many people including R. Anderson, M. Štan'ko, M. Bestvina, A. Dranishnikov and A. Chigogidze. It is treated in detail in [8].

Characterizations of ANR($\mathcal{C}$)s

(1) (Hanner [3, 14, 8]) Let $\mathcal{C}$ be a class of regular and fully normal spaces, containing every closed subset of its members, e.g., metric spaces. Then X is an ANE($\mathcal{C}$) if and only if X has an open cover of ANE($\mathcal{C}$)s.

(2) (Dugundji–Kuratowski–Wojdysławski [3, 14, 17]) A metric space X is an ANR($\mathcal{M}$) if and only if it is a retract of a closed, convex subset of some normed linear space.

(3) (Hanner [3, 14, 17, 8]) A metric space X is an ANR($\mathcal{M}$) if and only if for each open cover α of X there is an **α-domination** of X by an ANR($\mathcal{M}$). (That is, there is an ANR($\mathcal{M}$) Y and maps $u : X \to Y$ and $d : Y \to X$ and a homotopy $H : X \times I \to X$ such that for each $x \in X$, there is an $A \in \alpha$ containing $H(\{x\} \times I)$.)

(4) (G. Kozlowski [11, Chapter X]) Let $f : X \to Y$ be a proper surjective map, where $X \in$ ANR($\mathcal{M}$). If f is an hereditary shape equivalence, then $Y \in$ ANR($\mathcal{M}$). Furthermore, a ***cell-like map*** between compact metric spaces is an hereditary shape equivalence if the range is finite-dimensional. (f is a **hereditary shape equivalence** provided that for each open $U \subset Y$, $f \restriction f^{-1}(U) : f^{-1}(U) \to U$ is a ***shape equivalence***. A proper surjection is **cell-like** if each point inverse has the ***shape*** of a point.)

(5) (R. Cauty [7]) A metric space X is in ANR($\mathcal{M}$) if and only if each open subset of X is homotopy equivalent to a simplicial complex.

(6) (Borsuk–Kuratowski [3, 14], [8, Chapter 6]) A finite-dimensional metric space is an ANR($\mathcal{M}$) if and only if it is locally contractible; if X is n-dimensional this is equivalent to X being LC^{n-1}. (X is LC^k provided that for each $x \in X$ and each neighbourhood U of x there is a neighbourhood V of x such that for $0 \leqslant i \leqslant k$ each map $f : S^i \to V$ extends to a map of B^{i+1} into U.)

Many criteria for ANRs in infinite dimensions are based on giving some condition under which maps of simplicial complexes extend and then on the technique of replacing the domain by the ***nerve(s)*** of suitably chosen open cover(s) [14]. The one following is often used.

(7) (Lefschetz [3, 14]) A metric space X is in ANR($\mathcal{M}$) if and only if for each open cover α of X there is an open cover β of X such that if K is a simplicial complex with the ***weak topology*** and if L is a subcomplex of K, then any map $f : L \to X$ such that for each simplex $\sigma \subset K$, $f(\sigma \cap L) \subset B$ for some $B \in \beta$ extends to a map $F : K \to X$ such that for each simplex $\sigma \subset K$, $F(\sigma) \subset A$ for some $A \in \alpha$.

Properties of ANRs

BORSUK'S PASTING THEOREM. *If $X = X_1 \cup X_2$ is the union of two closed subsets with X_1, X_2, and $X_1 \cap X_2 \in$*

ANR($\mathcal{M}$), *then* $X \in$ ANR($\mathcal{M}$) [3, 14, 17, 8]. *More generally, let X, Y, and A be in* ANR($\mathcal{CM}$), *A closed in Y, and* $f : A \to X$ *be continuous. Then the adjunction space* $Y \cup_f X \in$ ANR($\mathcal{M}$) [3, 14]. ($Y \cup_f X = Y \cup X/\cong$, *where* $\cong$ *is the equivalence relation generated by* $a \cong f(a)$ *for each* $a \in A$.) *This has been extended to finite-dimensional metric spaces* [1]. *By the first Hanner theorem above, these extend to locally finite unions. Dranishnikov has shown that if A, X, Y, and Z are in* ANR($\mathcal{CM}, n$), *and if* $g : A \to Z$ *is a second map that is one-to-one on the singular set of f then the adjunction can be made both from A to X and from A to Z*: $(Y \cup_g Z) \cup_f X \in$ ANR($\mathcal{M}$).

Homotopy extension property

ANR($\mathcal{M}$)s have the important **Homotopy Extension Property**: If Y is metrizable, $f : Y \to X$ is continuous, $A \subset Y$ is closed, and $F : A \times I \to X$ is a continuous extension of $f \restriction A = F \restriction A \times \{0\}$, then F extends continuously to $Y \times I$ [3, 14]. Moreover, given an open cover α of X, if $F(A \times I)$ is an α-homotopy, F may be required also to be an α-homotopy. (This will hold in generality if X is an ANE($\mathcal{C}$), and $Y \times \{0\} \cup A \times I \in \mathcal{C}$.) The Homotopy Extension Property implies that a map $f : A \to X$ will extend to Y if it is homotopic to one that does [3].

Deformation retractions

A **deformation retraction** of a space X to a subspace A is a homotopy $F : X \times I \to X$ from the identity of X to a retraction of X onto A, which is said to be a **deformation retract** of X. A **strong deformation retraction** is a deformation retraction as above in which for each $t \in I$, $f_t \restriction A = F \restriction A \times I$ is the identity of A, which is termed a **strong deformation retract** of X. These two concepts are equivalent if $X \in$ ANR($\mathcal{M}$). It follows that if A is a retract of X and if the inclusion of A into X is a homotopy equivalence, then A is a strong deformation retract of X [14].

Homotopy types

Every ANR($\mathcal{M}$) X is homotopy equivalent to a ***simplicial complex*** K (with either the metric or the weak topology), for example, by Cauty's theorem above. If X is locally compact, it is homotopy equivalent to a locally finite (locally compact) simplicial complex. If X is compact, it is homotopy equivalent to a finite simplicial complex (compact polyhedron) (West [17], [vMR, Chapter 30], [8]). In all cases, the homotopy equivalence $f : K \to X$ may be taken to be an **α-equivalence**, meaning that there is a ***homotopy inverse*** $g : X \to K$ with $f \circ g$ α-homotopic to the identity of X and $g \circ f$ $f^{-1}(\alpha)$-homotopic to the identity of K, where $f^{-1}(\alpha)$ is the cover of K by the inverse images under f of the elements of α.

T. Chapman and West used Hilbert cube manifolds to extend simple-homotopy theory to compact and locally compact ANR($\mathcal{M}$)s (cf. "Cell-Like Maps" [vMR, Chapter 30] and [17]), but D. Henderson [12] showed it does not extend to Hilbert manifolds, and S. Ferry showed it does not even extend to metric compacta homotopy equivalent to simplicial complexes [vMR, Chapter 30].

Fixed points

A. Tychonoff proved that every compact convex set A in a locally convex topological vector space has the **fixed-point property**, i.e., for every continuous map $f : A \to A$ there is some point $a \in A$ for which $f(a) = a$ [14]. Hence, every power of the closed interval and all of their retracts, in particular all AR($\mathcal{CM}$)s, have the fixed-point property. In addition, the Lefschetz Fixed Point Theorem [3, 14] holds for ANR($\mathcal{CM}$)s X: a map $f : X \to X$ has a fixed point provided that $\Sigma(-1)^i \operatorname{Tr}_i(f) \neq 0$, where $\operatorname{Tr}_i(f)$ is the trace of the induced homomorphism $f_* : H_i(X; \mathbb{R}) \to H_i(X, \mathbb{R})$ on rational homology. (Since X is dominated by a finite simplicial complex [3], the singular homology modules with any coefficient group are finitely generated and almost all are zero.)

Pathologies

Even for compact metric spaces, local contractibility does not guarantee ANR($\mathcal{M}$) in the presence of infinite-dimensionality; Borsuk gave an example in the Hilbert cube of a compact set that has a base of contractible open sets but is not in ANR($\mathcal{CM}$) [3]. However, various people have proved that if a metric space X has a base with all non void intersections of finitely many base elements contractible, then X is in ANR($\mathcal{M}$), which follows from several of the characterizations above. A locally contractible metric space that is the union of a countable collection of finite-dimensional subsets is an ANR($\mathcal{M}$). (Cf. D. Addis and J. Gresham, F. Ancel, Geoghegan, Haver [vMR, Chapter 30].)

Borsuk also constructed a 2-dimensional compact AR($\mathcal{M}$) X in the 3-ball with the property that the only contractible set in X containing an open set of X is the entire space, which therefore cannot be the union of even a countably infinite collection of smaller compact AR($\mathcal{M}$)s [3].

Hanner showed that Borsuk's Pasting Theorem is false in ANR($\mathcal{CH}$) [14].

Dranishnikov [10] has found two 4-dimensional compact metric ANRs the Cartesian product of which is 7-dimensional, not 8 (see "Extensors"), but J. Bryant has recently shown that this phenomenon cannot occur in homogeneous ANR($\mathcal{M}$)s.

Cauty has given a metrizable topological vector space (necessarily not locally convex) that is not in AE($\mathcal{M}$) [7].

Relation with manifolds

The pathologies of ANR($\mathcal{M}$)s can be removed by embedding them in finite and infinite-dimensional manifolds and taking very well behaved closed neighborhoods of them that have particularly good retractions onto them.

Let $f : A \to B$ be a map. The **mapping cylinder** of f is the quotient space $M(f) = (A \times [0, 1] \cup B)/\equiv$, where $\equiv$ is the equivalence relation generated by the relation $(x, 0) \equiv f(x)$. There is a natural retraction $r : M(f) \to B$ with each $r^{-1}(b)$ being the cone on $f^{-1}(b)$. If X is a closed subset of Y, then a **mapping cylinder neighborhood** Z of X in Y is a closed neighborhood Z of X that is homeomorphic with the mapping cylinder of a map f from the boundary ∂Z of Z to X by a homeomorphism that is the identity on X and on the boundary of Z [9]. This allows the linear structure of $M(f)$ to be transported to Z.

Finite-dimensional locally compact ANR($\mathcal{M}$)s embed as closed subsets of Euclidean spaces, $\mathbb{R}^n$. The embedded copy, X', is said to be **k-LCC** (**locally k-co-connected**) in $\mathbb{R}^n$ provided that for each $x \in X'$ and each neighborhood U of x in $\mathbb{R}^n$, there is a neighborhood V of x in $\mathbb{R}^n$ contained in U such that each mapping $f: S^k \to V \setminus X'$ extends to a map $f': D^{k+1} \to U \setminus X'$.

R. Miller, R. Edwards, and F. Quinn showed that if $n > \dim X + 2$ and also $n \geqslant 5$ and if the ANR X is a closed 1-LCC subset of $\mathbb{R}^n$ then X has a mapping cylinder neighborhood $Z \subset \mathbb{R}^n$. This result is valid in suitable form for ANR($\mathcal{M}$)s in infinite-dimensional manifolds (West, Chapman, Toruńczyk) [17], [vMR, Chapter 30].

In 1963, R. Anderson solved a problem of Borsuk in the *Scottish Book* by proving that the Cartesian product with the Hilbert cube of a **triod** (three arcs with one end point of each identified together) is homeomorphic with the Hilbert cube. This was extended by West to all compact contractible polyhedra and by R. Edwards to all compact metric ARs, thus showing that the product with the Hilbert cube of any locally compact ANR($\mathcal{M}$) is a Hilbert cube manifold [17]. D. Henderson [13] showed that the product of any locally compact finite-dimensional ANR($\mathcal{M}$) with the direct limit $R_\infty = \lim \ldots R^n \to R^{n+1} \ldots$ of Euclidean spaces is homeomorphic with an open subset of R_∞ and thus to a simplicial complex with the Whitehead weak topology. Toruńczyk showed the product of any separable ANR($\mathcal{M}$) with an infinite-dimensional Hilbert space is a Hilbert manifold [vMR, Chapter 30].

The culmination of this development is the topological characterization of Hilbert and Hilbert cube manifolds by Toruńczyk [17], [vMR, Chapter 30]: An ANR($\mathcal{M}$), X, is a manifold modeled on the Hilbert cube provided that it is locally compact and for each n two maps from an n-cube into X may always be approximated by maps with disjoint images. It is a Hilbert manifold modeled on separable (infinite-dimensional) Hilbert space if X is complete, separable, and if for each n each map of the disjoint union of a countably infinite collection of n-cells may be approximated arbitrarily closely, by a closed embedding. There are analogs for Hilbert spaces of all weights. M. Bestvina characterized manifolds modeled on Menger's universal space μ^n as the ANE($\mathcal{M}, n$) spaces X for which each pair of maps of I^n into X may be approximated arbitrarily closely by disjoint embeddings [8].

For X a finite-dimensional locally compact ANR($\mathcal{M}$), the situation is much more complex. In order to be an n-manifold, it must first satisfy the local homology condition for each of its points, x, that $H_i(X, X \setminus \{x\}; \mathbb{Z}) = \mathbb{Z}$ if $i = n$ and 0 if $i \neq n$, in which case it is called a **homology n-manifold** or **generalized manifold**. Next, it must satisfy Cannon's **disjoint discs property**, which is the same as Toruńczyk's but only for $n = 2$ and two maps. These are called **DDP Homology Manifolds**. Edwards and Cannon proved that for $n \geqslant 5$, if a DDP homology n-manifold X is the image of a cell-like map with domain an n-manifold, then it is an n-manifold [9]. These properties turn out to be relatively easy to detect in many cases.

Bryant, Ferry, W. Mio and S. Weinberger [5] have shown that there exist ANR homology n-manifolds, even with the DDP, that are not the images of n-manifolds by cell-like maps. These generalized manifolds share many algebraic and geometric-topological properties of finite-dimensional manifolds. At present, these objects are the subject of intense investigation [4].

Still open is the well-known question asked by Borsuk "Is the one-point space the only homogeneous compact finite dimensional Absolute Retract?" A variant is "Is every homogeneous finite-dimensional Absolute Neighborhood Retract a homology manifold?" Another question of Borsuk that is still unsolved is "Does every n-dimensional compact Absolute Retract embed in R^{2n}?" It appears at this writing that Cauty has solved positively the conjecture of J. Schauder, that every compact convex subset C of any linear metric space, i.e., metrizable topological vector space, has the fixed point property. It is not known whether C must be an Absolute Retract, although Nguyen To Nhu has proved that the interesting examples [16] of N. Kalton, N. Peck and J. Roberts in L^p spaces, $0 < p < 1$, that have no extreme points are in fact Absolute Retracts. For many of those types of examples, Nguyen has shown that they are Hilbert cubes, but the question is in general open.

The extension of maps is so important in modern mathematics that many variations on Absolute Neighborhood Retracts and absolute neighbourhood extensors have been developed. In addition to restrictions on the spaces and maps involved, two very important ideas are the following: The first is to replace the space X with a map $p: E \to X$, and the second is to respect a group of symmetries (homeomorphisms) of the spaces. In the first case important questions ask about lifting of maps $f: Y \to X$ to ones $g': Y \to E$ such that $p \circ g' = g$ or, given g', about extending it to lift a homotopy $G: Y \times I \to X$ by one $G': Y \times I \to E$. This latter property, taken as an axiom, defines the **Hurewicz fibrations**.

The second idea is to let G be a fixed ***topological group***, i.e., where the group operations of multiplication and inversion are continuous. A **G-space** is a topological space equipped with an **action** of G, that is, a map $\alpha: G \times X \to X$ such that $\alpha(gh, x) = \alpha(g, \alpha(h, x))$ for all g and h in G and all x in X. The maps considered then should all be **equivariant**, i.e., such that, writing $g(x)$ for $\alpha(g, x)$, $f(g(x)) = g(f(x))$. Equivariant maps can increase the **isotropy subgroup**, $G_x = \{g \in G\colon g(x) = x\}$, of a point. In this category all of the questions treated above, e.g., equivariant retraction, are important, but the situation is complicated due to the structure of subgroups H of G and the stratification of X by their **fixed-point sets**. The **orbit** of $x \in X$ is $G(x) = \{g(x)\colon g \in G\}$. It is homeomorphic with G/G_x, the **orbit type** of X. (Some authors use the conjugacy class (G_x) to denote the orbit type.) There is a natural orbit map

$p : X \to X/G$, the **orbit space**, which is given the quotient topology.

An outstandingly useful result due to J. Jaworowski [15] is that if G is a compact ***Lie group***, then in the category $\mathcal{G}$ of finite-dimensional locally compact separable metric G-spaces with only finitely many orbit types and equivariant maps, $X \in \mathrm{ANR}(\mathcal{G})$ if and only if $X/G \in \mathrm{ANR}(\mathcal{M})$, that is, a G-space X is an equivariant ANR if and only if its orbit space is an inequivariant ANR. R. Lashof removed the local compactness assumption, but the finiteness of orbit types and finite-dimensionality assumptions remain, as does the compactness assumption on G [2].

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c-12 Extensors

The **extension problem** is considered one of the main problems of Topology. The problem asks whether a given continuous map $f : A \to Y$ admits a continuous extension to a space X, where $A \subset X$ is a closed subset. For reasonably nice spaces X and Y Algebraic Topology presents a solution to the extension problem in terms of a sequence of obstructions. In the General Topology extension problems are considered in families. K. Kuratowski introduced a notation $X\tau Y$ for the property that all extension problems $f : A \to Y$ on X can be resolved. In this notations the Tietze–Urysohn Extension theorem reads: A space X is ***normal*** iff $X\tau[0, 1]$ (cf. [E, 2.1.8]). Also Alexandroff's characterization of the ***covering dimension*** can be written as follows: $\dim X \leqslant n$ iff $X\tau S^n$, where S^n is an n-sphere.

Let $\mathcal{C}$ be a class of topological spaces. A space Y is called an **Absolute Extensor** (AE) for the class $\mathcal{C}$, $Y \in \mathrm{AE}(\mathcal{C})$, if the property $X\tau Y$ holds for all $X \in \mathcal{C}$. Note that the condition $X\tau Y$ is equivalent to $Y \in \mathrm{AE}(\{X\})$, where $\{X\}$ is the class which consists of one space X. A topological space Y is called an **Absolute Neighborhood Extensor** (ANE) for the class $\mathcal{C}$, $Y \in \mathrm{ANE}(\mathcal{C})$, if for every $X \in \mathcal{C}$ and every continuous map $f : A \to Y$ of a closed subset $A \subset X$ there is an open neighbourhood $W \supset A$ and a continuous extension $\bar{f} : W \to Y$ to W. It will be assumed that every space in a given class $\mathcal{C}$ is normal, otherwise every ANE($\mathcal{C}$) would consist of one point. The normality of a space can be defined in terms of ANE as follows: a space X is normal iff $S^0 \in \mathrm{ANE}(\{X\})$ [E, Chapter 2]. The property of being AE($\mathcal{C}$) is preserved under topological products. This implies that the ***Tychonoff cube*** I^τ is AE($\mathcal{C}$). It is known that a ***contractible*** ANE($\mathcal{C}$) is AE($\mathcal{C}$) [14]. Clearly, every open subset of an ANE($\mathcal{C}$)-space is ANE($\mathcal{C}$). This implies that $\mathbb{R}$ as well as $\mathbb{R}^\tau$ are AE($\mathcal{C}$). Since a retract of ANE($\mathcal{C}$) is ANE($\mathcal{C}$), the above implies that all finite polyhedra are ANE($\mathcal{C}$). One of the main properties of ANEs is the **Homotopy Extension Property**: Every continuous map $H : A \times [0, 1] \cup X \times \{0\} \to Y$, $X \times [0, 1] \in \mathcal{C}$, $Y \in \mathrm{ANE}(\mathcal{C})$, can be extended to a map $\overline{H} : X \times [0, 1] \to Y$. For a compact metric ANE space Y there is a number $\varepsilon > 0$ such that any two ε-close maps $f, g : X \to Y$ are homotopic. In the general case the number ε must be replaced by an open cover.

A topological space Y is called an **Absolute Retract** for the class $\mathcal{C}$ if for every embedding $Y \subset X$, $X, Y \in \mathcal{C}$, as a closed subset there is a **retraction**, i.e., a map $r : X \to Y$ such that the restriction of r to Y is the identity map id_Y. A topological space Y is called an **Absolute Neighborhood Retract** for the class $\mathcal{C}$ if for any closed embedding $Y \subset X$, $X \in \mathcal{C}$ there is a neighbourhood $W \subset Y$ and a retraction $r : W \to Y$. It is easy to see that A(N)E($\mathcal{C}$) $\subset$ A(N)R($\mathcal{C}$). In the class of all ***metrizable spaces*** we have the equality A(N)E = A(N)R. The proof of the opposite inclusion is an application of the Kuratowski–Wojdysławski Embedding Theorem and the Dugundji Extension Theorem [1]. The former states that for every metric space Y there is an embedding of Y into a ***Banach space*** such that the image of Y is a closed subset in its convex hull. The latter says that a convex subset of a ***locally convex topological vector space*** is an Absolute Extensor for the class of all metrizable spaces $\mathcal{M}$. The local convexity condition cannot be dropped here. R. Cauty constructed an example of a complete metric topological linear space which is not AE($\mathcal{M}$) [2].

Metrizable compact spaces are called **compacta**. In the class of compacta $\mathcal{CM}$ the same equality holds: A(N)E = A(N)R. Different examples of ANE-compacta can be constructed by means of the Borsuk Pasting Theorem [1]: The ***adjunction space*** $X \cup_{f : A \to Y} Y$, $A \subset X$, is an ANE provided all compacta X, Y and A are ANE. Finite-dimensional ANE-compacta are exactly those compacta which are ***locally connected*** in all dimensions (LC^∞). Precisely, the following conditions for an n-dimensional compactum Y are equivalent: (1) $Y \in \mathrm{ANE}$, (2) $Y \in LC^{n-1}$, (3) $Y \in LC^\infty$. Here LC^n denotes the class of spaces Z **locally connected in dimension** n, i.e., spaces Z having the property: For each point $z \in Z$ and for every neighbourhood $U \ni z$, there is a smaller neighbourhood V such that every continuous map $f : S^i \to V$ of an i-sphere, $i \leqslant n$, there is a continuous extension $\bar{f} : B^{i+1} \to U$ to the ball with the boundary $\partial B^{i+1} = S^i$. There are analogs of the above results in the category of all metrizable spaces [14].

It was already mentioned that covering dimension can be defined in terms of extensions: $\dim X \leqslant n$ iff $X\tau S^n$. Similarly, the **cohomological dimension** with respect to a coefficient group G can be defined as follows: $\dim_G X \leqslant n$ iff $X\tau K(G, n)$. Here $K(G, n)$ is an **Eilenberg–MacLane complex**, i.e., a CW-complex K with trivial ***homotopy groups*** $\pi_i(K)$ for $i \neq n$ and with $\pi_n(K) = G$. The extension property $X\tau M$ for finite dimensional compact spaces and a simple CW-complex M can be expressed in terms of cohomological dimension. A space Y is called **simple** if the natural action of the ***fundamental group*** $\pi_1(Y)$ on all homotopy groups $\pi_i(Y)$ is trivial. In particular, it implies that the fundamental group of Y is Abelian. For a simple complex M and a finite-dimensional compactum X the following conditions are equivalent: (1) $X\tau M$; (2) for every integer k, $\dim_{\pi_k(M)} X \leqslant k$; and (3) for every integer k, $\dim_{H_k(M)} X \leqslant k$ [6]. A **Moore space** $M(G, n)$ is a connected CW-complex with the following reduced homology groups: $H_i(M(G, n); \mathbf{Z}) = 0$, $i \neq n$, and $H_n(M(G, n); \mathbf{Z}) = G$. In particular this theorem implies that the property $X\tau M(G, n)$, $n > 1$, is equivalent to the inequality $\dim_G X \leqslant n$ for finite-dimensional X. This result was extended to metrizable spaces X by J. Dydak and to nilpotent complexes M by M. Cencelj and A.N. Dranishnikov. For infinite-dimensional X the implication $X\tau M(G,n) \Longrightarrow \dim_G X \leqslant n$ still holds and there are examples (Dranishnikov, Dydak–Walsh $G = \mathbf{Z}$, T. Miyata $G = \mathbf{Z_p}$) for which the reverse implication does not hold.

J. Dydak proved the following [13].

THE UNION THEOREM. *If a metric space X can be presented as $X = Y \cup Z$ with $Y\tau K$ and $Z\tau L$ for CW-complexes K and L, then $X\tau(K * L)$.*

Here $K * L$ denotes the **join of two spaces**, i.e., the space obtained from $K \times L \times I$ by identifying $\{x\} \times L \times \{0\}$ to a point for each $x \in K$, and identifying $K \times \{y\} \times \{1\}$ to a point for each $y \in L$.

THE DECOMPOSITION THEOREM [8]. *If a compact space X has the property $X\tau(K * L)$ for countable complexes K and L, then $X = Y \cup Z$ with $Y\tau K$ and $Z\tau L$ and either Y or Z can be assumed to be F_σ.*

This also holds when X is a metric separable space [11]. The proof of the Decomposition Theorem is based on the following theorem: *Let $X\tau(K * L)$ for a separable metric space X and countable complexes K and L. Then any extension problem $f : A \to K$ on X can be resolved over an open subset $U \subset X$ such that $(X \setminus U)\tau L$* [8, 11].

When $K = S^k$ and $L = S^l$, these theorems turn into the classic Urysohn–Menger Union Theorem, Urysohn–Tumarkin–Hurewicz Decomposition Theorem and Eilenberg–Borsuk Theorem, respectively.

Among other results on the extension property τ analogous to classical theorems in dimension theory are the following.

THE COMPLETION THEOREM (W. Olszewski). *If $X\tau L$ for a separable metric space X and a countable complex L, then there exists a completion $\overline{X}$ with $\overline{X}\tau L$.*

Various proofs of Olszewski's theorem can be found in [17, 4, 11, 12].

THE UNIVERSAL SPACE THEOREM (Olszewski). *There is a* ***universal space*** *in the class $\mathcal{SM}\tau L$ of separable metric spaces X with the property $X\tau L$ for a countable complex L.*

Moreover, there is a universal space in the class $\mathcal{SM}\tau L$ which is $\mathrm{AE}(\mathcal{SM}\tau L)$ (A. Chigogidze and M. Zarichnyi). For general metric spaces universal spaces were constructed first by M. Levin and by Chigogidze and V. Valov. In the class of compacta $\mathcal{CM}\tau L$ with the property $X\tau L$ there is a universal compactum provided L is ***homotopy dominated*** by a compact polyhedron [13]. When L is a compact polyhedron itself, there is a universal space X in the class $\mathcal{CM}\tau L$ such that $X \in \mathrm{AE}(\mathcal{C}\tau L)$ [4, 12].

THE HUREWICZ MAPPING THEOREMS. (1) *Let $f : X \to Y$ be a map of a finite-dimensional compactum and let $Y\tau K$ and $f^{-1}(y)\tau L$ for all $y \in Y$. Then $X\tau(K \wedge L)$* [15].

The case when f is the projection onto a factor is considered in [11].

(2) *Let $f : X \to Y$ be a map between compacta such that $Y\tau L$ and $|f^{-1}(y)| \leqslant k+1$ for all $y \in Y$. Then $X\tau\Omega^k L$ (Dranishnikov and V.V. Uspenskij).*

Here $\Omega^k L$ denotes the kth iterated ***loop space*** *on L, and $|f^{-1}(y)|$ denotes the cardinality of $f^{-1}(y)$.*

The above results form the basis for Extensional Dimension Theory. Here we define an **extensional dimension** for compact spaces (see [10] for generalizations). Let the relation $K \leqslant L$ on CW-complexes mean that for each compact space X the condition $X\tau K$ implies $X\tau L$. This relation defines an equivalence relation on CW-complexes. The equivalence class $[K]$ of a complex K is called its **extension type**. Every CW-complex has the extension type of a (infinite) wedge of countable complexes [9]. The set of all extension types $\mathcal{E}$ has a partial order inherited from the relation $\leqslant$. The extension dimension of a compact space X is defined as $e - \dim X = \min\{[K] \in \mathcal{E} \mid X\tau K\}$. The extension dimension is well-defined for every compact space, and for each $\alpha \in \mathcal{E}$ there is a compact space X with $e - \dim X = \alpha$ [7]. We note that the inequality $e - \dim X \leqslant [K]$ means exactly the extension property $X\tau K$ ($\Longleftrightarrow K \in \mathrm{AE}(\{X\})$). The set $\mathcal{E}$ admits the following operations: the ***wedge*** $\bigvee[K_s] = [\bigvee K_s]$, the product $[K] \times [L] = [K \times L]$, and the join $[K] * [L] = [K * L]$ [10]. Here the products $\times, *$ for uncountable CW-complexes are considered as the ***direct limits*** of corresponding products of all finite subcomplexes. The Mardešič Factorization theorem holds for the extensional dimension: Every map $f : X \to Z$ of a compact Hausdorff space with $e - \dim X \leqslant [K]$ to a metrizable space Z can be factored through a metrizable compact space Y with $e - \dim Y \leqslant [K]$. ([7] for countable complexes K and [16] in full generality, see also [9].)

The class of Absolute Extensors for compact Hausdorff spaces ($\mathcal{CH}$) with $\dim X \leqslant n$ is denoted as $\mathrm{AE}(\dim \leqslant n)$ or just as $\mathrm{AE}(n)$. According to Kuratowski's theorem for a compactum Y the property $Y \in \mathrm{AE}(n)$ is equivalent to $LC^{n-1} \cap C^{n-1}$, the local and global connectedness in dimensions $< n$. Thus the universal n-dimensional ***Menger space*** μ^n is an $\mathrm{AE}(n)$. For a compactum Y the following conditions are equivalent:

(1) $Y \in \mathrm{AE}(n)$;
(2) Y is an n-invertible image of μ^n;
(3) Y is an $(n-1)$-invertible image of μ^n;
(4) Y is an $(n-1)$-soft image of μ^n;
(5) Y is an $(n-1)$-invertible image of the Hilbert cube [5].

A map $f : X \to Y$ is called an ***n*-invertible map** if every map $g : Z \to Y$ with $\dim Z \leqslant n$ has a lift $\bar{g} : Z \to X$, $f \circ \bar{g} = g$. A map f is called an ***n*-soft map** if every partial lift $g' : A \to X$, for a closed subset $A \subset Z$, $\dim Z \leqslant n$, can be extended to a lift $\bar{g} : Z \to X$. All this terminology has been transferred to the extension dimension theory. Thus AE($[L]$) denotes the class of absolute extensors for compact Hausdorff spaces X with $e - \dim X \leqslant [L]$.

Every n-dimensional AE(n) space, $n > 0$, is metrizable [5]. This generalizes a theorem by E.V. Shchepin asserting that a finite-dimensional AE space is metrizable. Chigogidze and Zarichnyi generalized this result to the extensional dimension: Let L be a noncontractible connected CW-complex, Y be a Hausdorff compact space, and assume that Y is an ANE($[L]$)-space. If $e - \dim(Y) \leqslant [L]$, then Y is metrizable. Nonmetrizable $(n+1)$-dimensional AE(n) spaces do exist [5]. Also there are nonmetrizable 0-dimensional AE(0), for example, $D^{\omega} \in$ AE(0), where D is the two point set and ω is any ***cardinal number***. All compact metric spaces are AE(0).

The class of AE(0)-spaces coincides with the class of **Dugundji compact** spaces (R. Haydon). A compact space X is called Dugundji compact if for every embedding $X \subset Y$ to a compact space there is a linear operator $u : C(X) \to C(Y)$ on the spaces of continuous functions such that $u(f)|_X = f$, $\|u\| = 1$, $u(1) = 1$. Such operators are called **regular extension operators**. A regular extension operator defines a map $r : Y \to P(X)$ to the space of ***probability measures*** with the $*$-weak topology such that $r(x) = \delta_x$ is the Dirac measure for all $x \in X$. Note that the restriction $r|_X$ is an embedding. A map r with the above properties is called a **P-valued retraction**. Thus, AE(0) spaces are exactly absolute P-valued retracts. For a ***covariant functor*** $F : \mathcal{CH} \to \mathcal{CH}$ which contains the identity functor i as a subfunctor one can define an **F-valued retraction** of $Y \supset X$ onto X as a map $r : Y \to F(X)$ such that $r|_X = i_X$, where $i_X : X \to F(X)$ is the natural embedding. A space X is called an **absolute F-valued retract** if for every embedding $X \subset Y$ there is an F-valued retraction. The class of AE(1) spaces coincides with the class of absolute exp-valued retracts (V.V. Fedorchuk and G.M. Nepomnjashchii). Here exp denotes the ***hyperspace*** functor, i.e., $\exp(X)$ is the space of all closed subsets of X with the Vietoris topology. The base of the **Vietoris topology** consists of sets $\langle U_1, \dots, U_n \rangle = \{F \in \exp(X) \mid F \subset \bigcup_{i=1}^{n} U_i,\ F \cap U_i \neq \emptyset \text{ for all } i\}$. A compact ***topological group*** is a torus if and only if it is AE(1) (Chigogidze).

Functors exp and P preserve the property to be an Absolute Extensor in the case of metrizable spaces. For every uncountable cardinal τ the hyperspace $\exp(I^{\tau})$ is not an Absolute Extensor. The probability measure space $P(I^{\tau})$ is an AE for $\tau = \omega_1$ and it is not, if $\tau > \omega_1$ (L.B. Shapiro). If a normal functor preserves Absolute Extensors, then it is the functor $(\ )^{\tau}$ of taking τ-power for some cardinal τ (Shchepin–Zarichnyi). A covariant functor $F : \mathcal{CH} \to \mathcal{CH}$ is called a **normal functor** if it commutes with inverse limits, preserves monomorphisms and epimorphisms, preserves preimages i.e., $F(g^{-1}(A)) = (F(g))^{-1}(F(A))$ for every map $f : X \to Y$ and every closed subset $A \subset Y$, preserves singletons and empty set. The functors exp and P are normal. The superextension functor λ is an example of a functor which takes all AE to AE (A.V. Ivanov). This functor is not normal, since it does not preserve preimages. The **superextension** $\lambda(X)$ of a space X was defined by J. de Groot as the space of all maximal with respect to inclusion linked systems of closed subsets of X with the topology induced from $\exp(\exp(X))$. A system $\{F_\alpha\}_{\alpha \in A}$ is called **linked** if $F_\alpha \cap F_{\alpha'} \neq \emptyset$ for all $\alpha, \alpha' \in A$.

Let $F : \mathcal{CH} \to \mathcal{CH}$ be a covariant functor and let $\mu \in F(X)$. The **support** $\operatorname{supp}(\mu)$ of x is defined as the intersection $\bigcap Y$ for all closed subsets $Y \subset X$ for which $\mu \in F(i_Y)(F(Y))$, where $i_Y : Y \to X$ is the inclusion. The **degree of a covariant functor** $F : \mathcal{CH} \to \mathcal{CH}$ is the minimum number m such that for every $X \in \mathcal{CH}$ and every $\mu \in F(X)$ the cardinality of the set $\operatorname{supp}(\mu)$ does not exceed m. V.N. Basmanov found general conditions when a covariant functor of finite degree preserves A(N)E-compacta. In particular, the **symmetric power** SP^n takes A(N)E to A(N)E (J. Jaworowski). The nth symmetric power $SP^n Y$ of a space Y is the ***orbit space*** of the action on Y^n of the symmetric group S_n by permutations of coordinates. Other examples of functors of finite degrees can be obtained by taking subfunctors F_n of a functor F, where $F_n(X) = \{\mu \in F(X) \mid |\operatorname{supp}(\mu)| \leqslant n\}$. It is known that $\exp_3(S^1) = S^3$ (R. Bott), $\lambda_3(S^1) = S^3$ (Zarichnyi) and $P_2(S^1) = S^3$ (obvious). This implies contractibility of the naturally embedded circle S^1 in $\exp_3(S^1)$, $\lambda_3(S^1)$ and $P_2(S^1)$. Then it follows that every m-sphere S^m, $m \geqslant 1$, is an absolute $\exp_n$-valued retract and an absolute λ_n-valued retract for $n \geqslant 3$. Also, S^m ($m \geqslant 1$) is an absolute P_n-valued retract for $n \geqslant 2$. None of the spheres is an absolute SP^n-valued retract. An example of a nontrivial SP^2-valued Absolute Retract is any noncontractible acyclic polyhedron. Note that $SP^2 = \exp_2$. It is proved in [6] that the property $X \tau L$ implies $X \tau SP^n L$ for $n = 1, 2, \dots, \infty$. The importance of this result follows from the homotopy equivalence of $SP^{\infty} L$ to the weak product $\prod_i K(H_i(L), i)$. The implication $X \tau L \implies X \tau F(L)$ holds for all functors satisfying Basmanov's conditions [6], in particular for $\exp_n$, P_n, λ_n.

There are many different generalizations of the A(N)E-theory of compact spaces. Chigogidze considered ***C-embeddings*** to define the notions of A(N)E for Tychonoff spaces [3]. He developed a rich theory containing the A(N)E-theory of complete metrizable spaces. The concept of an absolute extensor has been intensively studied in shape theory (K. Borsuk, S. Mardešič, D. Edwards, R. Geoghegan, J. Dydak and others). There is an equivariant version of the theory of A(N)E (H. Abels, S. Antonian, J. de Vries, J. West and others). Extension of partition of unity problems give new insight into some concepts of General Topology (Dydak). Extension problems are of great importance in the Lipschitz category and in the coarse geometry (M. Gromov). We present here the following generalization. Let $\mathcal{A} \subset \mathcal{B}$ be a pair of classes of topological spaces. We say that Y is an **AE($\mathcal{A}, \mathcal{B}$)-space** if for each $X \in \mathcal{B}$ and every closed subset $A \subset X$, $A \in \mathcal{A}$, every continuous map $f : A \to Y$ has a continuous extension. It was shown in [5] that in the case of compacta AE($\dim \leqslant n, \dim \leqslant m$) = AE($n+1$) for any n and $m = n+1, n+2, \dots, \infty$. The equality is violated in the nonmetrizable case with $n = 0$ and $m = \infty$ [5]. J. van Mill showed that AE($\mathcal{CM}, \mathcal{M}$) $\neq$ AE($\mathcal{M}$) [vMR, Chapter 30].

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c-13 Generalized Continuities

There are many different properties of functions that can be considered as generalized continuity notions. In this article we will describe mainly those that arose from the study of real functions, concentrating first on the classes of functions known under the common name of Darboux-like functions. (See, e.g., survey articles [13, 14, 6, 3].)

For topological spaces X and Y, a function $f : X \to Y$ is a **Darboux function** (or has the **Darboux property**), $f \in \mathrm{D}(X, Y)$, provided the image $f[C]$ of C under f is a ***connected*** subset of Y for every connected subset C of X. In particular, $f : \mathbb{R} \to \mathbb{R}$ is Darboux provided f maps intervals onto intervals, that is, when it has the **intermediate value property**. The name comes after G. Darboux who showed in 1875 that every derivative (of a function from $\mathbb{R}$ to $\mathbb{R}$) has the intermediate value property, while there are derivatives discontinuous on a ***dense set***. (Some 19th century mathematicians thought that the intermediate value property could be taken as the definition of continuity. Some calculus teachers still think so.) One of the easiest examples of a discontinuous Darboux function is $f_0 : \mathbb{R} \to \mathbb{R}$ given by $f_0(x) = \sin(1/x)$ for $x \neq 0$ and $f_0(0) = 0$.

A function $f : X \to Y$ is **connectivity**, $f \in \mathrm{Conn}(X, Y)$, if the graph of the restriction $f \restriction Z$ of f to Z is connected in $X \times Y$ for every connected subset Z of X. It is easy to see that $f : \mathbb{R} \to \mathbb{R}$ is connectivity if and only if its graph is a connected subset of $\mathbb{R}^2$. However, there are functions $F : \mathbb{R}^2 \to \mathbb{R}$ with a connected graph which are not connectivity functions. For example, this is the case if $F(x, y) = \sin(1/x)$ for $x \neq 0$, and $F(0, y) = h(y)$, where $h : \mathbb{R} \to [-1, 1]$ is any function with a disconnected graph.

A function $f : X \to Y$ is **extendable**, $f \in \mathrm{Ext}(X, Y)$, provided there exists a connectivity function $F : X \times [0, 1] \to Y$ such that $f(x) = F(x, 0)$ for every $x \in X$. It is easy to see that

$$\mathrm{C}(X, Y) \subset \mathrm{Ext}(X, Y) \subset \mathrm{Conn}(X, Y) \subset \mathrm{D}(X, Y)$$

for arbitrary topological spaces, where $\mathrm{C}(X, Y)$ stands for the class of all ***continuous functions*** from X into Y.

A function $f : X \to Y$ is **almost continuous** (in the sense of Stallings), $f \in \mathrm{AC}(X, Y)$, provided each open subset of $X \times Y$ containing the graph of f also contains the graph of a continuous function from X to Y. This property was defined as a generalization of functions having the ***fixed-point property***. It is easy to see that if every function in $\mathrm{C}(X, X)$ has the fixed-point property, then so does every $f \in \mathrm{AC}(X, X)$.

A function $f : X \to Y$ is **peripherally continuous**, $f \in \mathrm{PC}(X, Y)$, if for every $x \in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset W of U such that $x \in W$ and $f[\mathrm{bd}(W)] \subset V$, where $\mathrm{bd}(W)$ is the ***boundary*** of W. For the functions $f : \mathbb{R} \to \mathbb{R}$ this means that f has the **Young property**, that is, for every $x \in \mathbb{R}$ there exist sequences $\{x_n\}_n$ and $\{y_n\}_n$ such that $x_n \nearrow x$, $y_n \searrow x$, and both $f(x_n)$ and $f(y_n)$ converge to $f(x)$. In 1907 J. Young showed that for the ***Baire class*** 1 functions, the Darboux property and the Young property are equivalent.

We will discuss the above mentioned classes only when $Y = \mathbb{R}$. If $X = \mathbb{R}^n$ and $n > 1$ the relations between these classes are given by the following chart, where arrows $\longrightarrow$ denote strict inclusions. (See [6].)

$$\mathrm{C} \longrightarrow \mathrm{Ext} = \mathrm{Conn} = \mathrm{PC} \longrightarrow \mathrm{AC} \cap \mathrm{D} \begin{array}{l} \nearrow \mathrm{AC} \\ \searrow \mathrm{D} \end{array}$$

Chart 1. Darboux-like functions from $\mathbb{R}^n$, $n > 1$, into $\mathbb{R}$.

The inclusion $\mathrm{Conn} \subset \mathrm{Ext}$ was proved by K. Ciesielski, T. Natkaniec, and J. Wojciechowski [8]. The containment $\mathrm{Conn} \subset \mathrm{PC}$ was proved by O.H. Hamilton and J. Stallings, and the inclusion $\mathrm{PC} \subset \mathrm{Conn}$ by M.R. Hagan. The relation $\mathrm{Conn} \subset \mathrm{AC}$ was proved by J. Stallings. It is important to notice that Chart 1 remains unchanged if we consider only Baire class 1 functions [6].

Classes presented in Chart 1 were also studied within the class $\mathrm{Add}(\mathbb{R}^n)$ of **additive functions**, that is, functions $f : \mathbb{R}^n \to \mathbb{R}$ for which $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$. In this case Chart 1 transforms to

$$\mathrm{C} = \mathrm{Ext} = \mathrm{Conn} = \mathrm{PC} = \mathrm{AC} \cap \mathrm{D} \begin{array}{l} \nearrow \mathrm{AC} \\ \searrow \mathrm{D} \end{array}$$

Chart 2. Additive Darboux-like functions from $\mathbb{R}^n$, $n > 1$, into $\mathbb{R}$.

The inclusion $\mathrm{AC} \cap \mathrm{D} \subset \mathrm{C}$ was proved by K. Ciesielski and J. Jastrzębski [6].

The Darboux-like functions were most intensively studied when $X = Y = \mathbb{R}$. In this setting, more classes are considered Darboux-like. Thus, a function $f : \mathbb{R} \to \mathbb{R}$ has: the **Cantor intermediate value property** if for every $x, y \in \mathbb{R}$ and for each ***perfect set*** K between $f(x)$ and $f(y)$ there is a perfect set C between x and y such that $f[C] \subset K$; the **strong Cantor intermediate value property** if for every $x, y \in \mathbb{R}$ and for each perfect set K between $f(x)$ and $f(y)$ there is a perfect set C between x and y such that $f[C] \subset K$ and $f \restriction C$ is continuous; the **weak Cantor intermediate value property** if for every $x, y \in \mathbb{R}$ with $f(x) < f(y)$ there exists a perfect set C between x and y such that

$f[C] \subset (f(x), f(y))$; the **perfect road** if for every $x \in \mathbb{R}$ there exists a perfect set $P \subset \mathbb{R}$ having x as a bilateral (i.e., two sided) limit point for which $f \restriction P$ is continuous at x. The classes of these functions are denoted by CIVP, SCIVP, WCIVP, and PR, respectively. The relations between them are as follows [13, 6].

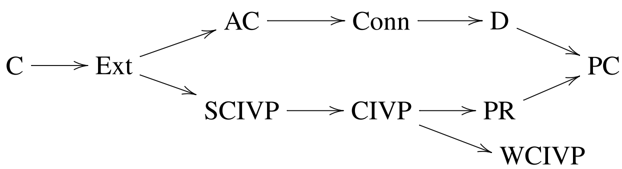

Chart 3. Darboux-like functions from $\mathbb{R}$ into $\mathbb{R}$.

The inclusions Ext ⊂ AC ⊂ Conn were proved by J. Stallings while the containment Ext ⊂ SCIVP was proved by H. Rosen, R.G. Gibson, and F. Roush.

The main interest in Darboux-like functions comes from the fact that the class Δ' of the derivatives from $\mathbb{R}$ into $\mathbb{R}$ is contained in all these classes, that is, $\mathrm{C} \subsetneq \Delta' \subsetneq \mathrm{Ext}$. This follows from the fact that every derivative is Darboux Baire class 1 (see, e.g., [2]) while within the Baire class 1 Chart 3 reduces to

$$\begin{aligned} \mathrm{C} \to \mathrm{Ext} &= \mathrm{AC} = \mathrm{Conn} = \mathrm{D} = \mathrm{PC} \\ &= \mathrm{SCIVP} = \mathrm{CIVP} = \mathrm{PR} \to \mathrm{WCIVP}. \end{aligned}$$

The proof that every peripherally continuous Baire class 1 function $f : \mathbb{R} \to \mathbb{R}$ is extendable is due to J. Brown, P. Humke and M. Laczkovich [1]. In fact, most of the properties used to define Darboux-like functions were introduced as characterizations of Darboux functions within the Baire class 1, in a form "a Baire class 1 function f is *Darboux* if and only if f satisfies the *given property*". But these properties make sense without the Baire class 1 restriction, so it was natural to study these various conditions on their own, and to find the interrelations. A number of mathematicians did just that in the latter part of the 20th century.

It is interesting that within the Baire class 2 Chart 3 has yet another, quite different form [6]:

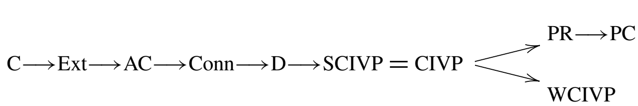

Chart 4. Darboux-like Baire class 2 functions from $\mathbb{R}$ into $\mathbb{R}$.

The most involved work in arguing for this chart is the nonreversability of the inclusions. (See [6, Theorem 1.2].) Chart 4 remains unchanged if we restrict Darboux-like functions to **Borel functions** (i.e., functions for which preimages of ***Borel sets*** are again Borel) in place of Baire class 2. Within the class of additive functions Chart 3 remains almost unchanged: the only difference is that in this case we have PR = WCIVP and that the example of additive function $f : \mathbb{R} \to \mathbb{R}$ from Conn \ AC (which is also CIVP) is known only under an extra set theoretical assumption that the union of less than continuum many meager subsets of $\mathbb{R}$ is meager in $\mathbb{R}$. (A subset of a topological space X is a **meager set**, or of the first category, if it is a countable union of ***nowhere dense*** subsets of X.)

The Darboux-like classes of functions are not closed under arithmetic operations. (See, e.g., surveys [13, 3].) For example, if f_0 is the $\sin(1/x)$ function defined above and $f_1 = f_0 + \chi_{\{0\}}$, where χ_A is the characteristic function of A, then both f_0 and f_1 are Darboux Baire class 1, so they are also extendable. However, $f_1 - f_0 = \chi_{\{0\}}$ is clearly not even in PC. In fact, in 1927 A. Lindenbaum noticed that every function $f : \mathbb{R} \to \mathbb{R}$ can be written as a sum of two Darboux functions, while H. Fast in 1959 proved that for every family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of cardinality continuum there is just one Darboux function $g \in \mathbb{R}^{\mathbb{R}}$ such that $g + F \stackrel{\mathrm{def}}{=} \{g + f : f \in \mathcal{F}\}$ is a subset of D. The result of H. Fast is a generalization of that of A. Lindenbaum, since it is easy to see that $\mathbb{R}^{\mathbb{R}} = \mathcal{F} - \mathcal{F}$ if and only if for every $f, f' \in \mathbb{R}^{\mathbb{R}}$ there exists a $g \in \mathbb{R}^{\mathbb{R}}$ such that $g + f, g + f' \in \mathcal{F}$. (See [3, Proposition 4.9].) This led T. Natkaniec [15] to study the following cardinal operator defined for every $\mathcal{F} \subset \mathbb{R}^X$, where $|X|$ stands for the cardinality of X:

$$\begin{aligned} \mathrm{A}(\mathcal{F}) = \min\bigl\{|H| : H \subset \mathbb{R}^X \ \&\ \neg\exists g \in \mathbb{R}^X \ g + H \subset \mathcal{F}\bigr\} \\ \cup \bigl\{\bigl|\mathbb{R}^X\bigr|^+\bigr\}. \end{aligned}$$

The values of the operator A for Darboux-like classes of functions from $\mathbb{R}$ to $\mathbb{R}$ are as follows (see, e.g., [3, Theorems 4.7 and 4.10]):

$$\mathfrak{c}^+ = \mathrm{A}(\mathrm{PR}) = \mathrm{A}(\mathrm{Ext}) \leqslant \mathrm{A}(\mathrm{AC}) = \mathrm{A}(\mathrm{D}) \leqslant \mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}},$$

where the value of A(D) between $\mathfrak{c}^+$ and $2^{\mathfrak{c}}$ can vary in different models of ZFC. Moreover, the monotonicity of the operator A implies that A(Ext) = A(SCIVP) = A(CIVP) = A(PR) = $\mathfrak{c}^+$ and A(AC) = A(Conn) = A(D).

The above discussion shows that unlike the derivatives, classes of Darboux-like functions are not closed under addition. It is also not difficult to see that none of these classes (including Δ', see [2, p. 14]) is closed under multiplication. Closure under composition gives a completely different picture. First of all, the derivatives are not closed under composition: by a theorem of I. Maximoff (see, e.g., [2, p. 26]), for every Darboux Baire class 1 function $g : \mathbb{R} \to \mathbb{R}$ (which does not need to be a derivative) there exists a homeomorphism h of $\mathbb{R}$ such that $f = g \circ h$ is a derivative; so, the composition $g = f \circ h^{-1}$ does not need to be a derivative. It is obvious from the definition that the class D of Darboux functions is closed under composition, and clearly so is C. The other classes from Chart 3, except for Ext, are not closed under composition. The problem of closure of Ext under composition remains open [14, Q. 9.1]. In fact, it is even not known whether the composition of two derivatives must be in Conn.

A partial positive result in this direction was recently obtained by M. Csörnyei, T.C. O'Neil, D. Preiss [10] and, independently, by M. Elekes, T. Keleti, V. Prokaj [11], who proved that the composition of two derivatives from [0, 1] into [0, 1] must have a fixed point. (So, we cannot exclude the possibility that $\Delta' \circ \Delta' \subset \mathrm{AC}$.)

The main reason for the studies of classes of functions related to derivatives comes from the fact that the class Δ' of all derivatives does not have any known nice characterization. (See [2].) One of the recent attempts of finding a characterization was to **topologize** it, that is to find two ***topologies*** τ_0 and τ_1 on $\mathbb{R}$ for which Δ' is equal to the class $\mathrm{C}(\tau_0, \tau_1)$ of all continuous functions from $\langle\mathbb{R}, \tau_0\rangle$ into $\langle\mathbb{R}, \tau_1\rangle$. Unfortunately, Δ' cannot be topologized, as shown by K. Ciesielski and, independently, by M. Tartaglia. (See [3, Corollary 5.5].) However K. Ciesielski [4] proved that it can be characterized by preimages of sets in the sense that there exist families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $\mathbb{R}$ with the property that $\Delta' = \{f \in \mathbb{R}^{\mathbb{R}}\colon f^{-1}(B) \in \mathcal{A} \text{ for every } B \in \mathcal{B}\}$. It is interesting, that if the ***Generalized Continuum Hypothesis*** holds then many classes of functions can be topologized [3, Section 5]. In particular, this is the case for any class $\mathcal{F}$ of functions containing all constant functions such that $\mathcal{F}$ is contained either in the class of analytic functions from $\mathbb{R}$ to $\mathbb{R}$ or in the class of harmonic functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.

The class Δ' is also closely related to the class Appr of **approximately continuous functions**, which was introduced by A. Denjoy in 1915. (See, e.g., [2, 7].) Recall that $f\colon\mathbb{R} \to \mathbb{R}$ is in Appr provided for every $x_0 \in \mathbb{R}$ the **approximate limit** $\operatorname{applim}_{x\to x_0} f(x)$ equals to $f(x_0)$, where $\operatorname{applim}_{x\to x_0} f(x) = L$ if there exists a set $S \subset \mathbb{R}$ such that x_0 is a (Lebesgue) **density point** of S (that is,

$$\lim_{h\to 0^+} \frac{\lambda(S \cap [x_0 - h, x_0 + h])}{2h} = 1,$$

with $\lambda(A)$ standing for the inner Lebesgue measure of A) and $\lim_{x\to x_0,\, x\in S} f(x) = L$. The interest in Appr comes from the fact that every bounded approximately continuous functions is a derivative. Also, each function in Appr is Darboux Baire class 1, so it belongs to every class of Darboux-like functions. It was not until 1952 that O. Haupt and C. Pauc defined the **density topology** $\tau_{\mathcal{N}}$ on $\mathbb{R}$ (which is the family of all $G \subset \mathbb{R}$ such that every $x \in G$ is a density point of G) and showed that Appr is equal to the class $\mathrm{C}(\tau_{\mathcal{N}}, \tau_O)$ of all functions continuous with respect to the density topology $\tau_{\mathcal{N}}$ on the domain and the ordinary topology τ_O on the range. Their paper seemed to have had almost no impact and the same results were rediscovered in 1961 by C. Goffman and D. Waterman. (See [7, Section 1.5].) This led to deep studies of the density topology, as well as to its category analog $\tau_{\mathcal{I}}$, known under the name of **$\mathcal{I}$-density topology**. Extensive research have been also conducted on the classes $\mathrm{C}(\tau_{\mathcal{I}}, \tau_O)$ of **$\mathcal{I}$-approximately continuous functions**, $\mathrm{C}(\tau_{\mathcal{N}}, \tau_{\mathcal{N}})$ of **density continuous functions**, and $\mathrm{C}(\tau_{\mathcal{I}}, \tau_{\mathcal{I}})$ of **$\mathcal{I}$-density continuous functions**. (See [7].) Classes $\mathrm{C}(\tau_{\mathcal{N}}, \tau_O)$ and $\mathrm{C}(\tau_{\mathcal{I}}, \tau_O)$ are closed under addition, while $\mathrm{C}(\tau_{\mathcal{N}}, \tau_{\mathcal{N}})$ and $\mathrm{C}(\tau_{\mathcal{I}}, \tau_{\mathcal{I}})$ are not.

A function $f\colon\mathbb{R} \to \mathbb{R}$ is **symmetrically continuous**, $f \in \mathrm{SC}$, provided $\lim_{h\to 0}[f(x + h) - f(x - h)] = 0$ for every $x \in \mathbb{R}$; it is **approximately symmetrically continuous**, $f \in \mathrm{ApprSC}$, if $\operatorname{applim}_{h\to 0}[f(x + h) - f(x - h)] = 0$ for each $x \in \mathbb{R}$. The theory of symmetrically continuous functions stems from the theory of trigonometric series and dates back to the beginning of the 20th century. Research in this area has been very active in the last several years (see [16]) after C. Freiling and D. Rinne in 1988 solved a long standing problem proving that every measurable function $f\colon\mathbb{R} \to \mathbb{R}$ having **approximate symmetric derivative**,

$$D^s_{ap} f(x) \stackrel{\text{def}}{=} \operatorname{applim}_{h\to 0} \frac{f(x + h) - f(x - h)}{2h},$$

equal 0 for all $x \in \mathbb{R}$ must be constant almost everywhere. C. Freiling [12] also proved that it is consistent with ZFC that in the above theorem the assumption of a measurability of f can be dropped. However, this cannot be proved in ZFC, as under the ***Continuum Hypothesis***, CH, W. Sierpiński constructed a nonempty proper subset X of $\mathbb{R}$ for which $f = \chi_X$ has approximate symmetric derivative equal 0 for all $x \in \mathbb{R}$. The above classes can be added to Chart 3 as follows.

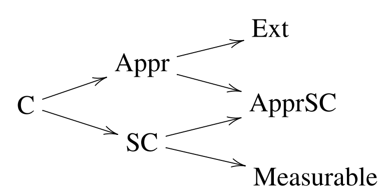

Chart 5. Approximately and symmetrically continuous functions.

The fact that every symmetrically continuous function is measurable follows from a theorem I.N. Pesin and D. Preiss [16, Theorem 2.26] asserting that if an f is symmetrically continuous then its set of points of discontinuity is meager and of measure zero. On the other hand, an approximately symmetrically continuous function need not be measurable, as witnessed by the function $f = \chi_X$ mentioned above. It is unknown whether there exists a ZFC example of a nonmeasurable function in ApprSC.

It is easy to see that all inclusions in Chart 5 are proper. For example, clearly $\chi_{\{0\}} \in \mathrm{SC} \setminus \mathrm{Appr}$. To get $g \in \mathrm{Appr} \setminus \mathrm{SC}$ take $a_0 < b_0 < a_1 < b_1 < \cdots < 0$ such that 0 is an accumulation point of $E = \bigcup_{n<\omega}[a_n, b_n]$ and a density point of $\mathbb{R} \setminus E$. Put $g(x) = 0$ for $x \in \mathbb{R} \setminus E$ and define $g(x) = (b_n - a_n)^{-1}\operatorname{dist}(x, \mathbb{R} \setminus [a_n, b_n])$ for $x \in [a_n, b_n]$. Finally $g + \chi_{\{0\}} \in \mathrm{ApprSC} \setminus (\mathrm{Appr} \cup \mathrm{SC})$.

Another interesting notion of generalized continuity is known as **countable continuity**. For topological spaces X, Y and $\mathcal{F} \subset Y^X$, we define $\mathbf{dec}(\mathcal{F})$ as the smallest infinite cardinal κ such that for every $f \in \mathcal{F}$ there is a family of at most κ-many continuous functions g from a subset of X into Y such that the graph of f is covered by the graphs of all these functions g. A function $f\colon X \to Y$ is **κ-continuous** provided $\operatorname{dec}(\{f\}) \leqslant \kappa$; it is **countably continuous** if f is

ω-continuous. (See [3, Section 4].) The study of these notions was initiated by a question of N. Luzin whether every Borel function from $\mathbb{R}$ into $\mathbb{R}$ is countably continuous. This question was answered negatively by P.S. Novikov and generalized by L. Keldyš. In fact we have already $\mathrm{dec}(\mathcal{B}_1) > \omega$, where $\mathcal{B}_1$ is the family of Baire class 1 functions from $\mathbb{R}$ to $\mathbb{R}$. The most general result in this direction was obtained by J. Cichoń, M. Morayne, J. Pawlikowski and S. Solecki [3, Theorem 4.1] who proved that $\mathrm{cov}(\mathcal{M}) \leqslant \mathrm{dec}(\mathcal{B}_1) \leqslant d$, where $\mathrm{cov}(\mathcal{M})$ is the smallest cardinality of a ***covering*** of $\mathbb{R}$ by meager sets, and d is the ***dominating number***. The consistency of $\mathrm{cov}(\mathcal{M}) < \mathrm{dec}(\mathcal{B}_1)$ and $\mathrm{dec}(\mathcal{B}_1) < d$ was proved by S. Steprāns [3, Theorem 4.2], and S. Shelah, S. Steprāns [3, Theorem 4.3], respectively. Number dec has been also studied by K. Ciesielski in [5], in which he proved that $\mathrm{cof}(\mathfrak{c}) \leqslant \mathrm{dec}(\mathrm{SC}) = \mathrm{dec}(\mathrm{SZ}) = \mathrm{dec}(\mathbb{R}^{\mathbb{R}}) \leqslant \mathfrak{c}$ and that each of the inequalities can be strict. $\mathrm{cof}(\mathfrak{c})$ stands for the ***cofinality*** of the continuum $\mathfrak{c}$ and SZ for the class of **Sierpiński–Zygmund functions** $f : \mathbb{R} \to \mathbb{R}$, that is, those whose restriction $f \restriction X$ is discontinuous for every $X \subset \mathbb{R}$ of cardinality $\mathfrak{c}$.

Finally, one can ask how much continuity an arbitrary function from a topological space X into Y must have. (See, e.g., [3, pp. 148, 149].) In 1922 H. Blumberg proved that for every $f : \mathbb{R} \to \mathbb{R}$ there exists a dense subset D of $\mathbb{R}$ such that $f \restriction D$ is continuous. This result was generalized by several authors to more general topological spaces. However, the most interesting discussion of Blumberg theorem remains in the case of functions from $\mathbb{R}$ to $\mathbb{R}$. Blumberg's set D is countable and in ZFC this is the best that can be proved, since under CH a restriction of a Sierpiński–Zygmund function to any uncountable set is discontinuous. A similar example can be also found in some models of ZFC (e.g., a ***Cohen model***) in the absence of CH as noticed by several authors. (See [3, Theorem 2.9].) At the same time S. Baldwin [3, Theorem 2.8] showed that under ***Martin's Axiom*** for every function $f : \mathbb{R} \to \mathbb{R}$ and every cardinal number $\kappa < \mathfrak{c}$ there exists a set $D \subset \mathbb{R}$ such that $f \restriction D$ is continuous and D is **κ-dense**, that is, $D \cap I$ has cardinality at least κ for every nondegenerated interval I. In the same direction S. Shelah [3, Theorem 2.10] showed that it is consistent with ZFC that for every function $f : \mathbb{R} \to \mathbb{R}$ there exists a set $D \subset \mathbb{R}$ such that $f \restriction D$ is continuous and D is **nowhere meager**, that is, $D \cap I$ is nonmeager for every nontrivial interval I. Most recently, A. Rosłanowski and S. Shelah (unpublished) also found a model of ZFC in which it is always possible to find the set D of positive outer measure, though in this case we cannot require that D is dense in $\mathbb{R}$. (See [3, Theorem 2.11].)

Its is easy to find a function $f : \mathbb{R} \to \mathbb{R}$ which has no points of continuity – the characteristic function of the set of rational numbers has this property. But what if we ask for points of continuity in weaker sense? For example, a function $f : \mathbb{R} \to \mathbb{R}$ is **weakly continuous** at x if it has the Young property at x, that is, if there are sequences $a_n \nearrow 0$ and $b_n \searrow 0$ such that $\lim_{n\to\infty} f(x + a_n) = f(x) = \lim_{n\to\infty} f(x + b_n)$. This notion is so weak that it is impossible to find a function $f : \mathbb{R} \to \mathbb{R}$ which is nowhere weakly continuous. More precisely, every $f : \mathbb{R} \to \mathbb{R}$ is weakly continuous everywhere on the complement of a countable set. (See [3, Theorem 2.16].) A natural symmetric counterpart of weak continuity is defined as follows: a function $f : \mathbb{R} \to \mathbb{R}$ is **weakly symmetrically continuous** at x provided there exists a sequence $h_n \to 0$ such that $\lim_{n\to\infty}(f(x + h_n) - f(x - h_n)) = 0$. The symmetric version of the theorem mentioned above badly fails: K. Ciesielski and L. Larson [3, Theorem 2.17] constructed a nowhere weakly symmetrically continuous functions $f : \mathbb{R} \to \{0, 1, 2, 3, \ldots\}$. It is unknown whether a nowhere weakly symmetrically continuous functions $f : \mathbb{R} \to \mathbb{R}$ can have finite range [3, Problem 2], though its range must have at least four elements. K. Ciesielski and S. Shelah [9] proved that such an f can have bounded countable range.

For functions from $\mathbb{R}$ to $\mathbb{R}$ many generalized continuities mentioned above can be viewed in the context of **path limit** $P\text{-}\lim_{x\to x_0} f(x) \stackrel{\mathrm{def}}{=} \lim_{x\to x_0,\, x\in P} f(x)$ where x_0 is in the closure of $P \cap (x_0, \infty)$ and of $P \cap (-\infty, x_0)$. Thus, for a continuous function the path P at x_0 must be an interval; for $f \in \mathrm{PR}$ a path must be a perfect set; for $f \in \mathrm{PC}$ (i.e., weakly continuous) any path P works; for an approximately continuous function x_0 must be a density point of a path P; in any symmetric version of these notions the paths must be symmetric with respect to x_0. Luzin's theorem implies that every bounded measurable function $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous almost everywhere. Blumberg's theorem and Sierpiński–Zygmund's example illustrate the extent to which arbitrary functions have sets of restricted continuity.

Certainly, the above discussion barely touches the tip of the iceberg of different notions of generalized continuities. From the notions not mentioned so far probably the most studied is that of quasi-continuity introduced in 1932 by Kempisty. (See [13, Section 6].) Thus, a function f from a topological space X into $\mathbb{R}$ is **quasi-continuous**, $f \in \mathrm{QC}(X)$, if for every $x \in X$ and open sets $U \ni x$ and $V \ni f(x)$ there exists a nonempty open $W \subset U$ with $f[W] \subset V$. The other two closely related classes are defined as follows. A function $f : X \to \mathbb{R}$ is **cliquish**, $f \in \mathrm{CLIQ}(X)$, if for every $x \in X$, open $U \ni x$, and $\varepsilon > 0$ there is a nonempty open $W \subset U$ such that $|f(y) - f(z)| < \varepsilon$ for all $y, z \in W$; f is **almost continuous in sense of Husain**, $f \in \mathrm{ACH}(X)$, if for every $x \in X$ and open $V \ni f(x)$ point x belongs to the interior of the closure of $f^{-1}(V)$. It is not difficult to see that $\mathrm{QC}(X) \subset \mathrm{CLIQ}(X)$ and that every $f \in \mathrm{CLIQ}(X)$ has the ***Baire property***. Quasi-continuous functions need not to be in PC, as witnessed by $\chi_{(0,\infty)}$. Also, $\mathrm{Ext} \not\subset \mathrm{CLIQ}(\mathbb{R})$, since $\mathrm{Ext} + \mathrm{Ext} = \mathbb{R}^{\mathbb{R}} \neq \mathrm{CLIQ}(\mathbb{R}) = \mathrm{CLIQ}(\mathbb{R}) + \mathrm{CLIQ}(\mathbb{R})$. The relation $\mathrm{Ext} \not\subset \mathrm{ACH}(\mathbb{R}) \not\subset \mathrm{D}$ is justified by a $\sin(1/x)$-function and the characteristic function of the set of rational numbers, respectively. However, we have $\mathrm{ACH}(\mathbb{R}) \subset \mathrm{PC}$. Also, $\mathrm{QC}(\mathbb{R}) \not\subset \mathrm{ACH}(\mathbb{R})$ and $\mathrm{ACH}(\mathbb{R}) \not\subset \mathrm{CLIQ}(\mathbb{R})$, where the second relation is justified by the characteristic function of a ***Bernstein set***.

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c-14 Spaces of Functions in Pointwise Topology

By a space in this article is understood a ***Tychonoff*** topological space. If X is a space, R^X is the ***topological product*** of X copies of the usual space R of real numbers. Elements of R^X can be interpreted as real-valued functions on X. The topology of the space R^X is called the **topology of pointwise convergence**, or the **pointwise topology**. The set $C(X)$ of all real-valued continuous functions on X is a dense subset of R^X. The topology of R^X generates the subspace topology on $C(X)$. Endowing $C(X)$ with the topology of a subspace of the space R^X, we obtain $C_p(X)$, the space of real-valued continuous functions on X in the pointwise topology. Spaces $C_p(X)$ are also called **C_p-spaces**.

Investigation of topological properties of function spaces originated in functional analysis. The concept of a ***convergent sequence*** played the key role at that stage.

One of the central positions in functional analysis is occupied by the concept of the ***weak topology*** of a Banach space. These topologies produce a natural strata of non-metrizable topological spaces. One of implications is that topological properties of Banach spaces with the weak topologies can vary along a wider scale than topological properties of Banach spaces with the standard (metrizable) topology. All separable Banach spaces are homeomorphic, according to the famous theorem of I. Kadec, which solved a problem of S. Banach (thus, their topological classification is trivial) [21].

Here is a general reason why it is beneficial to consider weak topologies or topologies which are weaker than a given one: the weaker topology one takes, the more ***compact*** subspaces one gets. Note that many powerful topological or linearly topological principles can be applied to compact spaces. Also the global compactness type properties of a space itself can only improve when its topology is weakened. Usually the weak topology of a Banach space has better compactness properties than the standard metrizable topology—it may be, for example, ***Lindelöf***, while the Banach space itself is not separable. Compact subspaces of a Banach space in the weak topology, called ***Eberlein compacta***, are playing an important role in functional analysis, and their most interesting topological properties are largely responsible for that.

Modifying a classic problem of Banach, we ask: When are two Banach spaces with the weak topologies homeomorphic? Since Banach spaces in the weak topologies are almost always neither metrizable, nor complete, this question differs very much from the question of Banach.

Every Banach space B with the weak topology is linearly homeomorphic to a closed subspace of the space $C_p(X)$, for some compact Hausdorff space X. Indeed, one can take X to be the unit ball in the dual Banach space B^* (consisting of continuous linear functionals on B and endowed with the topology of pointwise convergence on B). Then the natural **evaluation map** ($f \mapsto f(x)$) embeds B linearly as a closed subspace into $C_p(X)$, and this embedding is a homeomorphism with respect to the weak topology on B.

This construction suggests that the topological theory of Banach spaces with the weak topology can be reduced to the theory of C_p-spaces. Also, as a very natural object combining topological and algebraic structures, $C_p(X)$ can serve as an important technical tool in the study of topological spaces, providing a bridge between general topology and topological algebra and functional analysis. The main objects studied in C_p-theory are $C_p(X)$ itself and compact subspaces of $C_p(X)$, as well as relationships between X and $C_p(X)$. Observe, that the space $C_p([0,1])$ is not metrizable (though it is hereditarily ***separable***), is not complete (with respect to the natural uniformity), and does not have the ***Baire property***. Thus, $C_p([0,1])$ differs principally in its topological properties from the Banach space $C([0,1])$. Note, that if $C_p(X)$ has the Baire property, then every compact subset of X is finite (see [1]).

Main directions of research in C_p-theory correspond to the following general problems. How are topological properties of the space $C_p(X)$ related to topological properties of the space X itself? In particular, when, for a topological property $\mathcal{P}$, is there a topological property $\mathcal{Q}$ such that a space X has the property P if and only if the space $C_p(X)$ has the property $\mathcal{Q}$?

While X has only topological properties, being a topological space, $C_p(X)$ harbours algebraic properties as well, being a ***topological ring***. Hence, $C_p(X)$ can also be treated as a ***uniform space***. Besides, there is an obvious natural partial ordering on $C_p(X)$ with respect to which $C_p(X)$ is a ***topological lattice***. It is natural to ask what properties of X can be characterized by uniform properties of $C_p(X)$? What properties of X can only be characterized with the help of some algebraic restrictions on $C_p(X)$? In some cases, it is enough to consider $C_p(X)$ just as a ***topological group***, in others – to treat it as a linear topological space, etc. We can briefly describe this line of investigation in C_p-theory as an approach through duality theorems.

There is a slightly different, and, in fact, more general way to treat the same subject. Topological spaces X and Y are called **ℓ-equivalent** (notation: $X \overset{\ell}{\sim} Y$) if linear topological spaces $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. If $C_p(X)$ and $C_p(Y)$ are homeomorphic (uniformly homeomorphic), we write $X \overset{t}{\sim} Y$ (correspondingly, $X \overset{u}{\sim} Y$) and say that the spaces X and Y are **t-equivalent** (**u-equivalent**). The three equivalence relations are extensions of the topological equivalence.

Which topological invariants are preserved by these broader equivalences? For example, which properties are preserved by ℓ-equivalence? Such properties are called **ℓ-invariants**. The properties preserved by t-equivalence (i.e., **t-invariants**) can be considered as purely topological properties of a space X, since they do not depend on the algebraic structure of $C_p(X)$ at all.

The general foundation and motivation for this approach is provided by an important theorem of J. Nagata: If topological rings $C_p(X)$ and $C_p(Y)$ are topologically isomorphic, then the spaces X and Y are homeomorphic [1]. No compactness or completeness conditions are imposed on X and Y in this theorem, it is valid for all Tychonoff spaces X and Y. To make our notation uniform, we shall write $X \overset{h}{\sim} Y$ when X and Y are homeomorphic.

The four equivalences defined above are all distinct. Clearly, $X \overset{h}{\sim} Y \Rightarrow X \overset{\ell}{\sim} Y \Rightarrow X \overset{u}{\sim} Y \Rightarrow X \overset{t}{\sim} Y$. Can one reverse these implications? We will see below that the answer is "no".

Spaces X and Y are said to be **M-equivalent** (i.e., equivalent in the sense of A.A. Markov), if the ***free topological groups*** of these spaces are topologically isomorphic. M.I. Graev showed that M-equivalence implies ℓ-equivalence (see [20], [HvM, Chapters 1, 2]). Thus, $\overset{h}{\sim} \Rightarrow \overset{M}{\sim} \Rightarrow \overset{\ell}{\sim} \Rightarrow \overset{u}{\sim} \Rightarrow \overset{t}{\sim}$, where $\overset{M}{\sim}$ does not imply $\overset{h}{\sim}$.

The circle is not M-equivalent to the closed unit interval. On the other hand, D.S. Pavlovskij proved that every two compact ***polyhedra*** of the same finite positive dimension are ℓ-equivalent (see [2, 3]). Hence, the circumference and the segment are ℓ-equivalent and not M-equivalent, that is $\overset{\ell}{\sim} \not\Rightarrow \overset{M}{\sim}$. It is much more difficult to show that $X \overset{u}{\sim} Y$ does not imply, in general, that $X \overset{\ell}{\sim} Y$, and to demonstrate that t-equivalence is really weaker than u-equivalence. S.P. Gul'ko and T. Hmyleva proved that the spaces $C_p(R)$ and $C_p(I)$ are homeomorphic [12]. On the other hand, it was established by V.V. Uspenskij that compactness is preserved by u-equivalence [1]. It follows that the spaces $C_p(R)$ and $C_p(I)$ are not uniformly homeomorphic as uniform spaces. Thus, t-equivalence does not imply u-equivalence. To show this, we could use, alternatively, the following theorem of T. Dobrowolski, S. Gul'ko, and J. Mogilski [9]: All non-discrete countable metrizable spaces are t-equivalent. For any such space X, $C_p(X)$ is homeomorphic to σ^ω, where σ is the ***σ-product*** of ω copies of the space R of real numbers [9]. This theorem should be compared with yet another result of Gul'ko [10]: all infinite countable compact spaces are u-equivalent to each other. Note that, by Uspenskij's theorem, not all countable non-discrete metric spaces are u-equivalent, since some of them are compact and some are not.

A subset A of a space X is called a **bounded subset** (in X) if every continuous real-valued function on X is bounded on A. The set $C(X)$ with the **topology of uniform convergence on bounded subsets** of X will be denoted by $C_o(X)$; we shall also consider the space $C_k(X)$, which is $C(X)$ endowed with the ***compact-open topology***, which may also be called the **topology of uniform convergence on compact subsets** of X. In both cases a basic neighbourhood of f is of the form $\{g\colon (\forall x \in K)(|g(x) - f(x)| < \varepsilon)\}$, where $\varepsilon > 0$ and K is a bounded or compact set, respectively.

A space X is ***Dieudonné complete*** if it is complete with respect to the largest uniformity generating the topology of X. For any Dieudonné complete space X, the spaces $C_o(X)$ and $C_k(X)$ coincide, since all pseudocompact Dieudonné complete spaces are compact. It is easy to show that if X is an uncountable pseudocompact space such that all compact subsets of X are finite, then $C_k(X)$ coincides with $C_p(X)$ and $C_k(X)$ is not metrizable, since $\mathcal{X}(C_p(X)) = |X| > \aleph_0$, while $C_o(X)$ coincides with the Banach space $C_B(X)$, which is metrizable. Thus $C_o(X)$ and $C_k(X)$ are not always the same.

When X is compact, then $C_k(X)$, topologically, is the Banach space $C(X)$. Spaces X and Y are **B-equivalent** (notation: $X \overset{B}{\sim} Y$) if linear topological spaces $C_k(X)$ and $C_k(Y)$ are linearly homeomorphic. If $C_k(X)$ and $C_k(Y)$ are homeomorphic, we write $X \overset{k}{\sim} Y$ and say that X and Y are **k-equivalent**. If $C_o(X)$ and $C_o(Y)$ are linearly homeomorphic, the spaces X and Y are said to be **o-equivalent** (notation: $X \overset{o}{\sim} Y$).

Here is a basic result on ℓ-equivalence and o-equivalence [3]: if $X \overset{\ell}{\sim} Y$, then $X \overset{o}{\sim} Y$. In fact, every linear homeomorphism between spaces $C_p(X)$ and $C_p(Y)$ is also a linear homeomorphism between spaces $C_o(X)$ and $C_o(Y)$ [3]. Hence, if X and Y are Dieudonné complete spaces and $X \overset{\ell}{\sim} Y$, then $X \overset{B}{\sim} Y$.

There exist countable compacta X and Y which are not ℓ-equivalent, while $X \overset{u}{\sim} Y$ [10]. Hence, u-equivalence does not imply ℓ-equivalence, even for compact spaces. It is unknown, if one can find t-equivalent compact spaces X and Y which are not u-equivalent. Can such X and Y be, in addition, ***scattered***?

Given a topological property, there is no general rule or principle, how to guess whether this property is ℓ-invariant, or u-invariant or even t-invariant. For example, compactness is u-invariant and not t-invariant, while local compactness is not even ℓ-invariant. Neither the second axiom of countability, nor the first axiom of countability is ℓ-invariant. Metrizability is also not preserved in general by M-equivalence or ℓ-equivalence. Indeed, the general construction below shows, in particular, that the ***Fréchet–Urysohn fan*** $V(S_0)$ is ℓ-equivalent to the product $S \times N$ of the convergent sequence S and the discrete space N. The construction is based on the notion of a continuous extender, which proved to be of fundamental importance in C_p-theory.

In what follows, S denotes the simplest infinite compact space, that is, $S = \{0, \frac{1}{n}\colon n \in N^+\}$ is the convergent sequence; $\oplus$ stands for the topological sum. Let X be a normal topological space and A a non-empty closed subset of X. Consider the decomposition $\gamma_A = \{A, \{x\}\colon x \in X \setminus A\}$ of X, the only non-trivial element of which is A. In general, the quotient topology on the set γ_A need not be Tychonoff, however, it is Tychonoff in this case, since X is assumed to be normal. We denote by $X|_A$ the space obtained when γ_A is taken with the quotient topology.

Put $C_p(X; A) = \{f \in C_p(X)\colon f|_A \equiv 0\}$ and $C_p(X|_A)_0 = C_p(X|_A; \{A\}) = \{f \in C_p(X|_A)\colon f(A) = 0\}$. Obviously, $C_p(X; A)$ and $C_p(X|_A)_0$ are closed linear subspaces of $C_p(X)$ and $C_p(X|_A)$, respectively. Adding one new isolated point to a space X we obtain a space denoted by X^+.

Let Y be a subspace of X. A map $\varphi : C_p(Y) \to C_p(X)$ is called an **extender**, if for each $g \in C_p(Y)$ the restriction of the function $\varphi(g)$ to Y coincides with g.

A subspace Y of a space X is said to be **t-embedded** (**ℓ-embedded**) in X if there exists a continuous (a linear continuous) extender $\varphi : C_p(Y) \to C_p(X)$. Notation: $X \overset{t}{\subset} Y$ (respectively, $X \overset{\ell}{\subset} Y$). Below X, Y and Z are non-empty topological spaces.

In general, not every closed subspace even of a ***normal*** space X is t-embedded in X. For example, $\beta N \setminus N$ is not t-embedded in βN. On the other hand, every metrizable compact subspace of a space X is ℓ-embedded in X [1].

If $Y \subset X$ and Y is a ***retract*** of X, then Y is ℓ-embedded in X. It is quite reasonable to consider the concept of an ℓ-embedded subspace as a generalization of the concept of a retract. This generalization is a very bold one, since every metrizable compactum is ℓ-embedded in every larger space. Of course, if Y is t-embedded into X, then Y is closed in X.

Here are some key facts concerning ℓ- and t-embeddings. If a subspace Y of X is t-embedded (ℓ-embedded) in X, then:

(a) $C_p(X)$ is homeomorphic (linearly homeomorphic) to the space $C_p(X|_Y)_0 \times C_p(Y)$, and

(b) $X^+ \overset{t}{\sim} X|_Y \oplus Y$ (respectively, $X^+ \overset{\ell}{\sim} X|_Y \oplus Y$).

Formula (b) enables us to construct many non-trivial pairs of ℓ-equivalent spaces. Formula (b) also provides an elementary step playing a key role in the proofs of several important results on ℓ-equivalence. Obviously, formula (b) would become simpler if we could replace X^+ with X. When X is finite, this cannot be done. But what about infinite X? W. Marciszewski [16] constructed an infinite compact Hausdorff space X such that $C_p(X)$ is not linearly homeomorphic to $C_p(X) \times R$. It remains unknown if the space $C_p(X)$ is homeomorphic to $C_p(X) \times R$, for every infinite space X.

However, under very general conditions this is the case, in particular, if X contains a non-trivial convergent sequence, or if X is not pseudocompact. On the other hand, not every locally convex infinite-dimensional linear topological space L_1 over R is homeomorphic to its product with R. Indeed, Jan van Mill constructed a locally convex infinite-dimensional linear topological space L_1 such that every subspace of L_1 homeomorphic to L_1 is open in L_1 [18].

Let S be the "convergent sequence", and let A be the set of non-isolated points in the space $S \oplus S$. Obviously the space $(S \oplus S)^+$ is homeomorphic to $S \oplus S$, and the space $(S \oplus S)|_A \oplus S$ is homeomorphic to S. Also, being finite, A is ℓ-embedded in $S \oplus S$. Therefore, $S \oplus S \overset{\ell}{\sim} S$.

If X is a space containing a non-trivial convergent sequence S, then X^+ is ℓ-equivalent to X, that is, $C_p(X)$ is linearly homeomorphic to $C_p(X) \times R$. Indeed, S is ℓ-embedded in X, since S is a metrizable compactum. Besides, $S \oplus S$ is ℓ-equivalent to S. Hence, $X \oplus S$ is ℓ-equivalent to X. Since S^+ is homeomorphic to S, we have: $X^+ \overset{\ell}{\sim} X \oplus S^+ \overset{\ell}{\sim} X \oplus S \overset{\ell}{\sim} X$. In particular, if X is an infinite metrizable space, then $X^+ \overset{\ell}{\sim} X$. J. Dugundji showed that every closed subspace of a metrizable space X is ℓ-embedded in X ([E], [1]). It follows that, for every infinite metrizable space X and every closed subspace A of X, the next formula holds: $X \overset{\ell}{\sim} X|_A \oplus A$.

For example, let X be the product of the convergent sequence S with the discrete space N of natural numbers, and let A be the set of all non-isolated points in X. Then the quotient space $X|_A$ is the **Fréchet–Urysohn fan** $V(S_0)$, which is not first-countable, not metrizable, and not locally compact. However, by the above formula, $X|_A$ is ℓ-equivalent to the space X which is locally compact, metrizable, and countable. This simple example can serve to disprove many nice conjectures in C_p-theory.

Here is a similar example, which is a little easier to visualize. Let X be the Euclidean plane R^2 and $Y = \{(x, 0)\colon x \in R\}$. Then X is ℓ-equivalent to $X|_Y$. The space X is separable, metrizable and locally compact, while the space $X|_Y$ is neither first-countable nor locally compact (at the point $Y \in X|_Y$). The same example shows that ***Čech-completeness*** is not preserved by ℓ-equivalence, since $X|_Y$ is not Čech-complete, while X is obviously Čech complete.

In particular, the ***weight*** of a space is not preserved by ℓ-equivalence. On the other hand, the ***network weight*** of X is even t-invariant, since the network weight of X equals the network weight of $C_p(X)$ [1]. The ***density*** and the i-weight are preserved by t-equivalence, while the Souslin number $c(X)$ is not even ℓ-invariant [1]. Recall that the **i-weight** $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous map onto a space of the weight not greater than τ. A general method for proving that a certain property $\mathcal{P}$ is ℓ-invariant, u-invariant or t-invariant is to establish a duality theorem, characterizing this property in terms of $C_p(X)$. If the characterization involves only topological properties of $C_p(X)$, then $\mathcal{P}$ is clearly t-invariant; if only uniform properties of $C_p(X)$ are involved, then $\mathcal{P}$ is u-invariant, etc. This approach was used to show that compactness and pseudocompactness are u-invariant, and that density and network-weight are t-invariant [1].

However, one can prove that a property $\mathcal{P}$ is ℓ-invariant or t-invariant without finding a dual property. One of the most important and far reaching results of this type was obtained by S.P. Gul'ko [11]: If $X \overset{u}{\sim} Y$, then $\dim X = \dim Y$. Here $\dim X$ is understood as Lebesgue's (covering) dimension of βX. This statement generalizes an earlier basic result of V.G. Pestov on ℓ-invariance of Lebesgue's dimension [22]. Unfortunately, from the proofs Gul'ko and Pestov provided for their theorems no characterization of $\dim X$ in terms of linear topological or uniform properties of $C_p(X)$ emerges.

It is still unknown whether $\dim X = \dim Y$ when $X \overset{t}{\sim} Y$. Is this true at least when X and Y are compact? If we could find two compact spaces X and Y such that $C_p(X)$ is homeomorphic to $C_p(Y)$ and $\dim X \neq \dim Y$, this would imply that t-equivalence does not coincide with u-equivalence on the class of compact spaces. At present, we do not know if this is the case. We do not even know if $C_p(I)$ is homeomorphic to $C_p(I \times I)$ [vMR, Chapter 31]. However, R. Cauty [8] proved that the Hilbert cube I^ω is not t-equivalent to any finite-dimensional metrizable compactum. W. Marciszewski generalized this by showing that if X and Y are t-equivalent separable metrizable space, then one of them is countable-dimensional if and only if the other one is also countable-dimensional (see [17]). It is unknown, whether the Cantor set D^ω is t-equivalent to the interval I [vMR, Chapter 31].

On the contrary, Banach spaces $C_k(D^\omega)$, $C_k(I)$ and $C_k(I \times I)$ are all linearly homeomorphic, according to a theorem of A.A. Miljutin [21], while $C_p(I)$ and $C_p(I \times I)$ are not linearly homeomorphic, since $\dim I = 1 \neq 2 = \dim(I \times I)$ (by Pestov's theorem [22]). It follows that B-equivalence does not imply u-equivalence, even for compact spaces. On the other hand, we already noted that ℓ-equivalence implies B-equivalence in the class of compact spaces.

For some classes of spaces, the necessary condition for ℓ-equivalence, the equality of the dimension, turns out to be also sufficient. All uncountable zero-dimensional metrizable compacta are ℓ-equivalent to the Cantor set [5], see also [3]. We already noted that all compact polyhedra of the same positive dimension are ℓ-equivalent (Pavlovskij, see [3, 2]). Arhangel'skiĭ showed that all separable zero-dimensional non-σ-compact complete metric spaces are ℓ-equivalent [2, 3]. All separable metrizable non-compact CW-spaces of the same positive dimension are also ℓ-equivalent [2, 3].

Duality theorems are more rare than the results on ℓ- and t-equivalence. ***Density*** of X was characterized by a topological property of $C_p(X)$ (by the ***pseudocharacter*** of $C_p(X)$, [1]). Density of $C_p(X)$ was characterized by the ***i-weight*** of X. We also have: $nw(X) = nw(C_p(X))$ [1]. No crisp characterization of Lindelöf spaces in terms of $C_p(X)$ is available. However, N.V. Veličko proved that if X is Lindelöf and $X \overset{\ell}{\sim} Y$, then Y is also Lindelöf [25]. It remains unknown if the Lindelöf property is t-invariant. If X^n is Lindelöf, for each $n \in N^+$, and Y is t-equivalent to X, then Y^n is also Lindelöf, for each $n \in N^+$, and a similar result holds for Lindelöf degree in general. It is based on the next theorem (Arhangel'skiĭ and Pytkeev [1]): For any space X and any infinite cardinal number τ, the following conditions are equivalent:

(1) The ***Lindelöf degree*** of X^n does not exceed τ for every $n \in N^+$;
(2) The ***tightness*** of $C_p(X)$ does not exceed τ.

While compactness is not preserved by t-equivalence in general, O.G. Okunev established that if a space X is σ-compact and $Y \overset{t}{\sim} X$, then Y is also σ-compact. This follows from the next characterization [1]: A space X is σ-compact if and only if there exists a compact space Y such that $C_p(X)$ is homeomorphic to a subspace of $C_p(Y)$.

If X is a ***Lindelöf Σ-space*** and $X \overset{t}{\sim} Y$, then Y is also a Lindelöf Σ-space (O.G. Okunev [1]). An important subclass of the class of Lindelöf Σ-spaces is that of K-analytic spaces. A space X is said to be a **$K_{\sigma\delta}$-space** if it can be represented as the intersection of a countable family of σ-compact subspaces of some larger space $\widetilde{X}$. Continuous images of $K_{\sigma\delta}$-spaces are called K-analytic spaces. If a space X is K-analytic and $X \overset{t}{\sim} Y$, then the space Y is also K-analytic (O.G. Okunev [1]).

E.A. Rezničenko constructed an example showing that ***paracompactness*** is not preserved by ℓ-equivalence [HvM, Chapter 1].

It is unknown, whether the class of ***k_ω-spaces*** is u-invariant, though it is ℓ-invariant [3]. However, W. Marciszewski showed that a space t-equivalent to a countable compact space need not be a k-space and need not contain non-trivial convergent sequences (see [17]). V.V. Tkačuk observed that there exists a ***pseudocompact*** space such that all spaces t-equivalent to it are pseudocompact (see [4], [HvM, Chapter 1]).

While the class of separable metrizable spaces is not ℓ-invariant, the class of ***$\aleph_0$-space***s is ℓ-invariant. It follows that every space ℓ-equivalent to a separable metrizable space is an $\aleph_0$-space [3]. It is unknown, whether the class of $\aleph_0$-spaces is preserved by u-equivalence. Since every first-countable $\aleph_0$-space is metrizable, it follows that if a first-countable space Y is ℓ-equivalent to a separable metrizable space X, then Y is also separable and metrizable. It is unknown, whether every first-countable space t-equivalent to a separable metrizable space X must be metrizable. R.A. McCoy and I. Ntantu [19] established that if a paracompact space Y of ***point-countable type*** is ℓ-equivalent to a locally compact paracompact space X then Y is also locally compact.

This result is based on the following duality theorem [19]. Let X be a paracompact space of point-countable type. Then $C_k(X)$ has the Baire property if and only if X is locally compact. If Y is a first-countable space ℓ-equivalent to completely metrizable separable space then Y is also completely metrizable (J. Pelant, see [17]).

When X is countable and infinite, the space $C_p(X)$ is an infinite-dimensional separable linear metric space dense in R^X. Therefore, it can be studied using methods of infinite-dimensional topology. The technique of ***absorbing sets*** plays an important role in these investigations. There are also deep connections of this domain to descriptive set theory.

W. Marciszewski showed that there exists a countable space X with a closed subspace Y such that Y is not t-embedded in X and $C_p(Y)$ is not a factor of $C_p(X)$ [15]. This means, of course, that Dugundji's theorem cannot be extended to countable spaces.

If X is a space with a countable network, then $X \overset{\ell}{\sim} X^+$. In particular, $X \overset{\ell}{\sim} X^+$ if X is countable. If X is a σ-compact space that is not compact then also $X \overset{\ell}{\sim} X^+$.

For C_p-spaces, as in general for topological spaces with a nice algebraic structure, one might expect that some topological properties become productive. However, it is still unknown if, for every Lindelöf (normal) $C_p(X)$ the square $C_p(X) \times C_p(X)$ is also Lindelöf (respectively, normal). The above question is obviously related to the following one, which is also open: does there exists a continuous map from $C_p(X)$ onto $C_p(X) \times C_p(X)$, for every (compact) space X [HvM, Chapter 1]? Note that the space $C_p(\omega_1 + 1)$ is not homeomorphic to $C_p(\omega_1 + 1) \times C_p(\omega_1 + 1)$ (S.P. Gul'ko, see [17]). R. Pol showed that there exists an infinite compact metrizable space X such that $C_p(X)$ is not linearly homeomorphic with $C_p(X) \times C_p(X)$ [23]. This means that the space X is not ℓ-equivalent to the space $X \times \{0, 1\}$, that is, to the ***topological sum*** of two copies of the space X. J. van Mill, J. Pelant and R. Pol showed that there exists an infinite metrizable space X such that X is not u-equivalent to $X \times \{0, 1\}$ (see [17]). However, it remains unknown whether $C_p(X)$ is homeomorphic to $C_p(X) \times C_p(X)$, for every infinite (compact) metrizable space X [vMR, Chapter 31]. It is also unknown if $C_p(X)$ is homeomorphic to $(C_p(X))^\omega$, for every infinite countable space X [17].

Every uniform homeomorphism between uniform spaces $C_p(X)$ and $C_p(Y)$ can be extended to a uniform homeomorphism of the uniform spaces R^X and R^Y, since R^X and R^Y are uniform completions of the uniform spaces $C_p(X)$ and $C_p(Y)$, respectively. This leads to yet another equivalence, which lies in between the u-equivalence and the t-equivalence.

Spaces X and Y are **absolutely *t*-equivalent**, or ***at*-equivalent** (notation: $X \overset{at}{\sim} Y$) if there exists a homeomorphism g between the spaces R^X and R^Y such that $g(C_p(X)) = C_p(Y)$. If X is absolutely t-equivalent to Y and X is compact, then Y is also compact [HvM, Chapter 1]. If Y is a space u-equivalent (at-equivalent) to a ***perfectly normal*** compact space, then Y is also a perfectly normal compact space [HvM, Chapter 1]. These results suggest that at-equivalence is closer to u-equivalence than to t-equivalence. The real line R and the unit segment I are t-equivalent, but not at-equivalent. Hence, t-equivalence does not imply at-equivalence. It is unknown, whether u-equivalence and at-equivalence coincide on the class of all Tychonoff spaces. Do they coincide on the class of all compacta?

The above facts give rise to another natural question, which is still open: is the dimension $\dim X$ preserved by absolute t-equivalence? A negative answer to this question would mean that at-equivalence and u-equivalence are distinct.

A space Y is *t-dominated* (*ℓ-dominated, u-dominated*) by a space X, if $C_p(X)$ can be mapped continuously (linearly and continuously, uniformly continuously) onto $C_p(Y)$. Notation: $[Y \leqslant X]_t$ ($[Y \leqslant X]_\ell$, $[Y \leqslant X]_u$).

Given a cardinal-valued topological invariant φ, it is natural to consider the following general question. Let Y be t-dominated (u-dominated, ℓ-dominated) by X. Is then $\varphi(Y) \leqslant \varphi(X)$? Suppose that Y is u-dominated by X. Is then $\dim Y \leqslant \dim X$? What if Y is ℓ-dominated by X? Both questions were answered in negative by A. Leiderman, M. Levin, and V.G. Pestov. They established that, for every finite-dimensional metrizable compactum X, there exists a linear continuous surjection of $C_p(I)$ onto $C_p(X)$ [14]. They also proved that, for every finite-dimensional metrizable compactum Y, there exists a 2-dimensional metrizable compactum X such that $C_p(X)$ admits an open linear continuous map onto $C_p(Y)$ (see [14]). On the other hand, for an uncountable cardinal τ, $C_p(D^\tau)$ cannot be mapped by a linear continuous map onto $C_p(I^\tau)$ (Arhangel'skiĭ and Choban, see [HvM, Chapter 1], [3]).

Here is a positive result [2]: If $C_p(X)$ is linearly homeomorphic to the product space $C_p(Y) \times L$, where L is a linear topological space over R and Y is a compact space, then $\dim Y \leqslant \dim X$.

From $[Y \leqslant X]_\ell$ it does not follow in general that $d(Y) \leqslant d(X)$. Indeed, let Y be a closed non-separable subspace of a normal separable space X. Then, by the restriction map, $C_p(X)$ is mapped linearly and continuously onto $C_p(Y)$ and hence $[Y \leqslant X]_\ell$.

Let $X = I^{\omega_1}$, $x_0 \in X$ and $Y = X \setminus \{x_0\}$. Then X is the Stone-Cech compactification of Y, which implies that there exists a one-to-one continuous homomorphism of topological ring $C_p(X)$ onto topological ring $C_p(Y)$. Clearly X is homeomorphic to a subspace of Y. Since X is compact, it follows that the restriction map is an open continuous homomorphism of $C_p(X)$ onto $C_p(Y)$. The space Y is not Lindelöf, while X is compact. Thus the results on preservation of compactness cannot be generalized to the situation, when each of the topological rings $C_p(X)$ and $C_p(Y)$ can be mapped onto another by a continuous (ring) homomorphism.

The next general question is relevant to many situations in topology: Given a class $\mathcal{P}$ of spaces, and a space Y, when can Y be topologically embedded into $C_p(X)$, for some $X \in \mathcal{P}$? If $\mathcal{P}$ is the class of all Tychonoff spaces, then every space Y can be embedded into $C_p(X)$ for some X in $\mathcal{P}$. Indeed, it is enough to take a sufficiently large discrete space X. Hence, the ***Tychonoff embedding theorem*** is a result in this direction.

If f is a continuous map of a space X onto a space Y, then $C_p(Y)$ naturally embeds into $C_p(X)$ by the map dual to f that send φ to $\varphi \circ f$ [1]. If the map f is, in addition, quotient, then $C_p(Y)$ embeds in $C_p(X)$ as a closed subspace. A deep result on subspaces of C_p-spaces is the famous theorem of Grothendieck, which in it's simplest form states that *if X is countably compact, and A is a subset of $C_p(X)$ which is countably compact in $C_p(X)$, then the closure of A in $C_p(X)$ is compact* (see [1, 4]).

It follows from Grothendieck's theorem that ***Ψ-space*** cannot be embedded into $C_p(X)$, for any countably compact space X. An amazing fact is that Grothendieck's theorem remains valid far beyond the class of countably compact spaces [1, 4]. For example, it holds for all spaces of countable tightness. This is so, since $C_p(X)$ is

Dieudonné complete for any such X. Whether $C_p(X)$ satisfies Grothendieck's theorem hereditarily, strongly depends on tightness type properties of compacta in $C_p(X)$ (see [3]).

Much research was done to find out what properties $C_p(X)$ must have when X is compact. In particular, if X is compact, then the tightness of $C_p(X)$ is countable and $C_p(X)$ is monolithic [1]. Recall, that a space X is **monolithic**, if for every $A \subset X$, the network weight of the closure of A in X does not exceed the cardinality of A. It follows, that if X is compact, then every separable subspace of $C_p(X)$ has a countable network, and every separable compact subspace of $C_p(X)$ is metrizable. In particular, the ***Sorgenfrey line*** cannot be embedded into $C_p(X)$, for any compact space X. Since monolithicity and countable tightness are both hereditary properties, the above results establish very clear barriers to embeddings of spaces into $C_p(X)$, where X is compact.

Curiously, the space $C_p(X)$ does not have to be sequential when X is compact. Indeed, for a compact X, $C_p(X)$ is sequential if and only if X is scattered (see [1]). Thus, if X is compact, then the topology of $C_p(X)$ is determined by countable subsets of $C_p(X)$, but not necessarily by convergent sequences. On the other hand, not every countable subset can be embedded into $C_p(X)$ for a compact X: even the Fréchet–Urysohn fan $V(S_0)$ cannot be embedded like that. This follows from the next result: if X is compact, then $C_p(X)$ has countable fan-tightness [1]. We say that a space Y has **countable fan-tightness** if for any point $x \in Y$ and any countable family $\{A_n: n \in \omega\}$ such that $x \in \overline{A_n}$ for each $n \in \omega$, it is possible to select finite sets $K_n \subset A_n$, so that $x \in \overline{\bigcup\{K_n: n \in \omega\}}$.

E.G. Pytkeev established that if X is compact, and Y is a subspace of $C_p(X)$, then Y is Fréchet–Urysohn if and only if Y is a k-space [24]. D.P. Baturov proved that if X is compact, and Y is a subspace of X such that every closed discrete subspace of Y is countable, then Y is Lindelöf [1]. This implies that every countably compact subspace of $C_p(X)$, where X is compact, is compact (and therefore, closed in $C_p(X)$). This corollary also follows from Grothendieck's theorem. Recently R.Z. Buzyakova has established a fundamental result on the topological structure of $C_p(X)$ for compact X, bringing to the light the true nature of Baturov's and Grothendieck's theorems. She proved [6] that if X is compact then every subspace Y of $C_p(X)$ is a D-space, that is, for every neighbourhood assignment $\{V(y): y \in Y\}$ on Y there exists a closed discrete subspace A of Y such that $Y \subset \bigcup\{V(y): y \in A\}$. Clearly, every D-space of the countable extent is Lindelöf, and Baturov's theorem follows. Buzyakova has also shown that neither Baturov's theorem, nor her result extends to countably compact spaces [7]. According to a result of E.A. Reznichenko, if $C_p(X)$ is normal, then all closed discrete subspaces of $C_p(X)$ are countable [1]. Together, Baturov's and Reznichenko's theorems imply that, for a compact space X, $C_p(X)$ is normal if and only if $C_p(X)$ is Lindelöf.

Note that paracompactness of $C_p(X)$ is equivalent to the Lindelöf property of $C_p(X)$ for any space X. This is so, since $C_p(X)$ is dense in R^X and, therefore, the Souslin number of $C_p(X)$ is countable [1]. Note also, that if $C_p(X)$ is normal, then $C_p(X)$ is countably paracompact (V.V. Tkachuk, see [1]). A necessary condition for $C_p(X)$ to be Lindelöf is that the tightness of X is countable (M. Asanov, see [1]).

Compact subspaces of $C_p(X)$, when X is compact, are precisely Eberlein compacta. Various useful characterizations of Eberlein compacta are available, and many important properties of Eberlein compacta have been established (see [1]). In fact, C_p-theory brought into set-theoretic topology a whole variety of new classes of compacta: ***Corson compact***a, ***Talagrand compact***a, ***Gul'ko compact***a, ***Rosenthal compact***a, ***Radon–Nikodým compact***a. All these types of compact spaces are very interesting from purely topological point of view [1], [HvM, Chapter 1], [12, 3]. Such compacta, as well as scattered compacta and ***extremally disconnected*** compacta, were also applied by experts in functional analysis to construct non-trivial examples of Banach spaces in the weak topology (see, for example, [21, 13]).

However, very little is known about compact subspaces of $C_p(X)$, when X is a Lindelöf space. We might expect that compact parts of $C_p(X)$, where X is Lindelöf, should have much better structure than the whole of $C_p(X)$

The C_p-theory serves as a most valuable source of non-trivial examples of topological spaces. It has found applications to topological groups [4], to Banach spaces [4], to descriptive theory of sets [17], and in set-theoretic topology itself [1, 3]. It is rich in deep results and in most natural and attractive open questions.

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c-15 Radon–Nikodým Compacta

Let (X,τ) be a ***topological space*** and let φ be a non-negative function on $X\times X$ such that $\varphi(x,y)=0$ if and only if $x=y$. Then the space (X,τ) is said to be **fragmented by** φ or φ**-fragmented** if, whenever A is a non-empty subset of X and $\varepsilon>0$, there is a τ-open set U in X such that $\emptyset\neq U\cap A$ and $\varphi\text{-diam}(U\cap A)\equiv\sup\{\varphi(x,y)\colon x,y\in U\cap A\}<\varepsilon$. This term, in case φ is a metric, is due to Jayne and Rogers [7] in connection with their work on Borel selectors for set-valued maps taking values among subsets of certain Banach spaces. A ***compact Hausdorff*** space K is called **Radon–Nikodým compact** (**RN-compact**) if K is fragmented by a ***lower semi-continuous metric*** ρ on K. Recall that the function $\rho\colon K\times K\to[0,\infty)$ is lower semi-continuous if and only if $\{(x,y)\colon\rho(x,y)\leqslant r\}$ is closed for each $r\in\mathbb{R}$.

The above definition of RN-compactness obscures the genesis of the notion and of the terminology. In order to describe it, we need additional definitions. Let E be a ***Banach space*** and let $(\Omega,\mathcal{F},\mu)$ be a finite ***measure space***, i.e., $\mu(\Omega)<\infty$. A map $F\colon\mathcal{F}\to E$ is called an E**-valued measure** if, for each pairwise disjoint sequence $\{A_n\colon n\in\mathbb{N}\}$ in $\mathcal{F}$,

$$F\left(\bigcup_{i=1}^{\infty}A_i\right)=\sum_{i=1}^{\infty}F(A_i),$$

where the series on the right-hand side converges relative to the ***norm topology***. Given an E-valued measure F, the **total variation** $|F|(B)$ of F on $B\in\mathcal{F}$ is defined by $|F|(B)=\sup\{\sum_{i=1}^{n}\|F(B_i)\|\colon B_i\in\mathcal{F},\ B_i\subset B,\ B_i\cap B_j=\emptyset$ for $i\neq j$ and $n\in\mathbb{N}\}$. The function $|F|$ is a measure on $(\Omega,\mathcal{F})$ which is not necessarily finite. If $|F|(\Omega)<\infty$, then F is said to be of **bounded variation**. The E-valued measure F is called **absolutely continuous** with respect to μ (written $F\ll\mu$) if $F(A)=0$ whenever $A\in\mathcal{F}$ and $\mu(A)=0$, or equivalently $|F|\ll\mu$ in the usual sense. A Banach space E is said to have the **Radon–Nikodým property** (**RNP**) if, whenever $(\Omega,\mathcal{F},\mu)$ is a finite measure space and $F\colon\mathcal{F}\to E$ is an E-valued measure of bounded variation such that $F\ll\mu$, there is a Bochner integrable function $\varphi\colon\Omega\to E$ such that

$$F(A)=\int_A\varphi\,\mathrm{d}\mu$$

for each $A\in\mathcal{F}$. (See [5] for further details.)

Let E be a Banach space. Recall that the ***dual (space)*** E^* of E is the Banach space of all scalar-valued linear functions f on E with the norm given by $\|f\|=\sup\{|f(x)|\colon x\in E,\ \|x\|\leqslant 1\}$. The ***weak topology*** (respectively ***weak* topology***) in E (respectively E^*) is the weakest topology that makes the map $x\mapsto f(x)$ (respectively $f\mapsto f(x)$) continuous for each $f\in E^*$ (respectively $x\in E$). The ***unit ball*** B_E of the Banach space E is defined to be $B_E=\{x\in E:\|x\|\leqslant 1\}$. It is known that the dual Banach space E^* has the RNP if and only if $(B_{E^*},weak^*)$ is norm-fragmented, i.e., fragmented by the metric defined by the norm, and this is the case if and only if each separable Banach subspace of E has norm-separable dual. (See e.g., [3].) The original definition of RN-compactness goes as follows: A compact Hausdorff space K is RN-compact if and only if it is homeomorphic to a compact subset of $(E^*,weak^*)$ for some dual Banach space E^* with the RNP.

The equivalence of the two definitions of RN-compact spaces is given in [8], where it is also shown that a compact Hausdorff space is RN-compact if and only if it is homeomorphic to a norm-fragmented weak*-compact subset of some dual Banach space. The earliest occurrence of the class of RN-compact spaces is in [10], under different terminology. There, Reynov calls a weak*-compact subset K of a dual Banach space an **RN-set** if each finite ***Radon measure*** on $(K,weak^*)$ is supported on a norm-σ***-compact*** subset of K, and a compact Hausdorff space is said to be a "compact of RN-type" if it is homeomorphic to an RN-set in some dual Banach space. Since it can be shown that a weak*-compact subset of a dual Banach space is an RN-set if and only if it is norm-fragmented [6], Reynov's class of "compacta of RN-type" is exactly the class of RN-compact sets as given above.

The following characterization of RN-compactness is an abstraction of all the above and some more (see [8] and [4] for the proof). A compact Hausdorff space is RN-compact if and only if, for some index set Γ, it is homeomorphic to a compact subset K of the product space $[0,1]^{\Gamma}$ (provided with the product topology) that satisfies one (hence, all) of the following mutually equivalent conditions:

(i) K is fragmented by the uniform metric d, where $d(x,y)=\sup\{|x(\gamma)-y(\gamma)|\colon\gamma\in\Gamma\}$.

(ii) For each countable subset A of Γ, (K,d_A) is separable, where d_A is a ***pseudometric*** on K given by $d_A(x,y)=\sup\{|x(\alpha)-y(\alpha)|\colon\alpha\in A\}$.

(iii) K is ***Lindelöf*** with respect to the topology of uniform convergence on the family of countable subsets of Γ, i.e., the topology generated by the family $\{d_A\colon A\subset\Gamma,$ A is countable$\}$ of pseudometrics on K.

The class of RN-compact spaces includes the class of ***Eberlein compact*** spaces and that of ***scattered*** compact Hausdorff spaces. We list the properties of RN-compact spaces.

(a) A closed subspace of an RN-compact space is RN-compact.

(b) The product of countably many RN-compact spaces is RN-compact.
(c) An RN-compact space is ***sequentially compact***.
(d) An RN-compact space contains a metrizable dense ***G_δ-subset***.
(e) If K is an RN-compact space, then the unit ball of the dual $C(K)^*$ of $C(K)$ is RN-compact relative to the weak*-topology.
(f) A Hausdorff ***quotient*** of an RN-compact space is fragmented by some metric.
(g) If a compact Hausdorff space is RN-compact and ***Corson compact***, then it is Eberlein compact.

Properties (a)–(f) are proved in [8], and property (g) is due to Orihuela, Schachermayer and Valdivia [9]. The proof in [9] relies heavily on Banach space techniques. In connection with property (f), we mention that it is yet unknown whether a Hausdorff quotient of an RN-compact space is again RN-compact.

We mention two notions closely related to RN-compactness. The first one is due to Ribarska [11]. A topological space X is said to be **fragmentable** if it is fragmented by some metric on X, or equivalently X is fragmented by a non-negative function λ on $X \times X$ such that $\lambda(x, y) = 0$ if and only $x = y$. The class of fragmentable compact spaces is properly larger than that of RN-compact spaces (see, e.g., [9]); yet the former possesses many of the good properties of the latter. For instance, properties (a)–(e) remain true when "RN" is replaced by "fragmentable". The analog of property (f) is strengthened: the image of a fragmentable space under a ***perfect map*** is again fragmentable [11].

In [1, p. 104], our second notion is attributed to Reznichenko. A topological space (X, τ) is said to be **strongly fragmentable** if it is fragmented by a metric ρ that satisfies the following condition: whenever $x, y \in X$ and $x \neq y$, there are τ-neighbourhoods U and V of x and y respectively such that $\inf\{\rho(u, v)\colon u \in U,\ v \in V\} > 0$. Using some results in [8], it can be shown that the space X is strongly fragmented if and only if the space is fragmented by a lower semi-continuous non-negative function γ on $X \times X$ such that $\gamma(x, y) = 0$ if and only if $x = y$. Independently of Reznichenko, Arvanitakis [2] defines a compact Hausdorff space K to be **quasi Radon–Nikodým** (or simply, **quasi RN**) if it is fragmented by a lower semi-continuous function $f : K \times K \to [0, 1]$ such that $f(x, y) = 0$ if and only if $x = y$ and that $f(x, y) = f(y, x)$ for all $(x, y) \in K \times K$. From what is said above it is clear that the class of quasi RN-compact spaces is identical with the class of strongly fragmentable compact Hausdorff spaces. Each Hausdorff quotient of a quasi RN-compact space is again quasi RN-compact and consequently a Hausdorff quotient of an RN-compact space is quasi RN-compact (for a proof, see [8] or [2]). The result above, as well as the next one, is reported to be due to Reznichenko in [1]. Property (g) of RN-compact spaces is generalized as follows: Each quasi RN-compact space which is also Corson compact is Eberlein compact. The proof of this theorem in [2] is purely topological and shorter than the proof of property (g) in [9]. The main step of the proof in [2] is the following result, which is of independent interest: Let K be a quasi RN-space and let $\{(C_\gamma, U_\gamma)\colon \gamma \in \Gamma\}$ be an indexed family of pairs of subsets of K with C_γ compact, U_γ open and $C_\gamma \subset U_\gamma$. If $\{U_\gamma\colon \gamma \in \Gamma\}$ is point-countable, then $\{C_\gamma\colon \gamma \in \Gamma\}$ is σ-point-finite. Here, recall that an indexed family $\{A_\gamma\colon \gamma \in \Gamma\}$ of subsets of K is ***point-finite*** (respectively ***point-countable***) if for each $x \in K$ the set $\{\gamma \in \Gamma\colon x \in A_\gamma\}$ is finite (respectively countable) and the family is ***σ-point-finite*** if Γ can be written as $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ with $\{A_\gamma\colon \gamma \in \Gamma_n\}$ point-finite for each $n \in \mathbb{N}$. Finally we mention that it is unknown if a quasi RN-compact space is necessarily RN-compact.

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c-16 Corson Compacta

All spaces considered in this article are assumed to be Hausdorff. Every compact Hausdorff space is homeomorphic to a (closed) subspace of some ***Tychonoff cube*** I^A, where $I = [0, 1]$ is the closed unit interval, with the usual topology. The ***Σ-product*** ΣI^A of A copies of the closed interval $[0, 1]$ is the subspace of I^A consisting only of the points with (at most) countably many non-zero coordinates [E]. Notice that there is no restriction on the set A in this definition. In particular, no restriction is imposed on the cardinality of A.

A **Corson compact**um is a compact space which can be topologically embedded in ΣI^A, for some set A. The Σ-products of separable metrizable spaces were introduced by L.S. Pontryagin in his famous book on topological groups; he was looking for natural examples of countably compact, non-compact topological groups. Corson compacta were first considered by H.H. Corson [3].

This class of compacta has numerous links to other classes of compacta naturally arising in functional analysis. In particular, this class extends the class of ***Eberlein compacta***, and hence, contains all metrizable compacta [2].

The class of Corson compacta behaves nicely with regards to natural operations. In particular, the product of any countable family of Corson compacta is a Corson compactum, a closed subspace of a Corson compactum is again a Corson compactum, and every continuous image of a Corson compactum is a Corson compactum (S.P. Gul'ko, and E. Michael and M.E. Rudin, see [2]).

Corson compacta are very interesting from a purely topological point of view. Not only have they a natural and transparent definition in topological terms, but also the list of their most natural topological properties is very impressive.

Every Corson compactum is a ***Fréchet–Urysohn*** space: every point x in the closure of a set A can be reached by a sequence of points in A converging to x. This is so since the whole Σ-product has this property [2]. Every Corson compactum X is ***monolithic***. In particular, the closure of every countable subset A of X is a metrizable compactum. These two properties imply the third one: each Corson compactum is first countable at a dense set of points. However, while every Eberlein compactum has a dense metrizable subspace [1], not every Corson compactum has such a subspace [KV, Chapter 6]. The above statements imply that a Tychonoff cube I^A is Corson if and only if the set A is countable. It follows that the class of Corson compacta is not closed under arbitrary products.

N.N. Yakovlev [16] established a strong hereditary paracompactness type property of Corson compacta: every subspace of a Corson compactum is ***meta-Lindelöf***. G. Gruenhage showed that a compact space X is Corson if and only if the complement of the diagonal Δ in $X \times X$ is a meta-Lindelöf space [5]. This result should be compared to the following theorem of A.P. Kombarov [8]: a compact space X is Corson if and only if $(X \times X) \setminus \Delta$ is the union of a point-countable family of rectangular open subsets of $X \times X$. Of course, **rectangular** subsets of $X \times X$ are the sets of the form $U \times V$. A compactum X is Corson if and only if X has a point-countable ***T_0-separating*** cover by open F_σ-sets (see [2]).

Several important theorems on Corson compacta involve spaces of continuous functions. The fact, that a compact Hausdorff space X is Corson depends only on the topological structure of the space $C_p(X)$. Indeed, if X and Y are Hausdorff compacta, and $C_p(X)$ is homeomorphic to $C_p(Y)$, then X is Corson if and only if Y is Corson [2].

S.P. Gul'ko and, somewhat later, K. Alster and R. Pol proved that, for any Corson compactum X, the space $C_p(X)$ of continuous real-valued functions on X in the topology of pointwise convergence is Lindelöf (see [2]). Compare this statement with the following easy to prove fact: for a compact space X, the space $C(X)$ of continuous real-valued functions on X in the topology of uniform convergence is Lindelöf if and only if X is metrizable. Gul'ko also established that, for every Corson compactum X, all the iterated function spaces $C_p(X)$, $C_p(C_p(X)), \dots, C_{p,n}(X), \dots$ are Lindelöf [6]. Note, that this theorem is parallel to Sipacheva's theorem that, for any Eberlein compactum X, all the iterated function spaces $C_{p,n}(X)$ are Lindelöf Σ-spaces [14]. G.A. Sokolov established that $C_p(X)$ need not be a Lindelöf Σ-space for a Corson compactum X [9]. On the other hand, S.P. Gul'ko proved that if $C_p(X)$ is a Lindelöf Σ-space, and X is Hausdorff and compact, then X is Corson [6]. Because of this remarkable result, compact spaces X such that $C_p(X)$ is a Lindelöf Σ-space are called **Gul'ko compact**a.

After this Gul'ko's theorem, it was natural to conjecture that the Lindelöf property of $C_p(X)$ characterizes Gul'ko compacta among all compact spaces. However, R. Pol showed that this is not true [12]. With this goal in mind, he considered the following space constructed by M. Wage. For every limit ordinal $\alpha \in \omega_1$, fix a sequence ξ_i of isolated ordinals α_i converging to α. Taking away from ξ_i arbitrary finite subsets K, we obtain a neighbourhood system of the point α in a new topology. All non-limit ordinals are declared isolated with respect to this topology. The locally compact Hausdorff space obtained in this way is compactified by one point. The resulting space W is monolithic and has the countable tightness. It was shown by Pol that $C_p(W)$ is Lindelöf [12]. However, W is not a Corson compactum.

After this example, and keeping in mind Gul'ko's theorem that all iterated function spaces over a Corson compactum are Lindelöf, it was natural to ask whether the last condition guarantees that a compactum is Corson. Answering this question of S.P. Gul'ko, G.A. Sokolov constructed a delicate example of a compact space X such that X is not Corson while all iterated function spaces $C_{p,n}(X)$ are Lindelöf [15]. Note a result of O.G. Okunev: if X is a Gul'ko compactum then all iterated function spaces $C_{p,n}(X)$ are Lindelöf Σ-spaces [10, 2]. This generalizes a similar statement about Eberlein compacta [14]. A powerful method for constructing examples of Corson compacta with a desired combination of properties, based on the notion of an adequate family of sets, was developed by M. Talagrand and later was put to practice by A.G. Leiderman and G.A. Sokolov [9].

The above results on function spaces have various applications. For example, they show that Corson compacta cannot be characterized as monolithic compacta of countable tightness. Indeed, there exists a non-metrizable first-countable monolithic linearly ordered compact space X (Aronszajn continuum, see [KV, Chapter 6]). For this X, the space $C_p(X)$ is not Lindelöf, since every linearly ordered compact space X such that $C_p(X)$ is Lindelöf is metrizable, by a theorem of L.B. Nachmanson (see [2]). The above argument, incidentally, shows that every linearly ordered Corson compactum is metrizable [2].

There are several interesting results on Corson compacta and continuous maps. There exists a Corson compactum X such that $C_p(X)$ cannot be represented as a continuous image of a Lindelöf k-space [2, Chapter 4]. G. Debs [4] proved that if X is a Corson compactum then every continuous map f from any Baire space Y into $C_p(X)$ also remains continuous at every point of some G_δ-subset of Y when $C(X)$ is given the topology of the uniform convergence. There exists an non-trivial connection between Corson compacta and compacta of countable tightness, expressed in terms of maps: for every compactum X of countable tightness there exists an ***irreducible*** continuous map of X onto a Corson compactum Y (B.E. Shapirovskij, see [KV, Chapter 23]). This implies that every separable compact space of countable tightness can be mapped by an irreducible continuous map onto a metrizable compactum.

E.A. Reznichenko constructed a Corson compactum B with a remarkable combination of properties [13]. Here are some of them. The space $C_p(B)$ is a Lindelöf Σ-space, and even more, $C_p(B)$ is ***K-analytic***. Compact spaces X such that $C_p(X)$ is K-analytic are called **Talagrand compact**a [2]. Hence, X is a Gul'ko compactum, and even a Talagrand compactum. There is a point $b \in B$ such that B is the Stone–Čech compactification of the space $Y = B \setminus \{b\}$. The subspace $Y = B \setminus \{b\}$ is also pseudocompact and not closed in B. Compare this to the fact that in any Eberlein compactum every pseudocompact subspace is closed and compact. Since Y is pseudocompact, the space B is not ***bisequential*** (at b). Finally, B is a Talagrand compactum which is not a ***Radon–Nikodým compact***um. This can be seen from the following theorem of Reznichenko (see also [11]): a compact space X is Eberlein if and only if X is Radon–Nikodým and Corson [13]. This implies an earlier result of K. Alster: every ***scattered*** Corson compactum is Eberlein. Reznichenko also established that every hereditarily meta-Lindelöf ***fragmentable*** compactum X is Corson [13].

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c-17 Rosenthal Compacta

A real-valued function $f : X \to \mathbb{R}$ on a metrizable space X is of the **first Baire class** (or of **Baire class** 1) if f is a pointwise limit of a sequence of continuous functions on X. If X is completely metrizable then, by a theorem of Baire, the following conditions are equivalent for a function $f : X \to \mathbb{R}$:

(a) f is of the first Baire class,
(b) $f^{-1}(U)$ is an F_σ-set in X for every open $U \subset \mathbb{R}$,
(c) $f|K$ has a point of continuity for every non-empty closed $K \subset X$.

For a metrizable space X, by $B_1(X)$ we denote the space of all real-valued first Baire class functions on X, endowed with the ***topology of pointwise convergence***. A ***compact*** space K is called **Rosenthal compact** if K can be embedded in the space $B_1(X)$ for some separable ***completely metrizable*** space X; equivalently, K is homeomorphic to a subspace of the space $B_1(\omega^\omega)$, where ω^ω is the space of irrational numbers. The term Rosenthal compact was introduced by G. Godefroy in [3]. This class of spaces has some important relationships with Banach space theory. Odell and Rosenthal showed that, for a separable Banach space X, the unit ball $B_{X^{**}}$ in the ***second dual space*** X^{**} equipped with the ***weak* topology*** is a separable Rosenthal compactum if and only if X contains no subspace isomorphic to ℓ_1, see [KV, p. 1065]. Moreover, if the ***dual space*** X^* is not (norm) separable, then $B_{X^{**}}$ is a non-metrizable Rosenthal compactum.

All metrizable compacta are Rosenthal compact spaces. The following subsets of the space $B_1([0,1])$ are standard examples of separable non-metrizable Rosenthal compacta. The **Helly space** of all nondecreasing functions $f : [0,1] \to [0,1]$. The **split interval** S which can be described as a subset of the Helly space consisting of $\{0,1\}$-valued functions (also known as Alexandroff's **double-arrow space**). The space of all functions $f : [0,1] \to [0,1]$ of total variation at most 1.

Pol [9] constructed an example of a separable Rosenthal compact space which cannot be embedded into $B_1(M)$ for any metrizable compact space M.

Separable Rosenthal compacta form an important subclass of the class of Rosenthal compact spaces; there are numerous examples of such spaces and they behave in a more regular way than non-separable ones. Each separable Rosenthal compactum K is, in some sense, generated by a sequence of continuous functions on ω^ω. More precisely, K is homeomorphic to a compact subset of $B_1(\omega^\omega)$ which is the closure of a countable set consisting of continuous functions on ω^ω.

The ***Alexandroff duplicate*** $D(M)$ of every metrizable compactum M is a Rosenthal compact space; $D(M)$ is non-separable for uncountable M.

It is clear from the definition that every Rosenthal compactum has the cardinality and ***weight*** at most 2^ω. The class of Rosenthal compact spaces is closed under taking closed subsets and countable products.

A deep result of Bourgain, Fremlin and Talagrand [2, Theorem 3F], based on earlier work of Rosenthal [11], shows that the topology of Rosenthal compacta is determined by convergent sequences, all such compacta are ***Fréchet*** topological spaces. Pol proved that all separable Rosenthal compact spaces possess stronger sequential property; they are ***bisequential***, cf. [10].

A ***Hausdorff*** continuous image of a Rosenthal compactum may not be Rosenthal compact (Godefroy [3]). Let A be a subset of $[0,1]$ such that $[0,1] \setminus A$ is not an **analytic set**, i.e., $[0,1] \setminus A$ is not a continuous image of the irrationals ω^ω. If we identify all pairs of split points from A in the split interval S (i.e., we identify the characteristic function of the set $[t,1]$ with the characteristic function of $(t,1]$ for every $t \in A$), then the resulting quotient space is a ***first-countable*** compact space which is not Rosenthal compact. More generally, if every Hausdorff continuous image of a separable Rosenthal compactum K is Rosenthal compact, then K is metrizable (Godefroy and Talagrand [4]). However, a Hausdorff image of a separable Rosenthal compactum under an open continuous map is Rosenthal compact (Godefroy [3]).

Godefroy and Talagrand proved that, for a Rosenthal compactum K, the unit ball $B_{C(K)^*}$ of the dual $C(K)^*$ of the Banach space $C(K)$, endowed with the weak*-topology is Rosenthal compact. They also proved that a separable Rosenthal compactum K is metrizable if and only if $(B_{C(K)^*}, weak^*)$ contains no discrete subspace of the cardinality 2^ω, cf. [8, Theorem 3.2].

Godefroy [3] found an interesting characterization of the class of separable Rosenthal compacta in terms of function spaces. Given a space X and a dense subset $D \subset X$, by $C_D(X)$ we denote the space of all real-valued continuous functions on X equipped with the topology of pointwise convergence on D. We use standard notation $C_p(X)$ for $C_X(X)$. A separable Hausdorff compact space K is Rosenthal compact if and only if, for every countable dense set $D \subset K$, the space $C_D(K)$ is analytic. Also in the non-separable case, the property of being Rosenthal compact is, in some sense, determined by the pointwise or the ***weak topology*** of the function space $C(K)$. If L is a compact space such that the function space $C_p(L)$ is a continuous image of the space $C_p(K)$, for a Rosenthal compact space K, then L is also Rosenthal compact (Marciszewski [5]). The same result is true for the function spaces equipped with the weak topology instead of the pointwise convergence topology [5].

Todorčević [12] has proved the following important property of Rosenthal compacta. All such spaces have dense metrizable subsets. This, in particular, implies that every ***ccc*** Rosenthal compact space is separable. Another consequence of Todorčević's theorem is an earlier result of Bourgain that each Rosenthal compactum has a dense subspace consisting of points with countable base of neighbourhoods. Let us mention that the split interval S is an example of a Rosenthal compactum without a metrizable dense G_δ-subset because every metrizable subspace of S is countable.

In his paper [12] Todorčević made a deep insight into the structure of Rosenthal compacta, especially separable ones. He proved that every non-metrizable Rosenthal compactum contains either an uncountable discrete subspace or a topological copy of the split interval S. Each Rosenthal compact space without an uncountable discrete subspace is an at most two-to-one continuous preimage of a metrizable compactum (Todorčević calls such a preimage as a **premetric compactum of degree at most 2**).

For every separable Rosenthal compactum K one of the following holds [12]:

(a) K contains a discrete subspace of cardinality 2^ω,
(b) K is a premetric compactum of degree at most 2.

Each separable premetric compactum of degree at most 2 which is Rosenthal compact satisfies one of the following alternatives [12]:

(a) K is metrizable,
(b) K contains a topological copy of the split interval S,
(c) K contains a topological copy of the Alexandroff duplicate $D(C)$ of the Cantor set C.

Let us note that the above result cannot be generalized for all separable premetric compacta of degree at most 2 (a quotient space of the split interval S, similar to the space described before, may serve as a counterexample).

For a separable Rosenthal premetric compactum of degree at most 2 the following conditions are equivalent [12]:

(a) K is **hereditarily separable**, i.e., every subspace of K is separable,
(b) K is ***perfectly normal***, i.e., every closed subset of K is a G_δ-set in K,
(c) K contains no copy of $D(C)$.

In the above result, the equivalence of (a) and (b) belongs to Pol [8, Theorem 3.3].

Todorčević [12] also proved that if x is a non-G_δ-point of a separable Rosenthal compactum K, then K contains a copy of the ***Alexandroff compactification*** $A(2^\omega)$ of a discrete space of size 2^ω, with x as its point at infinity. It follows that the ***character*** of a point of a separable Rosenthal compactum is either countable or is equal to 2^ω (previously mentioned results demonstrate that the same alternative holds for the cardinality and the weight of K).

Finally, let us indicate some relationships between the class of Rosenthal compacta and other classes of compact spaces connected with Banach space theory, the classes of ***Eberlein compact***a, ***Talagrand compact***a, ***Gul'ko compact***a and ***Corson compact***a (listed by the inclusion ordering) and the class of ***Radon–Nikodým compact***a. Mercourakis [6] proved that every Gul'ko compactum of weight less than or equal to 2^ω is Rosenthal compact. There exist Corson compacta which are Rosenthal compact but not Gul'ko compact (Argyros–Mercourakis–Negrepontis and Alster–Pol, see [1, Theorem 4.4]). Todorčević [KV, p. 289] constructed a Corson compactum K of weight 2^ω without a dense metrizable subspace, hence K is not Rosenthal compact. Every separable non-metrizable Rosenthal compactum (like the split interval S) is not Corson compact. The space $\omega_1 + 1$ with the order topology is Radon–Nikodým compact but not Rosenthal compact (it is not a Fréchet space). The split interval S is not Radon–Nikodým compact.

Recently, Mercourakis and Stamati [7] have introduced a new class of Rosenthal–Banach compact spaces, which extends the class of Rosenthal compacta and, at the same time, the class of Gul'ko compacta. For a metrizable space X and a Banach space E, let $B_1(X, E)$ be the family of all first Baire class maps from X into E (i.e., the pointwise limits of sequences of continuous maps $f_n : X \to (E, \|\cdot\|)$). We call a compactum K **Rosenthal–Banach compact** if there is a separable completely metrizable space X and a Banach space E such that K is homeomorphic to a subspace of $B_1(X, E)$ equipped with the Tychonoff product topology of $(E, weak)^X$. Separable Rosenthal–Banach compact spaces are Rosenthal compact. In general, this new class is essentially larger (there is no cardinal restriction on the weight of these compacta). Yet, Rosenthal–Banach compact spaces possess many important properties of Rosenthal compacta, see [7].

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c-18 Eberlein Compacta

Eberlein compacta were introduced to mathematics as compact subspaces of Banach spaces with the ***weak topology***. They have a natural description in terms of spaces of functions with the ***pointwise topology***: a compact space Y is an Eberlein compactum if and only if there exists a compact (Hausdorff) space X such that Y is homeomorphic to a subspace of $C_p(X)$. An equivalent description [2]: a compact space X is an Eberlein compactum if and only if there exists a compact subspace F of $C_p(X)$ such that the functions in F separate the points of X. Yet another characterization [1]: a compact space X is an Eberlein compactum if and only if in $C_p(X)$ there exists a dense σ-compact subspace. The advantage of this criterion is that it describes when a compactum is an Eberlein compactum by a topological property of $C_p(X)$.

Arguments involving compact sets of functions play a key role in functional analysis. It turns out that compact parts of $C_p(X)$, where X is compact, have much better convergence properties than $C_p(X)$ itself. In particular, convergence in any Eberlein compactum can be described in terms of usual (countable) convergent sequences, that is, a point x is in the closure of a set A if and only if some sequence $(a_n: n \in \omega)$ of points of A converges to x. Such spaces are called ***Fréchet–Urysohn***. On the other hand, for a compact space X, $C_p(X)$ is Fréchet–Urysohn if and only if the space X is ***scattered*** [2]. Since every dyadic compactum of countable tightness is metrizable [E], it follows that every Eberlein compactum which is also a dyadic compactum must be metrizable [2].

Another important property of Eberlein compacta: the closure of any countable subset in an Eberlein compactum is metrizable. Moreover, if A is any subset of an Eberlein compactum X, then the weight of the closure of A in X does not exceed the cardinality of A. Spaces with this property are called **monolithic** [2]. The importance of monolithicity can be seen from the following simple observation: every separable monolithic compactum is metrizable. Therefore, every separable Eberlein compactum is metrizable. It follows that any Tychonoff or Cantor cube of uncountable weight can serve as an example of a non-Eberlein compactum. In fact, H. Rosenthal proved that every Eberlein compactum with the countable ***Souslin number*** is metrizable [2] (see also [3]). This considerably improves the statement on metrizability of separable Eberlein compacta and implies that every non-metrizable compact topological group is a non-Eberlein compactum.

Every metrizable compactum is Eberlein, though this is not immediately clear from the definition [2]. The simplest example of a non-metrizable Eberlein compactum is the ***one-point compactification*** of an uncountable discrete space. The class of Eberlein compacta is "categorically" nice. Indeed, any closed subspace of an Eberlein compactum is an Eberlein compactum, and the product of any countable family of Eberlein compacta is an Eberlein compactum. We already noted above that the last statement does not expand to uncountable products.

Every Hausdorff continuous image of an Eberlein compactum is again an Eberlein compactum. This is a rather deep result, not so easy to prove (see [7]).

Monolithicity of Eberlein compacta has another nontrivial corollary. It is known that if a monolithic compact Hausdorff space X is Fréchet–Urysohn, then X is ***first-countable*** at a dense set of points [2]. It follows that every Eberlein compactum is first-countable at a dense set of points. Hence, each ***homogeneous*** Eberlein compactum X is first-countable and the cardinality of such X does not exceed 2^{ω}. The ***Alexandroff double circle*** is an example of a non-metrizable Eberlein compactum satisfying the first axiom of countability at every point. J. van Mill constructed a homogeneous non-metrizable Eberlein compactum [8]. Note that the double circle is the union of two metrizable subspaces (the two individual circles). M.E. Rudin proved that every compactum which is the union of two metrizable subspaces is an Eberlein compactum. However, a compact Hausdorff space which is the union of three metrizable subspaces need not be Eberlein (consider the one-point compactification of a ***Ψ-space***; the space obtained is not monolithic).

Eberlein compacta also enter the picture in connection with the following theorem of Grothendieck: if X is a countably compact space then, for every relatively countably compact subset A of $C_p(X)$, the closure of A in $C_p(X)$ is compact. Note that a subset A of a space Z is said to be **relatively countably compact** or **countably compact in Z** if every infinite subset of A has a point of accumulation in Z. It turns out that, under these assumptions, the closure of A in $C_p(X)$ is an Eberlein compactum [4, 11, 2]. From the fact that every Eberlein compactum is Fréchet–Urysohn it follows that every countably compact subset of $C_p(X)$ is compact whenever X is countably compact. However, even more is true: if P is a pseudocompact subspace of $C_p(X)$ where X is countably compact then P is compact and, therefore, closed in $C_p(X)$. A proof of this result is based on the following convergence property of Eberlein compacta: for each point x of any Eberlein compactum X there exists a sequence $(U_n: n \in \omega)$ of non-empty open subsets of X converging to x (D. Preiss and P. Simon [10]).

How "compact" can $C_p(X)$ be for a compact X? Obviously, $C_p(X)$ is compact only if X is empty. N.V. Velichko showed (see [2]) that $C_p(X)$ is σ-compact if and only if X is finite (then, and only then, $C_p(X)$ is locally compact as well). In this connection the next result is especially interesting: for every Eberlein compactum X, the space $C_p(X)$ is a $K_{\sigma\delta}$, that is, $C_p(X)$ can be represented as the intersection of a countable family of σ-compact subspaces of R^X

[1, 2]. Therefore, for every Eberlein compactum X, $C_p(X)$ is a ***Lindelöf Σ-space*** and even a ***K-analytic space*** (M. Talagrand [13]). Unfortunately, the $K_{\sigma\delta}$ property of $C_p(X)$ does not characterize Eberlein compacta; M. Talagrand constructed a compact Hausdorff space X such that $C_p(X)$ is a $K_{\sigma\delta}$, and X is not an Eberlein compactum [14]. However, a zero-dimensional compactum X is Eberlein if and only if the space $C_p(X, D)$ of all continuous maps of X in the discrete two-point space $D = \{0, 1\}$, in the pointwise topology, is σ-compact [2, 4.6.4]. From Talagrand's theorem (see above) it follows that every linearly ordered compactum which is also an Eberlein compactum must be metrizable. Indeed, it was shown by L.B. Nahmanson that every linearly ordered compact space X, such that $C_p(X)$ is Lindelöf, is metrizable (see [2]).

O.G. Okunev constructed a Tychonoff space X such that X is the union of a countable family of Eberlein compacta and the space $C_p(X)$ is not Lindelöf [9].

An original description of Eberlein compacta was found by D. Amir and J. Lindenstrauss (see [6, 2, 3]): a compact space X is an Eberlein compactum if and only if there exists a compact subspace F of $C_p(X)$ such that functions in F separate points of X, and F is homeomorphic to the one-point (Alexandroff) compactification of some discrete space. From this characterization it follows that a compact space Y is an Eberlein compactum if and only if it can be embedded in a **Σ_*-product** of real lines. Recall that the Σ_*-product of real lines is the set of all points x of R^τ (for some τ) such that, for every $\varepsilon > 0$, the number of coordinates of x not in the interval $(-\varepsilon, \varepsilon)$ is finite. This "concrete" description of the class of Eberlein compacta is instrumental in establishing an "inner" criterion for a compact space to be an Eberlein compactum. A family γ of subsets of a space X is called **T_0-separating** if whenever x and y are distinct points of X, there exists $V \in \gamma$ containg exactly one of the points x and y. Rosenthal proved (see [2, 3]) that a compact space X is an Eberlein compactum if and only if there exists a T_0-separating σ-point-finite family of ***cozero sets*** in X.

It is well known that a compact Hausdorff space X is metrizable if and only if the diagonal $\Delta_X = \{(x, x): x \in X\}$ is a G_δ-subset of $X \times X$ [E]. Clearly, the last condition is equivalent to σ-compactness of the space $X \times X \setminus \Delta_X$. G. Gruenhage [5] gave a parallel characterization of Eberlein compacta: a compact Hausdorff space X is an Eberlein compactum if and only if the space $X \times X \setminus \Delta_X$ is σ-metacompact.

An interesting natural question is: when does a Tychonoff space X have a Hausdorff compactification which is an Eberlein compactum? It was shown in [2] that every metrizable space has such compactification. On the other hand, not every ***Moore space*** has an Eberlein compactification, since there are separable non-metrizable Moore spaces.

With every space X one can associate a sequence of function spaces in pointwise topology. Indeed, put $X = C_{p,0}(X)$, and $C_{p,n}(X) = C_p(C_{p,n-1}(X))$, for positive $n \in \omega$. The spaces $C_{p,n}(X)$ are called **iterated function spaces** over X. A general impression is that with the growth of n the complexity of the space $C_{p,n}(X)$ grows. However, if X is a metrizable compactum, then each $C_{p,n}(X)$ has a countable network and is, therefore, a hereditarily Lindelöf, hereditarily separable space. O.V. Sipacheva established a similar, but much deeper, result for Eberlein compacta: if X is an Eberlein compactum, then every iterated function space $C_{p,n}(X)$ is a Lindelöf Σ-space [12] (see also [2]).

The last statement should be compared to the following fact: if $C_{p,n}(X)$ is a $K_{\sigma\delta}$-space (or a continuous image of such a space), for some $n \geqslant 2$, then the space X is finite [2].

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c-19 Topological Entropy

Let X be a ***compact metric space***, and let $f : X \to X$ be a ***continuous map***. By iterating this map one obtains a **dynamical system** (or rather a **discrete dynamical system**). Denote the nth **iterate** of f by f^n (where f^0 is the identity). An **orbit** is the set $\{x, f(x), f^2(x), f^3(x), \ldots\}$ (sometimes one considers it a sequence rather than a set). It is important to be able to measure how complicated the dynamics of f is: how many very different orbits it has, how fast it "mixes" together various sets, etc. To some extent this can be measured by **topological entropy**.

The setup chosen above is the simplest one. In general, the space need not be metric, the iterates of one map may be replaced by an action of a fairly general semigroup or group (for instance, $\mathbb{Z}^n$ or $\mathbb{R}^n$), etc. However, the basic ideas are similar in all cases.

The notion of topological entropy is analogous to the notion of metric entropy (see, e.g., [4, 8, 6]; it is also called **measure entropy** or **measure theoretical entropy**). If μ is a ***probability measure*** on X, invariant for f (i.e., such that $\mu(f^{-1}(A)) = \mu(A)$ for every measurable set A), then the metric entropy measures how complicated the system is from the measure theoretical point of view. In particular, the metric entropy does not "see" parts of the dynamics concentrated on sets of measure zero. This is very different from topological entropy, which can be influenced by the parts of the dynamics concentrated on very small sets. It is known that the topological entropy of f is equal to the supremum of the metric entropies of f over all probability f-invariant measures (see, e.g., [4, 8, 6]; this is the so-called **Variational Principle**). In fact, the definition of topological entropy was constructed in order to satisfy the Variational Principle, although it took several years to prove that this is really so. Sometimes positive topological entropy is used as a definition of **chaos**, although there are other definitions, not equivalent to this one.

The original definition of the topological entropy $h(f)$ of $f : X \to X$, given by R.L. Adler, A.G. Konheim and M.H. McAndrew [1], which is valid on all compact topological spaces, is as follows. For ***open covers*** $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ of X denote by $\bigvee_{i=1}^{n} \mathcal{A}_i$ the family of all nonempty intersections $A_1 \cap A_2 \cap \cdots \cap A_n$, where $A_i \in \mathcal{A}_i$ for all i. Note that $\bigvee_{i=1}^{n} \mathcal{A}_i$ is also an open cover.

For an open cover $\mathcal{A}$ of X denote $f^{-i}(\mathcal{A}) = \{f^{-i}(A): A \in \mathcal{A}\}$ and $\mathcal{A}^n = \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A})$. For each i, $f^{-i}(\mathcal{A})$ is an open cover, so $\mathcal{A}^n$ is also an open cover. Next, denote by $\mathcal{N}(\mathcal{A})$ the minimal possible cardinality of a **subcover** chosen from $\mathcal{A}$ (i.e., a subset of $\mathcal{A}$ which is also a cover of X).

It is customary to use logarithms in the definition. Usually the base of logarithms is chosen either as 2 or as e. The limit

$$h(f, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}\big(\mathcal{A}^n\big) \tag{1}$$

exists and is equal to the infimum of $(1/n) \log \mathcal{N}(\mathcal{A}^n)$. Clearly, $h(f, \mathcal{A}) \geqslant 0$. The number $h(f, \mathcal{A})$ is the (topological) entropy of f on the cover $\mathcal{A}$. Now we can define the (topological) entropy of f as

$$h(f) = \sup h(f, \mathcal{A}), \tag{2}$$

where supremum is taken over all open covers $\mathcal{A}$ of X. The entropy of f is non-negative. In this definition the ***metric*** in X was not used, so if one replaces the metric by an equivalent one, the entropy will not change.

There are also other definitions of topological entropy, equivalent to the one given above. They were introduced by R. Bowen [3]. Denote the metric in X by d. Then for each $n \geqslant 1$ the function d_n given by

$$d_n(x, y) = \max_{0\leqslant i\leqslant n-1} d\big(f^i(x), f^i(y)\big)$$

is a metric, equivalent to d. A finite set $E \subset X$ is (n, ε)**-separated** if $d_n(x, y) > \varepsilon$ for all $x, y \in E$. It is (n, ε)**-spanning** if for every $x \in X$ there exists $y \in E$ such that $d_n(x, y) \leqslant \varepsilon$. Define $s_n(f, \varepsilon)$ as the maximal cardinality of an (n, ε)-separated set and $r_n(f, \varepsilon)$ as the minimal cardinality of an (n, ε)-spanning set. Then set

$$\overline{s}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(f, \varepsilon),$$

$$\overline{r}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r_n(f, \varepsilon).$$

It is not difficult to show that each of the quantities defined above is equal to the topological entropy of f.

The definition through the (n, ε)-separated sets offers a nice interpretation of topological entropy. Suppose that we have a magnifying glass through which we can distinguish two points if and only if they are more than ε-apart. If we know the pieces of the orbits of points $x, y \in X$ of length n then we can distinguish between x and y using our magnifying glass if and only if $d_n(x, y) > \varepsilon$. Therefore $s_n(f, \varepsilon)$ counts how many distinct points of the space we can see if we know pieces of orbits of length n. Then we take the exponential growth rate with n of this quantity, and finally go to the limit as we take better and better magnifying glasses. The result we get is the topological entropy of f.

Let us give several examples. If f is an isometry then $h(f) = 0$. If the space X is finite, the entropy of any map $f : X \to X$ is 0. To give an example of a map with positive entropy, let us take the set $S = \{1, 2, \ldots, s\}$ with $s > 1$. Define $\Sigma = \prod_{-\infty}^{\infty} S$ and $\Sigma_+ = \prod_{0}^{\infty} S$. More precisely, $\Sigma = \prod_{i=-\infty}^{\infty} S_i$ and $\Sigma_+ = \prod_{i=0}^{\infty} S_i$, where $S_i = S$ for each i. Thus, the elements of Σ are the doubly infinite sequences $(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$ with $x_i \in S$ for all i, and the elements of Σ_+ are the usual one-sided sequences $(x_0, x_1, x_2, \dots)$ with $x_i \in S$ for all i. Those spaces are considered with the ***product topology***, where the topology in S is discrete. One can easily see that both spaces are homeomorphic to the ***Cantor set***.

Define a **shift** σ on Σ and Σ_+ as the shift by one to the left. This means that $\sigma(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$. To write the formula for σ on Σ is more difficult. For this it is necessary to introduce notation for the points of Σ which shows where the 0th coordinate is. Namely, we shall write $x = (\dots, x_{-2}, x_{-1}, \overset{*}{x}_0, x_1, x_2, \dots)$ if the 0th coordinate of x is x_0. With this notation we can write $\sigma(\dots, x_{-2}, x_{-1}, \overset{*}{x}_0, x_1, x_2, \dots) = (\dots, x_{-1}, x_0, \overset{*}{x}_1, x_2, x_3, \dots)$. Usually σ is called the **full s-shift**.

The entropy of the full s-shift is $\log s$. One gets this number as a limit of $\frac{1}{n}\log c_n$, where c_n is the number of possible sequences of length n. If X is a closed **invariant subset** (that is, $\sigma(X) \subset X$) of Σ or Σ_+, then $\sigma|_X : X \to X$ is a **subshift**. Its entropy can be computed in the same way, that is as the limit of $\frac{1}{n}\log c_n$. This time to get c_n one counts only those sequences of length n that appear in points of X (remember that those points are infinite sequences).

An important class of subshifts are **subshifts of finite type** (also called **topological Markov chains**). Such a subshift is determined by an $s \times s$ matrix M with entries 0 and 1. The space X consists of those sequences (x_i) for which the (x_i, x_{i+1})th entry of M is 1 for every i. Then the number c_n is equal to the sum of entries of M^{n-1}, so the entropy is equal to the logarithm of the **spectral radius** of M (the largest modulus of an eigenvalue of M).

Let us list the basic properties of topological entropy. For any integer n we have $h(f^n) = |n| h(f)$ (if f is a homeomorphism, one can consider also $n < 0$). If Y is a closed invariant subset of X (where $f : X \to X$) then $h(f|_Y) \leqslant h(f)$. There is an important special case where equality holds. A point $x \in X$ is **wandering** if there exists a neighbourhood U of x such that $U \cap f^n(U) = \emptyset$ for every $n > 0$; otherwise x is **nonwandering**. Denote the set of nonwandering points of f by $\Omega(f)$. It is closed and invariant. We have $h(f|_{\Omega(f)}) = h(f)$. This makes it possible to produce many examples of maps of zero entropy. For instance, if f is a homeomorphism of a compact interval then every nonwandering point of f is a ***fixed point*** of f^2, and thus $h(f) = 0$.

If X, Y are compact metric spaces and $f : X \to X$, $g : Y \to Y$ are continuous maps then f and g are **conjugate** if there is a homeomorphism $\varphi : X \to Y$ such that $\varphi \circ f = g \circ \varphi$. Conjugate maps have the same dynamical properties, and in particular $h(f) = h(g)$. Thus, topological entropy is an ***invariant*** of conjugacy. If φ is not necessarily a homeomorphism, but is continuous and onto, then g is a **factor** of f and f is **semiconjugate** to g. In this case $h(f) \geqslant h(g)$. If $f \times g$ is the **product** of f and g, that is $f \times g : X \times Y \to X \times Y$ is given by $(f \times g)(x, y) = (f(x), g(y))$, then $h(f \times g) = h(f) + h(g)$ (although the proof of this fact is rather easy, the first two proofs in the literature were incorrect).

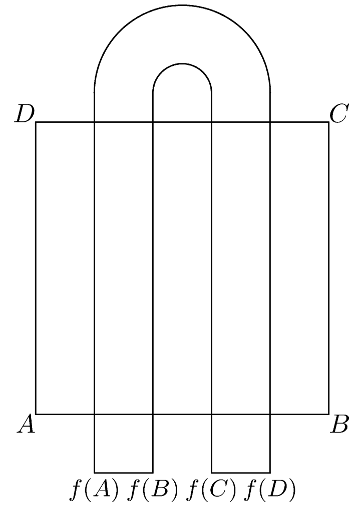

Fig. 1. Smale's horseshoe

Further important examples of maps with positive entropy are those with horseshoes. The classical **Smale horseshoe** looks as follows. A square $ABCD$ is stretched, bent as a horseshoe and reinserted as in Figure 1.

This construction is local; one can extend this map to a homeomorphism of a larger compact set into itself. However, the set K of points whose whole orbit (forward and backward; that is for f and for f^{-1}) stays in the square $ABCD$ is compact and invariant. Moreover, our homeomorphism restricted to K is conjugate to the full 2-shift. It follows that its entropy is at least $\log 2$. A general **s-horseshoe** is a similar map. The essence of this construction is that there are s disjoint subsets of the set Z (Z is the square in Smale's horseshoe) that are mapped across Z in such a way that the set of those points for which the pieces of the orbits of length n are contained in Z, consists of s^n disjoint closed sets. In the limit one gets a closed invariant set such that the map restricted to this set is semiconjugate to the full s-shift. This type of construction makes it possible to produce examples of homeomorphisms (or even diffeomorphisms of class C^∞) with arbitrarily large entropy. If we do not care about smoothness, we can even get infinite entropy. With smoothness this is impossible. If f maps a d-dimensional ***manifold*** into itself and is ***Lipschitz continuous*** with constant $L \geqslant 1$ then $h(f) \leqslant d \log L$.

This notion of horseshoes works very well for maps of an interval into itself (see, e.g., [2]). Namely, if I is a compact interval and $f : I \to I$ a continuous map, then there exist sequences of numbers (s_n) and (k_n) such that f^{k_n} has an s_n-horseshoe and $\lim_{n\to\infty}(1/k_n)\log s_n = h(f)$. In other words, the entropy of f is given by horseshoes for iterates of f. The same is true for $C^{1+\varepsilon}$ diffeomorphisms in dimension 2. Note that horseshoes persist under small perturbations, so if the entropy is given by horseshoes then it is ***lower semi-continuous*** as a function of a map. Without any special restrictions on the class of maps considered, one cannot prove it ***upper semi-continuous***, since small local horseshoes can be created by small perturbations. As an example of such special conditions we can mention the case when the map is a **unimodal** interval map (with one interior local extremum) and has positive entropy. Then small perturbations in the class of unimodal maps cannot create jumps of entropy. For more about topological entropy for interval maps, see for instance [2]. On manifolds, topological entropy is upper semi-continuous in the C^∞ topology.

One can also ask how the global properties of the map influence its topological entropy. A relevant example here is an **algebraic toral endomorphism**. If $L_A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation given by a matrix A with integer entries then $L_A(\mathbb{Z}) \subset \mathbb{Z}$, so L_A induces a continuous map f_A of the n**-dimensional torus** $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ into itself. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A (complex, repeated as many times as their multiplicities are), then $h(f_A) = \sum_{i=1}^n \max(0, \log|\lambda_i|)$. This number is equal also to the logarithm of the maximum of spectral radii of the transformations induced in ***homology groups*** of $\mathbb{T}^n$ by f_A. This led to the so called **Entropy Conjecture**, that for all maps of class C^1 on compact manifolds the topological entropy is larger than or equal to the logarithms of spectral radii of the induced transformations in homologies (M. Shub [7]). This conjecture has been verified for some classes of maps and some homologies, for instance maps of class C^∞, the first homology group (even the ***fundamental group***; here one does not need smoothness), the highest homology group (then the spectral radius is the degree of the map), or maps of tori (here one does not need smoothness either). In general, smoothness is important. A simple example of a continuous map of the sphere with degree 2 and entropy 0 can be obtained by taking the complex map $z \mapsto 2z^2/|z|^2$ on the Riemann sphere (each point except 0 and ∞ is wandering, so the entropy is 0). More on the Entropy Conjecture can be found, e.g., in [5] or [6].

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c-20 Function Spaces

Given topological spaces X and Y we use $C(X,Y)$ to denote the set of all ***continuous maps*** from X to Y. One can define various topologies on $C(X,Y)$ – one can always take the ***discrete*** and ***indiscrete*** topologies – but for a workable theory of function spaces there should be some relation with the given topologies on X and Y.

1. Desirable topologies

One approach starts with the set Y^X of *all* maps from X to Y and the observation that there is a natural bijection between $Y^{X\times Z}$ and $(Y^X)^Z$: to $f:X\times Z\to Y$ associate the map $\Lambda(f):Z\to Y^X$, defined by

$$\Lambda(f)(z)(x)=f(x,z).$$

The best one can hope for is that Λ induces a bijection between $C(X\times Z,Y)$ and $C(Z,C(X,Y))$. This problem splits naturally into two subproblems and hence into two definitions.

A topology on $C(X,Y)$ is said to be **proper** [E] or **splitting** [2] if for every space Z the map Λ maps $C(X\times Z,Y)$ into $C(Z,C(X,Y))$. If, conversely, Λ^{-1} always maps $C(Z,C(X,Y))$ into $C(X\times Z,Y)$ then the topology is called **admissible** [E], **jointly continuous** [Ke, N] or **conjoining** [2]. A topology that has both properties is called **acceptable** [E].

Every topology ***weaker*** than a proper topology is again proper and every topology ***stronger*** than an admissible topology is again admissible. Every proper topology is weaker than every admissible topology, hence there can be only one acceptable topology.

Also, a topology on $C(X,Y)$ is admissible iff it makes the evaluation map $(f,x)\mapsto f(x)$ from $C(X,Y)\times X$ to Y continuous. Furthermore, the join of all proper topologies is proper, hence there is always a largest proper topology on $C(X,Y)$.

2. The topology of pointwise convergence

The **topology of pointwise convergence** τ_p is simply the ***subspace topology*** that $C(X,Y)$ receives from the ***product topology*** on Y^X. The name comes from the fact that a ***net*** $(f_\alpha)_{\alpha\in D}$ converges with respect to τ_p iff it converges pointwise. In keeping with the other articles on this topology, we write $C_p(X,Y)$ to indicate that we use τ_p. This topology is proper but in general not admissible: $g\in C(Z,C_p(X,Y))$ means that $\Lambda^{-1}(g)$ is ***separately continuous***, whereas $\Lambda^{-1}(g)\in C(X\times Z,Y)$ means that it is ***jointly continuous***.

3. The topology of uniform convergence

If Y is a ***metric space*** or, more generally, a ***uniform space*** then one can define on $C(X,Y)$ the **topology of uniform convergence** τ_u, which can be defined by stipulating that a net of functions converges with respect to τ_u iff it **converges uniformly**, i.e., in the metric case $f_\alpha\to f$ iff for every $\varepsilon>0$ there is an α such that $d(f_\beta(x),f(x))<\varepsilon$ whenever $\beta\geqslant\alpha$ and $x\in X$. In the case of a uniform space this becomes: for every ***entourage*** U there is an α such that $(f_\beta(x),f(x))\in U$ whenever $\beta\geqslant\alpha$ and $x\in X$. A ***local base*** at f is given by sets of the form $B(f,U)=\{g\colon (f(x),g(x))\in U \text{ for all } x\}$, where U runs through the family of entourages.

This topology is admissible but in general not proper: Consider $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by $f(x,y)=xy$; the map $\Lambda(f):\mathbb{R}\to C_u(\mathbb{R},\mathbb{R})$ is not continuous.

4. The compact-open topology

The previous topologies are, in general, not acceptable. This is, to some extent, to be expected as neither depends on the topology of X (though the *set* $C(X,Y)$ does depend on X). The **compact-open topology** τ_c does depend on the topologies of both X and Y. It is defined by specifying a ***subbase***: we take all sets of the form

$$[K,O]=\{f\colon f[K]\subseteq O\}$$

where K runs through the ***compact*** subsets of X and O through the open sets of Y. We write $C_c(X,Y)$ to indicate that $C(X,Y)$ carries the compact-open topology.

The compact-open topology is always proper; it is admissible and hence acceptable if X is ***locally compact*** and ***Hausdorff***. Local compactness is the crucial property as one can show that $C(\mathbb{Q},\mathbb{R})$ carries no acceptable topology and, stronger, if X is ***completely regular*** and $C(X,\mathbb{R})$ carries an acceptable topology then X is locally compact.

5. Properties and relations

Any property that is ***productive*** and ***hereditary*** is transferred from Y to $C_p(X,Y)$, this includes the separation axioms up to and including complete regularity. The same holds for $C_c(X,Y)$; this follows because $\tau_p\subseteq\tau_c$ (for properties below ***regularity***) but requires extra proof for the other properties.

If X is discrete then $C(X,Y)=Y^X$, and τ_p and τ_c coincide with the product topology; this shows that normality and stronger properties in general do not carry over.

In case Y is a metric (or uniform) space then τ_c may also be described as the **topology of uniform convergence on compact subsets**: for every f the sets of the form

$$B(f, K, \varepsilon) = \{g\colon (\forall x \in K)(d(f(x), g(x)) < \varepsilon)\}$$

where K is compact in X and $\varepsilon > 0$ form a local base for τ_c at f (with obvious modifications in case of a uniform space). In particular, if X is compact the compact-open topology and the topology of uniform convergence coincide.

One obtains a local base also if K runs through a ***cofinal*** family of compact sets. This means that, if Y is a metric space, τ_c is a ***first-countable*** topology if X is **hemicompact**, which means that there is a countable cofinal family of compact sets; in fact τ_c is even metrizable. In general the ***character*** of τ_c is the product of the cofinality of the family of compact sets of X and the ***weight*** of the uniform space Y.

The compact-open topology makes $C(X, \mathbb{R})$ into a ***topological group***; this group is ***complete*** in its natural ***uniformity*** iff the space X is a k_R***-space***.

6. Compactness in function spaces

Compactness is a very useful property and it is desirable to have characterizations of compactness for subsets of $C(X, Y)$. The classical **Arzèla–Ascoli Theorem** states that a subset F of $C([0, 1], \mathbb{R})$ has a compact closure with respect to τ_u if it is equicontinuous and bounded. This theorem admits various generalizations.

If Y is a uniform space we define a subset F of $C(X, Y)$ to be **equicontinuous** if for every entourage U and every $x \in X$ there is one neighbourhood O of x such that $(f(x), f(y)) \in U$ for all $f \in F$ and $y \in O$. Then, if X is a ***k-space*** a closed subset F of $C_c(X, Y)$ is compact iff it is equicontinuous and for each x the set $\{f(x)\colon f \in F\}$ has compact closure.

A similar result holds for regular Y; one has to replace equicontinuity by even continuity, where a subset F of $C_c(X, Y)$ is **evenly continuous** if for every $x \in X$, every $y \in Y$ and every neighbourhood V of y there are neighbourhoods U and W of x and y respectively such that for every f the implication $f(x) \in W \Longrightarrow f[U] \subseteq W$ holds.

As noted above, if Y is metrizable and X is hemicompact then $C_c(X, Y)$ is metrizable. To define a metric one takes a bounded metric d on Y and a cofinal family $\{K_n\colon n \in \mathbb{N}\}$ of compact sets in X. For each n the formula $d_n(f, g) = \sup\{d(f(x), g(x))\colon x \in K_n\}$ defines a ***pseudometric*** in $C(X, Y)$. The sum $\rho = \sum_n 2^{-n} d_n$ is a metric on $C(X, Y)$ that induces the compact-open topology. This shows that the metric topology on the set $H(U)$ of holomorphic functions on a domain U that one encounters in complex function theory is the compact-open topology. Thus the Arzèla–Ascoli theorem provides an inroad to Montel's theorem on normal families of analytic functions.

7. The Stone–Weierstrass Theorem

The familiar **Stone–Weierstrass Theorem** states that for a compact space X every subring of $C(X, \mathbb{R})$ that contains all constant functions and separates the points of X is ***dense*** with respect to the topology of uniform convergence.

This theorem characterizes compactness among the completely regular spaces: if X is not compact then take a point x in X and a point z in $\beta X \setminus X$; the subring $\{f\colon \beta f(z) = f(x)\}$ of $C(X, \mathbb{R})$ is not dense, though it satisfies the conditions in the Stone–Weierstrass Theorem.

There is also a version of this theorem for general spaces: any subring as in the Stone Weierstrass theorem is always dense in $C_c(X, \mathbb{R})$.

8. More topologies

There are many more ways to introduce topologies in $C(X, Y)$, we mention two.

Variations on the compact-open topology
Given spaces X and Y one can first specify families $\mathcal{K}$ and $\mathcal{O}$ of subsets of X and Y respectively and then simply decree that

$$\{[K, O]\colon K \in \mathcal{K},\ O \in \mathcal{O}\}$$

be a subbase for a topology on $C(X, Y)$. The compact-open topology arises when $\mathcal{K}$ is the family of compact subsets of X and $\mathcal{O}$ the topology of Y; changing $\mathcal{K}$ to the finite subsets of X then yields the topology of pointwise convergence.

Hyperspace topologies
If Y is Hausdorff then the graph of every continuous map from X to Y is a closed subset of $X \times Y$, so $C(X, Y)$ is in a natural way a subset of the ***hyperspace*** $2^{X \times Y}$. Thus, any hyperspace topology immediately gives rise to a function space topology.

The volume [1] contains systematic studies of the somewhat bewildering array of topologies obtained in this way.

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D: Fairly general properties

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d-1 The Low Separation Axioms T_0 and T_1

A binary relation $\preceq$ on a set is a **quasiorder** if it is ***transitive*** and ***reflexive***. It is called a **trivial quasiorder** if allways $x \preceq y$, and is defined to be a **discrete relation** if it agrees with equality; it is said to be an ***order*** (sometimes simply called a ***partial order***) if it is **antisymmetric**, that is, $(\forall x, y)((x \preceq y \text{ and } y \preceq x) \iff (x = y))$. A subset Y of a quasiordered set X is a **lower set** if $(\forall x, y)((x \preceq y$ and $y \in Y) \Longrightarrow x \in Y)$; the definition of an **upper set** is analogous. On a ***topological space*** X set $x \leqslant y$ iff every ***neighbourhood*** of x is a neighbourhood of y. This definition introduces a quasiorder $\leqslant$, called the **specialisation quasiorder** of X. The terminology originates from algebraic geometry, see [18, II, p. 23]. The specialisation quasiorder is trivial iff X has the ***indiscrete topology***. The ***closure*** $\overline{Y}$ of a subset Y of a topological space X is precisely the lower set $\downarrow Y$ of all $x \in X$ with $x \preceq y$ for some $y \in Y$. The singleton closure $\overline{\{x\}}$ is the lower set $\downarrow\{x\}$, succinctly written $\downarrow x$. The intersection of all neighbourhoods of x is the upper set $\uparrow x = \{y \in X \colon x \preceq y\}$.

A topological space X is said to satisfy the **separation axiom** T_0 (or to be a T_0**-space**), and its topology $\mathcal{O}_X$ is called a T_0**-topology**, if the specialisation quasiorder is an order; in this case it is called the **specialisation order**. The space X is said to satisfy the **separation axiom** T_1 (or to be a T_1**-space**), and its topology $\mathcal{O}_X$ is called a T_1**-topology**, if the specialisation quasiorder is discrete. The terminology for the hierarchy T_n of separation axioms appears to have entered the literature 1935 through the influential book by Alexandroff and Hopf [3] in a section of the book called "**T**rennungsaxiome" (pp. 58 ff.). A space is a T_0-space iff

(0) for two different points there is an open set containing precisely one of the two points,

and it is a T_1-space iff

(1) every singleton subset is closed.

Alexandroff and Hopf call postulate (0) „das nullte Kolmogoroffsche Trennungsaxiom“ and postulate (1) „das erste Frechetsche Trennungsaxiom“ [3, pp. 58, 59], and they attach with the higher separation axioms the names of Hausdorff, Vietoris and Tietze. In Bourbaki [4], T_0-spaces are relegated to the exercises and are called ≪espaces de Kolmogoroff≫ (see §1, Ex. 2, p. 89). Alexandroff and Hopf appear to have had access to an unpublished manuscript by Kolmogoroff which appears to have dealt with quotient spaces [3, pp. 61, 619]. It is likely to have been the origin of this terminology to which Alexandroff continues to refer in later papers (see, e.g., [2]). Fréchet calls T_1-spaces ≪espaces accessibles≫ [7, p. 185]. From an axiomatic viewpoint, the postulate (1) is a natural separation axiom for those who base topology on the concept of a closure operator (Kuratowski 1933); Hausdorff's axiom, called T_2 by Alexandroff and Hopf, is a natural one if the primitive concept is that of neighbourhood systems (Hausdorff [9] 1914). In [14] Kuratowski joins the terminology by referring to ≪espaces T_1≫ (loc. cit. p. 38).

While today we call a topological space ***discrete*** if every subset is open, we now call a space **Alexandroff-discrete** if the intersection of open sets is open, or, which amounts to the same, that every upper set with respect to the specialisation quasiorder is open; in [3] this is applied to the ordered set of cells of a simplicial complex, the order being containment. (Alexandroff himself called these spaces "diskret", see [2].) On any ordered set the set of all upper sets is an Alexandroff discrete T_0-topology. In [2] Alexandroff associates with each Alexandroff discrete space a complex; this remains a viable application of T_0-spaces even for finite topological spaces. While trivially T_1 implies T_0, the Alexandroff discrete topology on any nondiscretely ordered set is a T_0-topology that is not a T_1-topology. Any set supports the so-called ***cofinite topology*** containing the set itself and all complements of finite sets; the cofinite topology is always a ***compact*** T_1-topology (failing to be a ***Hausdorff*** topology whenever the underlying set is infinite).

On any topological space X with topology $\mathcal{O}_X$, the binary relation defined by $x \equiv y$ iff $x \leqslant y$ and $y \leqslant x$ (with respect to the specialisation quasiorder $\leqslant$) is an equivalence relation. The ***quotient space*** $X/{\equiv}$ endowed with its ***quotient topology*** $\mathcal{O}_{X/\equiv}$ is a T_0-space, and if $q_X : X \to X/{\equiv}$ denotes the ***quotient map*** which assigns to each point its equivalence class, then the function $U \mapsto q_X^{-1}(U) \colon \mathcal{O}_{X/\equiv} \to \mathcal{O}_X$ is a bijection. Moreover, if $f : X \to Y$ is any continuous function into a T_0-space, then there is a unique continuous function $f' : X/{\equiv} \to Y$ such that $f = f' \circ q_X$. As a consequence of these remarks, for most purposes it is no restriction of generality to assume that a topological space under consideration satisfies T_0. Thus the ***category*** of T_0-spaces is reflective in the category of topological spaces (cf. [1, p. 43]). The topology $\mathcal{O}_X$ of a topological space is a ***complete lattice*** satisfying the (infinite) distributive law $x \wedge \bigvee_{j \in J} x_j = \bigvee_{j \in J} (x \wedge x_j)$. After Dowker [6] such a lattice is called a **frame**; other authors speak of a **Brouwerian lattice** or a **complete Heyting algebra** (cf., e.g., [13, 8]). The **spectrum** $\operatorname{Spec} L$ of a frame L is the set of all **prime element**s of L [8], and its topology is the so-called **hull-kernel topology** $\mathcal{O}(\operatorname{Spec} L)$. For a T_0-space X with topology $\mathcal{O}_X$, the function $x \mapsto X \setminus \overline{\{x\}} \colon X \to \operatorname{Spec}(\mathcal{O}_X)$ is a well-defined open embedding. The prime elements of $\mathcal{O}_X$ are of the form $X \setminus A$ where A is a closed set on which the ***induced topology*** is a ***filter base***; such sets are called **irreducible closed subsets**. Every singleton closure is one of these; the space X is called a **sober space** if the singleton closures are the only closed irreducible subsets. It is exactly for these T_0-spaces that the map $X \to \mathrm{Spec}(\mathcal{O}_X)$ is a homeomorphism; the topology of a sober space determines the space. An infinite set X given the cofinite topology is itself an irreducible closed subset which is not a singleton closure; thus T_1-spaces need not be sober. The Hausdorff separation axiom T_2 quickly implies that every closed irreducible subset is singleton; that is, Hausdorff spaces are sober. The position of the property (SOB) of being sober in the separation hierarchy thus is as follows

$$T_0 \Longleftarrow (\mathrm{SOB}) \Longleftarrow T_2 \Longleftarrow \cdots,$$

where none of these implications can be reversed. But (SOB) behaves more like a completeness than as a separation property. Indeed, if X is a T_0-space then the set X^s of all closed irreducible subsets of X carries a unique topology $\mathcal{O}(X^s)$ making the resulting space homeomorphic to $\mathrm{Spec}(\mathcal{O}_X)$. The space X^s is sober and the function $s_X : X \to X^s$, defined by $s_X(x) = \overline{\{x\}}$, is a continuous embedding with dense image such that the function $U \mapsto \sigma_X^{-1}(U) : \mathcal{O}(X^s) \to \mathcal{O}_X$ a bijection. Moreover, if $f : X \to Y$ is any continuous function into a sober space, then there is a unique continuous function $f' : X^s \to Y$ such that $f = f' \circ s_X$. The category of sober spaces is reflective in the category of T_0-spaces. According to these remarks, for many purposes it is no restriction of generality to assume that a T_0-space and indeed any topological space under consideration is a sober T_0-space. The space X^s is called the **sobrification** of X.

If L is a frame, the natural frame map $x \mapsto \mathrm{Spec}(L) \setminus {\uparrow} x : L \to \mathcal{O}(\mathrm{Spec}(L))$ is surjective; but it is an isomorphism iff and only if for two different elements in L there is a prime element which is above exactly one of the two elements. A complete ***Boolean algebra*** without atoms (such as the set of (equivalence classes modulo null sets of) Lebesgue measurable subsets of the unit interval) provides an example of a frame without prime elements. The class of frames is the backdrop for a theory of T_0-spaces "without points" (cf. [13, 16]).

In a ***topological group*** G, the axiom T_0 implies all separation axioms up to the separation axiom $T_{3\frac{1}{2}}$, that is, Hausdorff separation plus ***complete regularity***. In fact, the singleton closure $\overline{\{1\}}$ is a closed characteristic and thus normal subgroup N; and the factor group $G/N = G/{\equiv}$ is a Hausdorff topological group with the universal property that every morphism from G into a Hausdorff topological group factors through G/N. Every open set U of G satisfies $NUN = U$ and thus us a union of N-cosets. In topological group theory, therefore, the restriction to Hausdorff topological groups is no loss of generality most of the time.

There is a serious watershed between the lower separation axioms T_0 and T_1 and the higher separation axioms T_n with $n \geqslant 2$. All of classical general and algebraic topology, classical topological algebra and functional analysis is based on Hausdorff separation T_2, and Bourbaki [4] shows little interest in anything but Hausdorff spaces, relenting a bit in the second edition. On the other side, in the early history of topology, the theory of T_0-spaces played a comparatively subordinate role relating mostly to axiomatic matters, but later it developed a momentum of its own as a link between topology, order theory, combinatorics, finite topological spaces, logic, and theoretical computer science. Its significance in algebra and functional analysis is evident in algebraic geometry and the spectral theory of rings, lattices and operator algebras.

Algebraic Geometry

If R is a commutative ring with identity, then the space $\mathrm{spec}(R)$ of prime ideals in the hull-kernel or **Zariski topology** is called the spectrum of R; the spectrum of a commutative ring is a compact sober T_0-space, and by a theorem of Hochster every compact sober T_0-space is so obtained, [10]. In the sense of the spectrum of a frame L mentioned earlier, $\mathrm{spec}(R) = \mathrm{Spec}(L)$ for the lattice L of radical ideals (i.e., those which are intersections of prime ideals, see, e.g., [18, p. 147]). The spectrum is not readily functorial, but Hofmann and Watkins exhibited a suitable category allowing it to become functorial [12]. The subspace $\max(R) \subseteq \mathrm{spec}(R)$ of maximal ideals is dense; it is a T_1-space in the Zariski topology, and indeed $\mathrm{Spec}(R)$ is a T_1-space iff every prime ideal is maximal. The rings that are relevant in algebraic geometry are the polynomial rings $k[X_1, \ldots, X_n]$ in n variables over a field k of their homomorphic images. A singleton set in an algebraic variety over an algebraically closed field is an irreducible algebraic variety; hence it is closed. Accordingly, such algebraic varieties are compact T_1-spaces in their Zariski topology. In particular, the Zariski topology on an algebraic group G over an algebraically closed field is a compact T_1-topology. (One should note that G is not a topological group with respect to this topology, since the algebraic variety $G \times G$ does not carry the product topology).

Operator theory

The spectrum of a C^*-algebra A (see, e.g., [5], p. 58ff.) is the space of all primitive two-sided ideals (i.e., those which are kernels of irreducible representations) in the hull-kernel topology. It is a T_0-space which in general fails to be a T_1-, let alone a Hausdorff space; but it is a ***Baire space***. If the algebra is ***separable***, then the primitive spectrum is a sober space. N. Weaver has constructed a C^*-algebra which is prime but not primitive; thus its primitive spectrum is not sober [17].

Directed completeness

An ordered set is **directed complete** if every directed subset D has a least upper bound $\sup D$. In the semantics of programming languages such an ordered set is called a **DCPO** ("directed complete partially ordered set"). On a DCPO L, two new structures emerge: (i) an additional transitive relation $\ll$ defined by $x \ll y$ (read "x is **way below** y") iff for all directed subsets D

$$\text{the relation } \sup D \in {\uparrow} y \quad \text{implies} \quad D \cap {\uparrow} x \neq \emptyset,$$

and (ii) a T_0-topology $\sigma(L)$, where a subset $U \subseteq L$ is in $\sigma(U)$ iff for each directed subset D

$$\text{the relation } \sup D \in U \quad \text{implies} \quad D \cap U \neq \emptyset.$$

The topology $\sigma(L)$ is called the **Scott topology** of L. A function between DCPOs is continuous with respect to the Scott topologies iff it preserves directed sups. The theory of T_0-spaces arising as DCPOs with their Scott topologies is particularly rich for those DCPOs in which the set $\Downarrow x = \{u \in L: u \ll x\}$ is directed and satisfied $\sup \Downarrow x = x$. Such a DCPO is called a **domain** (cf. [8, new edition, Definition I-1.6]). The Scott topology of a domain is sober. For the details see [8]. If a domain L is a lattice, then it is called a **continuous lattice**; this terminology is due to D.S. Scott [15].

Injective T_0-spaces

A T_0-space L is called **injective** according to D.S. Scott [15], if for every subspace Y of a T_0-space X, any continuous function $f: Y \to L$ extends to a continuous function $F: X \to L$. Scott proved in 1972 that a T_0-space is injective iff it is a continuous lattice L endowed with its Scott topology $\sigma(L)$ (cf. [8, II-3]). The theory of continuous lattices has both a T_0 aspect and a T_2 aspect: An ordered set is a continuous lattice if and only if it supports a compact Hausdorff topology with respect to which it is a **Lawson semilattice**, that is, a topological semilattice with identity in which every element has a base of multiplicatively closed neighbourhoods. The topology $\mathcal{O}_X$ of a sober space X is a continuous lattice iff X is locally compact. The order theoretical theory of locally compact sober spaces relies thoroughly on the theory of domains. The category of domains and Scott continuous functions is ***Cartesian closed***, and Scott produced a canonical construction of continuous lattices L which have a natural homeomorphism onto the space $[L \to L]$ of continuous self maps; in such a T_0-space as "universe", every "element" is at the same time a "function" whence expressions, like $f(f)$, are perfectly consistent. In this fashion Scott produced the first models of untyped λ-calculus of Church and Curry in the form of certain compact T_0-spaces. Hofmann and Mislove proved that it is impossible to base a model of the untyped λ-calculus on a compact Hausdorff space [11]. For the many beautiful properties of continuous lattices and domains and their relation to general topology in the area of T_0-spaces see [8].

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d-2 Higher Separation Axioms

All ***spaces*** are assumed to be T_1***-spaces***, i.e., each singleton is closed.

1. Definitions and basic facts

Three familiar separation axioms are defined by requiring that certain pairs of disjoint sets have disjoint neighbourhoods: a **Hausdorff space** (or a T_2**-space**) is one in which distinct points have disjoint neighbourhoods; a **regular space** (or a T_3**-space**) requires this for closed sets and points outside them and in a **normal space** (or a T_4**-space**) disjoint closed sets should have disjoint neighbourhoods. According to **Urysohn's Lemma** in a normal space disjoint closed sets are **completely separated**, i.e., if A and B are closed and disjoint in the normal space X then there is a continuous function $f : X \to \mathbb{I}$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$. This leads to the definition of **completely regular** or **Tychonoff spaces**: if F is closed and $x \notin F$ then $\{x\}$ and F are completely separated. Tychonoff spaces are also called $T_{3\frac{1}{2}}$**-space**s because they clearly sit between T_3- and T_4-spaces.

Every subspace of a T_i-space is again a T_i-space, for $i \leqslant 3\frac{1}{2}$. Closed subspaces of normal spaces are readily seen to be normal; this holds even for F_σ-subspaces. A space is **hereditarily normal** (or a T_5**-space**) if every subspace is normal; this can be characterized by "every open subspace is normal" and "separated subsets have disjoint neighbourhoods"; the sets A and B are separated if $\operatorname{cl} A \cap B = A \cap \operatorname{cl} B = \emptyset$.

A **perfectly normal** or T_6**-space** is a space that is a normal and **perfect space**, i.e., closed sets are G_δ***-sets*** or open sets are F_σ***-sets***. From the above it is clear that perfectly normal spaces are hereditarily normal. A space is perfectly normal iff every closed set is a **zero set**, that is, of the form $f^{-1}(0)$ for some continuous real-valued function f.

A collection $\mathcal{F}$ of subsets of a space X is a **discrete family** if for every point $x \in X$ there is a neighbourhood U of x that intersects at most one element of the collection. A space X is called κ**-collectionwise normal** (κ**-CWN** for short), where κ is a cardinal, if for every discrete collection of $\mathcal{F}$ with $|\mathcal{F}| \leqslant \kappa$, there is a pairwise disjoint collection $\mathcal{U} = \{U(F): F \in \mathcal{F}\}$ of open sets such that $F \subset U(F)$ for each $F \in \mathcal{F}$. It is not difficult to show "normal iff ω-CWN", where ω is the smallest infinite cardinal. A space is **collectionwise normal** (**CWN** for short) if it is κ-CWN for all cardinals κ.

Of course, CWN spaces are normal; ***compact Hausdorff*** (more generally regular ***Lindelöf***) spaces are CWN and hence normal. Moreover ***GO-spaces*** are CWN, therefore subspaces of ordinal numbers with the usual order topology are CWN. In particular, ω_1 is hereditarily CWN, but not perfect. Indeed, the subset of all limit ordinals in ω_1 is closed but not a G_δ-set. In [E, 1.5] one finds examples of non-regular Hausdorff spaces, regular non-Tychonoff spaces and non-normal Tychonoff spaces.

For $i = 4, 5, 6$ the image of a T_i-space by a continuous ***closed map*** is again a T_i-space; this does not hold if $i \leqslant 3\frac{1}{2}$: if X is not normal and A and B are disjoint closed sets without disjoint neighbourhoods then by identifying A and B to points one creates a closed continuous map onto a non-Hausdorff space. The properties T_2 and T_3 are invariant under ***perfect maps*** but the property $T_{3\frac{1}{2}}$ is not [E, 3.7.20].

Products of T_i-spaces are again T_i-spaces when $i \leqslant 3\frac{1}{2}$, but as is witnessed by the ***Sorgenfrey line***, the product of two perfectly normal, CWN spaces need not be normal. The same is true for function spaces: for $i \leqslant 3\frac{1}{2}$, if Y is a T_i-space then so is the space Y^X of continuous maps from X to Y, with respect to the ***topology of pointwise convergence*** and the ***compact-open topology***.

Imposing separation axioms on the space 2^X of all non-empty closed subsets of the space X usually leads to stronger properties of the space X itself. If 2^X is given the ***Vietoris topology*** then for every T_1-space X: (1) 2^X is T_1; (2) 2^X is T_2 iff X is regular; (3) 2^X is regular iff X is normal; and (4) 2^X is normal iff X is compact Hausdorff, see [E]. If 2^X is given the ***Fell topology*** then (1) 2^X is regular iff 2^X is Tychonoff iff X is locally compact [7]; and (2) 2^X is normal iff 2^X is paracompact iff 2^X is Lindelöf iff X is locally compact Lindelöf [14].

2. Tychonoff spaces

Clearly every subspace of a normal space is Tychonoff. Conversely, every Tychonoff space can be embedded in a compact Hausdorff space. If X is Tychonoff and $\mathcal{F}$ is the collection of all continuous functions from X to $\mathbb{I}$ then the diagonal map $F : X \to \mathbb{I}^{\mathcal{F}}$ is an embedding. Thus the Tychonoff spaces are exactly the subspaces of compact Hausdorff spaces.

Tychonoff spaces are also exactly the ***uniformizable*** spaces, i.e., X is Tychonoff iff its topology is generated by a ***uniformity***. For more details, the reader should refer to the article on uniform spaces in this Encyclopedia.

To every space X one can associate a Tychonoff space Y as follows. Two points x and y in X are equivalent if $f(x) = f(y)$ for all continuous real-valued functions f on X. The corresponding ***quotient space*** Y is Tychonoff and the rings $C(X)$ and $C(Y)$ of real-valued continuous functions are isomorphic; the same holds for the rings $C^*(X)$ and $C^*(Y)$ of bounded real-valued continuous functions. Thus, when studying rings of these form one may as well assume that the spaces under consideration are Tychonoff.

3. Normal spaces

An important result on normal spaces is the **Tietze–Urysohn theorem**, which states that a space X is normal iff for every closed subspace A of X every continuous function $f : A \to \mathbb{R}$ can be extended to a continuous function $\hat{f} : X \to \mathbb{R}$. One can demand in addition that the extension respects upper and lower bounds, i.e., if $f(x) < c$ for all $x \in A$ then $\hat{f}(x) < c$ for all $x \in X$ (and likewise for $>$). This leads to the notions of **C-embedding** and **C^*-embedding**: the set A is C-embedded (C^*-embedded) in X if every (bounded) real-valued function on A can be extended to a (bounded) real-valued function on X. Thus X is normal iff every closed subset is C-embedded iff every closed subset is C^*-embedded. In general a subset is C-embedded iff it is C^*-embedded and completely separated from every zero set in its complement.

Another structural result on normal spaces is **Jones' Lemma** [E, 1.7.12(c)]: If X is normal, S a closed and discrete subset, and D a dense subset then $2^{|S|} \leqslant 2^{|D|}$. This result implies readily that the ***Niemytzki Plane*** N and the ***Sorgenfrey plane*** $\mathbb{S}^2$ are not normal: both spaces are separable and have a closed discrete subspace of cardinality $\mathfrak{c}$ (the x-axis and the antidiagonal '$y = -x$', respectively).

In this connection it is of interest to note that Przymusinski [20] showed that assuming ***Martin's Axiom*** (MA for short) and $\omega_1 < \mathfrak{c}$, if $Q \subseteq \mathbb{S}$ and $\omega < |Q| < \mathfrak{c}$, then Q^2 is normal but not CWN. A similar example is obtained by taking a subset Q of $\mathbb{R}$ with $\omega < |Q| < \mathfrak{c}$ and deleting the points $(x, 0)$ with $x \notin Q$ from the Niemytzki plane N.

There is a large literature on normality. This largely due to the facts that subspaces of normal spaces need not be normal – the best-known example is the ***Tychonoff plank***, obtained by deleting the point (ω_1, ω) from the compact product $(\omega_1 + 1) \times (\omega + 1)$ – and that products of normal spaces need not be normal.

4. Normality in products

Normality of a product usually imposes extra structure on the factors.

In 1962, H. Tamano established: (1) X is paracompact iff $X \times \beta X$ is normal iff $X \times \alpha X$ is normal for every ***compactification*** αX of X; (2) X is separable metrizable iff $X \times \alpha X$ is perfectly normal for some compactification αX of X; and (3) X is CWN iff $F \times \beta X$ is C^*-embedded in $X \times \beta X$ for every closed subspace F of X.

These theorems can be given parametrized versions. Dowker proved that X is normal and ***countably paracompact*** iff $X \times (\omega + 1)$ is normal iff $X \times Y$ is normal for every (or some) infinite compact metrizable space Y. This gave rise to the search for ***Dowker spaces*** (normal spaces for which $X \times \mathbb{I}$ is not normal); these are discussed in a separate article in this Encyclopedia.

A space is called **κ-paracompact** if every open cover of size at most κ has a locally finite open refinement; so a space is ***paracompact*** if it is κ-paracompact for each cardinal κ. In analogy with the Dowker Theorem, Morita and Kunen proved that for an infinite cardinal κ one has: X is normal and κ-paracompact iff $X \times (\kappa + 1)$ is normal iff $X \times \mathbb{I}^\kappa$ is normal iff $X \times 2^\kappa$ is normal iff $X \times Y$ is normal for each compact space Y with $w(Y) = \kappa$, see [KV, Chapter 18]. Since the open cover $\{[0, \alpha): \alpha < \omega_1\}$ of ω_1 witnesses that ω_1 is not ω_1-paracompact, this result shows the non-normality of $\omega_1 \times (\omega_1 + 1)$.

Another parallel of the Morita and Kunen's result is theorem of Alas and Rudin that X is countably paracompact and κ-CWN iff $X \times A(\kappa)$ is normal iff $X \times Y$ is normal for some compact space Y with $w(Y) = \kappa$, where $A(\kappa)$ denotes the one point compactification of the discrete space of size κ. Since ω_1 is countably paracompact and ω_1-CWN, $\omega_1 \times A(\omega_1)$ is normal.

The product space X of a normal space and a non-discrete metric space is normal iff X is countably paracompact [E, 5.5.18]. Moreover, the product space of a perfectly normal space and a metric space is normal. Another simple result is that the product space of a countably compact normal space and a metric space is normal. In 1964, Morita characterized the normal spaces whose product with every metrizable space is normal. That is, a space is a **P-space (in the sense of Morita)** if for every cardinal $\kappa \geqslant 1$ and for every collection $\{F(s): s \in \bigcup_{n\in\omega} \kappa^n\}$ of closed sets such that $F(s) \supseteq F(t)$ whenever $s \subseteq t$, there exists a collection $\{U(s): s \in \bigcup_{n\in\omega} \kappa^n\}$ of open sets with $U(s) \supseteq F(s)$ for each s, and $\bigcap_{n\in\omega} F(f|n) = \emptyset$ implies $\bigcap_{n\in\omega} U(f|n) = \emptyset$ for each $f \in \kappa^\omega$. Observe that P-spaces are countably metacompact, therefore normal P-spaces are countably paracompact. Morita asked whether any one of the following is true:

(1) if $X \times Y$ is normal for every metric space Y, then X is discrete;
(2) if $X \times Y$ is normal for every normal P-space Y, then X is metrizable; and
(3) if $X \times Y$ is normal for every countably paracompact normal space Y, then X is metrizable and σ-locally compact.

Atsuji [4] showed that the answer to (1) is "yes" if there is a κ-Dowker space for each infinite cardinal κ, where a **κ-Dowker space** is a normal space having an open cover $\{U_\alpha: \alpha < \kappa\}$ which does not have a closed cover $\{F_\alpha: \alpha < \kappa\}$ such that $F_\alpha \subseteq U_\alpha$ for each $\alpha < \kappa$. Finally Rudin [21] solved (1) affirmatively by showing that κ-Dowker spaces do exist. Chiba, Przymusinski and Rudin [10] showed that assuming $V = L$, the answers to (2) and (3) are "yes". Finally Balogh [6] proved that the answer to (3) is "yes" in ZFC.

Normality of a product space of a normal space with the space of irrationals $\mathbb{P}$ has been also investigated. Historically, Michael constructed a normal space M, so-called

Michael line [E, 5.1.32], whose product space with $\mathbb{P}$ is not normal. A **Michael space** is a Lindelöf space whose product with $\mathbb{P}$ is not normal and the **Michael problem** is: is there a Michael space in ZFC? The existence of a Michael space follows from $\omega_1 = \mathfrak{c}$ [18] or MA [1]. Lawrence [17] showed that Michael problem is equivalent to the existence of a Lindelöf space and a separable completely metrizable space with a non-normal product, and that it is not possible in ZFC to construct a Michael space of weight ω_1. Afterwards he proved that there is a ZFC example of a Lindelöf space and a completely metrizable (but not separable) space whose product is non-normal and weight ω_1.

The product ω^{ω_1} is not normal and this implies that if an infinite product space $X = \prod_{\alpha<\kappa} X_\alpha$ is normal, then all X_α's, except for at most countably many, are countably compact. A theorem of Noble asserts that a space X is compact Hausdorff iff all of its powers are normal (in fact normality of $X^{\aleph_1 \cdot w(X)}$ suffices).

Thus if an infinite product space $X = \prod_{\alpha<\kappa} X_\alpha$ of metric spaces is normal, then all X_α's, except for at most countably many, are compact. So in this sense, countable product spaces are essential for discussing infinite products. Zenor and Nagami established that if all finite subproducts of a product space $X = \prod_{n\in\omega} X_n$ are normal, then X is normal iff it is countably paracompact. Aoki [2] proved that if a product space $X = \prod_{\alpha<\kappa} X_\alpha$ is κ-paracompact, then X is normal iff all finite subproducts of X are normal. Moreover Bešlagić [8] proved that if $X = \prod_{\alpha<\kappa} X_\alpha$ is normal, then it is κ-paracompact. Thus these results extend the Zenor and Nagami's one to uncountable products.

5. Collectionwise normality

Normal spaces are ω-CWN, but Bing constructed an example of a normal not ω_1-CWN space (***Bing's example G***). A good article on collectionwise normality is [KV, Chapter 15]. Moreover good articles on covering properties and metrization theory are [KV, Chapter 9] and [KV, Chapter 16]. Most of results below are found in these articles.

Collectionwise normality has been studied in connection with metrization theorems. Historically, Jones conjectured that normal ***Moore spaces*** are metrizable. This conjecture was called the **Normal Moore Space Conjecture** (NMSC for short). Observe that Moore spaces are first-countable and ***subparacompact*** and that metrizable spaces are paracompact, hence CWN, and Moore. Assuming $2^\omega < 2^{\omega_1}$, Jones showed in 1937 that separable normal spaces have no uncountable closed discrete subspaces, and as a corollary, that separable normal Moore spaces are metrizable. Afterwards, Bing established that a space is metrizable iff it is a CWN Moore space. Therefore the problem on the NMSC was focused on the difference between normality and collectionwise normality of Moore spaces. In 1964, Heath showed that there is a separable normal non-metrizable Moore space iff there exists an uncountable ***Q*-set** in the reals $\mathbb{R}$, where $E \subset \mathbb{R}$ is a Q-set if every subset $E' \subset E$ is a G_δ-set in E. The "if" part of the Heath's result was shown by considering a subspace of the Niemytzki plane (see above). Around 1967–1968, Silver and Tall proved that $\mathrm{MA} + \omega_1 < \mathfrak{c}$ yields a non-metrizable separable normal Moore space. So the existence of a separable normal non-metrizable Moore space is consistent with and independent of ZFC.

For the non-separable case, the situation was more complicated. Nyikos [19] proved that the **Product Measure Extension Axiom** (PMEA for short, i.e., for every cardinal κ, the product measure on 2^κ extends to a $\mathfrak{c}$-additive full measure) implies that normal first-countable spaces are CWN, therefore normal Moore spaces are metrizable. It is known that the consistency of PMEA follows from the existence of a ***strongly compact cardinal*** and that the consistency of PMEA implies the existence of a ***measurable cardinal***. A remarkable result of this line is, by Fleissner [12], that the statement "normal Moore spaces are metrizable" implies the consistency of the existence of a measurable cardinal. In a sense, this completes the NMSC problem. An interesting open problem is whether NMSC implies the statement "normal first-countable spaces are CWN".

The problem whether normal locally compact spaces are collectionwise normal is also interesting. Arkhangelskiĭ [3] showed that locally compact ***metacompact*** perfectly normal spaces are paracompact. Paracompact spaces are subparacompact and metacompact, moreover subparacompact spaces and also metacompact spaces are ***submetacompact***. On the other hand, CWN submetacompact spaces are paracompact. Then Watson [22] proved that $V = L$ implies that locally compact normal spaces are collectionwise normal with respect to compact sets (i.e., every discrete collection of compact sets are separated by disjoint open sets), hence locally compact submetacompact normal spaces are paracompact. In ZFC, it is known that locally compact submetacompact normal spaces are subparacompact [16]. Tall asked if there exists a locally compact normal non-CWN space under various additional set theoretical or topological assumptions. Then Daniels and Gruenhage [11] proved that assuming $V = L$, there exists a locally compact collectionwise Hausdorff perfectly normal non-CWN space. Afterwards, Balogh [5] proved that if the existence of a ***supercompact cardinal*** (a notion stronger than a strongly compact cardinal) is consistent with ZFC, then so is "locally compact normal spaces are CWN".

6. Hereditary and perfect normality

Metrizable spaces are perfectly normal, perfectly normal spaces are hereditarily normal and there is a hereditarily normal but not perfectly normal space. Evidently subspaces of a hereditarily normal space are hereditarily normal, and it is not difficult to show that subspaces of a perfectly normal space are also perfectly normal.

If the product space $X \times Y$ is hereditarily normal, then the factor spaces X and Y have stronger properties, that is, either X is perfectly normal or countable subsets of Y are closed discrete. This is due to Katětov [15]. In case that both X and Y are compact, this means that both X and Y are perfectly normal. Another result of Katětov is: if X is compact and X^3 is hereditarily normal, then X is metrizable. This follows from Šneĭder's theorem [E, 4.2.B] that a compact space X is metrizable iff its diagonal $\{\langle x, x\rangle \in X \times X: x \in X\}$ is a G_δ-set in X^2. In particular, a compact space X is metrizable whenever X^2 is perfectly normal. Chaber proved that compactness can be relaxed to ***countable compactness***. So one can ask whether, if X^2 is hereditarily normal and X is compact, X is metrizable, and whether, if X^2 is hereditarily normal and X is countably compact, X is compact. The former was first asked by Katětov. For the latter, Bešlagić [9] constructed, assuming $\diamondsuit$, a countably compact non-compact space such that X^2 is hereditarily normal. For **Katětov's problem**, Nyikos and Gruenhage constructed a compact non-metrizable space such that X^2 is hereditarily normal assuming $\mathsf{MA} + \neg\mathsf{CH}$, or $\omega_1 = \mathfrak{c}$ [13]. Recently Larson and Todorčcević has shown a consistently affirmative answer to Katětov's problem. An interesting open problem is: if the product space $X \times Y$ of compact spaces X and Y is perfectly normal, then is at least one of X and Y metrizable?

Non-normality of ω^{ω_1} yields that the product space of uncountably many spaces having at least two points is not hereditarily normal. For perfect normality or hereditary normality of countable product spaces, the following are known: (1) a countable product space is perfect iff its all finite subproducts are perfect, (2) a countable product space is hereditarily normal iff it is perfectly normal iff it is hereditarily countably compact.

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d-3 Fréchet and Sequential Spaces

Let X be a ***topological space***, $\langle x_n : n \in \omega \rangle$ a sequence in X, and x a point in X. The sequence $\langle x_n : n \in \omega \rangle$ **converges** to x and the point x is a limit of the sequence $\langle x_n : n \in \omega \rangle$, if each neighbourhood of x contains all but finitely many points x_n. It may happen that the topology of a space X is fully determined by the collection of all convergent sequences in the space, such spaces are called **Fréchet spaces**[1] (synonymously, **Fréchet–Urysohn spaces**) or **sequential**, depending on how the topological closure can be obtained from convergent sequences. For a set $M \subseteq X$, let the **sequential closure** of M, denoted $\operatorname{seqcl}(M)$, be the set $\{x \in X : x$ is a limit of some sequence $\langle x_n : n \in \omega \rangle$ having all values in $M\}$. The space X is Fréchet, if $\operatorname{seqcl}(M) = \overline{M}$ for each $M \subseteq X$. Put $\operatorname{seqcl}_1(M) = \operatorname{seqcl}(M)$, and for an ordinal $\alpha > 1$, let $\operatorname{seqcl}_\alpha(M) = \operatorname{seqcl}(\bigcup_{\beta<\alpha} \operatorname{seqcl}_\beta(M))$. Clearly, $\operatorname{seqcl}(\operatorname{seqcl}_{\omega_1}(M)) = \operatorname{seqcl}_{\omega_1}(M)$ and the space X is called sequential if $\overline{M} = \operatorname{seqcl}_{\omega_1}(M)$ for every $M \subseteq X$. Thus the space X is Fréchet, if the closure of any set agrees with its sequential closure, and sequential, if the closure of any set is obtained by the iteration of a sequential closure. An equivalent description of a sequential space reads as follows: A set $F \subseteq X$ is closed if each convergent sequence ranging in F has its limit in F as well.

At the dawn of general topology, M. Fréchet [6] considered sets equipped with a list of all convergent sequences together with their limits as a promising alternative to topological spaces, and P.S. Urysohn [19] continued his research; nowadays these structures are called **convergence spaces**. The importance of convergent sequences in topological spaces was fully recognized after 1960 by many authors, let us mention here just a few: A.V. Arkhangel'skiĭ, R.M. Dudley, S.P. Franklin, J.L. Kelley and M. Venkataraman.

Every ***first-countable*** space is Fréchet and every Fréchet space is sequential. The reverse implications do not hold.

The class of sequential spaces is closed under taking quotients and disjoint topological sums, and consequently under inductive limits. Closed and open subspaces of sequential spaces are sequential too, but in general, sequentiality is not hereditary. The easiest example is Arens space A_2, i.e., $A_2 = (\omega + 1) \times \omega \cup \{\infty\}$, where $(\omega + 1) \times \omega$ has its product topology and where U is a neighbourhood of the point ∞ if $\infty \in U$ and U is a neighbourhood of all but finitely many points (ω, n). The space A_2 is sequential, but its subspace $\omega \times \omega \cup \{\infty\}$ is not. The class of sequential spaces is not productive, there are even two Fréchet spaces whose product is not sequential [5]. The Fréchet fan F_ω is the quotient space $\omega \times (\omega + 1)/(\omega \times \{\omega\})$, i.e., the set $\omega \times \{\omega\}$ is collapsed to a point ∞_1. Let Y be the metrizable space consisting of the discrete set $\omega \times \omega$ with one non-isolated point ∞_2, where the base at ∞_2 consists of all sets $\{\infty_2\} \cup ((p, \omega) \times \omega)$, for $p \in \omega$. Both F_ω and Y are Fréchet, but the diagonal $\Delta = \{((n, k), (n, k)) : n \in \omega,\ k \in \omega\}$ satisfies $\operatorname{seqcl}(\Delta) = \Delta \subset \overline{\Delta} = \Delta \cup \{(\infty_1, \infty_2)\}$ in $F_\omega \times Y$. So $F_\omega \times Y$ is not sequential.

A space is sequential if and only if it is a quotient of a topological sum of copies of a convergent sequence $\omega + 1$. Thus the following are equivalent: (a) the space X is sequential, (b) X is a quotient of a zero-dimensional, locally compact, complete metric space, (c) X is a quotient of a first-countable space, (d) X is a quotient of a metric space. If X is Hausdorff, then the list of equivalent conditions may be extended by (e) each $F \subseteq X$ having all intersections with compact metrizable subspaces of X closed, is itself closed.

The class of Fréchet spaces is closed under topological sums and closed under pseudo-open quotient maps where a map $f : X \to Y$ is **pseudo-open** if for every $y \in Y$ and every open set $U \supset f^{-1}(y)$ one has $y \in \operatorname{int} f[U]$. The last condition may be used for a characterization of Hausdorff Fréchet spaces: A Hausdorff space if Fréchet if and only if it is an image of a topological sum of convergent sequences under a pseudo-open quotient map. In contrast to sequential spaces, the class of Fréchet spaces is hereditary, and a sequential space is Fréchet iff it is hereditary sequential. We have already presented above an example showing that the class of Fréchet spaces is not productive.

Compactness or some of its weaker forms has a strong influence on sequential spaces and vice versa. It is well-known that a product of two ***countably compact*** spaces need not be countably compact. However, if X is sequential and countably compact, then it is ***sequentially compact***. Consequently, the product of two countably compact spaces is countably compact, provided one factor is sequential. On the other hand, the product of two countably compact sequential spaces is sequential, though sequentiality is not a productive property in general. Similarly, the product of a first-countable space with a countably compact Fréchet space is Fréchet.

E. Michael [9] and F. Siwiec [18] defined a space X to be **strongly Fréchet** if for each $x \in X$ and each decreasing family $\{A_n : n \in \omega\}$ of subsets of X with $x \in \bigcap_{n\in\omega} \overline{A_n}$, there is a sequence $\langle x_n : n \in \omega \rangle$ converging to x with $x_n \in A_n$ for all $n \in \omega$. Michael proved that a space X is strongly Fréchet if and only if $X \times [0, 1]$ (equivalently, $X \times (\omega + 1)$) is Fréchet.

A detailed analysis of the strange behavior of products of Fréchet spaces led A.V. Arkhangel'skiĭ [1] to introduce

[1] The term 'Fréchet space' is sometimes used in a different sense to mean ***topological vector spaces*** satisfying certain conditions. Thus 'Fréchet–Urysohn space' is more precise.

a finer classification of them. Call a **sheaf of convergent sequences** any family $\{\varphi_n: n \in \omega\}$ of one-to-one maps from ω to the space X such that some point $x \in X$ is a limit of $\langle \varphi_n(k): k \in \omega\rangle$ for all $n \in \omega$. One defines a space to be an α_i-space ($i = 1, 2, 3, 4$) if for each sheaf there is a one-to-one sequence $\psi : \omega \to X$ that converges to x and is such that

α_1: $|\operatorname{rng}\varphi_n \setminus \operatorname{rng}\psi| < \omega$ for all $n \in \omega$,
α_2: $|\operatorname{rng}\varphi_n \cap \operatorname{rng}\psi| = \omega$ for all $n \in \omega$,
α_3: $|\operatorname{rng}\varphi_n \cap \operatorname{rng}\psi| = \omega$ for infinitely many $n \in \omega$,
α_4: $\operatorname{rng}\varphi_n \cap \operatorname{rng}\psi \neq \emptyset$ for infinitely many $n \in \omega$.

Clearly, $\alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4$ and every first-countable space is Fréchet and α_1. The fundamental theorems from [1] are the following. If a space X is Fréchet α_3, Y Fréchet and countably compact, then $X \times Y$ is Fréchet. For a Fréchet space X, the product $X \times [0, 1]$ is Fréchet if and only if X is α_4.

Let us state several consequences of these statements. The product of a first-countable space and a Fréchet countably compact space is Fréchet. Every countably compact Fréchet space is α_4. A space X is Fréchet α_4 if and only if it is strongly Fréchet.

Not all Fréchet spaces are strongly Fréchet, F_ω is such an example. A rich source of examples here are so-called **Ψ-spaces**: If $\mathcal{A}$ is an almost disjoint family on ω, then $\Psi(\mathcal{A})$ denotes the space, whose underlying set is $\omega \cup \mathcal{A}$, all points from ω are isolated, a basic neighbourhood of $A \in \mathcal{A}$ is $\{A\} \cup A \setminus K$, where $K \subseteq A$ is finite. Each $\Psi(\mathcal{A})$ is a locally compact, first-countable Hausdorff space; denote by $\Psi(\mathcal{A}) + \infty$ its one-point compactification. If $\mathcal{A}$ is a ***MAD family*** on ω, then $\Psi(\mathcal{A}) + \infty$ is sequential, but not Fréchet. P. Simon showed that there are two almost disjoint families $\mathcal{A}_0$, $\mathcal{A}_1$, such that $\mathcal{A}_0 \cup \mathcal{A}_1$ is a MAD family and both $\Psi(\mathcal{A}_0) + \infty$ and $\Psi(\mathcal{A}_1) + \infty$ are Fréchet [16]. So there are two compact, Fréchet α_4-spaces, whose product is not Fréchet, and consequently, there are compact Fréchet α_4-spaces that are not α_3. If we identify ω with $\bigcup_{n\in\omega} {}^n2$ and for $f \in {}^\omega 2$ we put $A_f = \{f \restriction n: n \in \omega\}$, then for $T \subseteq {}^\omega 2$ we get an almost disjoint family $\mathcal{A}_T = \{A_f: f \in T\}$. For $T = {}^\omega 2$, the space $\Psi(\mathcal{A}_T) + \infty$ is Fréchet α_3, but not α_2. If T is an uncountable λ'-set in ${}^\omega 2$, then $\Psi(\mathcal{A}_T) + \infty$ is Fréchet α_2, but not first-countable. Both these examples are due to P.J. Nyikos [13]. (Recall that a **λ'-set** is a subset T of reals such that for every countable $K \subseteq \mathbb{R}$, K is a relative G_δ-set in $T \cup K$. A **λ-set** is an uncountable set in which every countable set is a relative G_δ-set.) Nyikos also gave consistent examples of a Fréchet α_2-space which is not α_1 and of a countable Fréchet α_1-space, which is not first-countable. However, there is no hope for a ZFC result here, since A. Dow with J. Steprāns proved that both statements “Every Fréchet α_2-space is α_1” and “Every countable Fréchet α_1-space is first-countable” are consistent with ZFC [3, 4].

Arkhangel'skiĭ's theorem does not characterize Fréchet α_3-spaces. T. Nogura gave under CH an example of a Fréchet space X such that $X \times Y$ is Fréchet whenever Y is regular Fréchet and countably compact, still X is not α_3 [11]. Nogura also proved that the classes of α_i spaces, $i = 1, 2, 3$, are countably productive [10]. The situation with α_4 is again messy: It is possible to find spaces X and Y, both Fréchet and α_4, such that the product $X \times Y$ is (a) neither Fréchet nor α_4 [10], (b) α_4 but not Fréchet [2], and under CH, also (c) Fréchet but not α_4 [17].

Let us turn back to sequential spaces. Given a subset A of a sequential space X, then there is some $\alpha \leqslant \omega_1$ such that $\operatorname{seqcl}_{\alpha+1} A = \operatorname{seqcl}_\alpha A$. The **sequential order** of a space X is $\min\{\alpha \leqslant \omega_1: (\forall A \subseteq X)(\operatorname{seqcl}_{\alpha+1} A = \operatorname{seqcl}_\alpha A)\}$. Notice that this definition makes sense even if the space is not sequential. As an example, observe that the sequential order of the space $\Psi(\mathcal{A}) + \infty$ with $\mathcal{A}$ a MAD family on ω, equals 2. Another frequently used example is the space *Seq*. The underlying set of *Seq* is the set $\bigcup {}^n\omega$ of finite sequences of natural numbers, and a set U is open iff for each $s \in U$, the set $\{n \in \omega: s^\frown n \in U\}$ is cofinite. *Seq* is the simplest example of a sequential space whose sequential order equals to ω_1.

The question about values of the sequential order becomes really intriguing if one assumes some special structure of the space. Recent research concentrated on three situations: The space in question is a product, a topological group or a space of real-valued continuous functions. T. Nogura and A. Shibakov [12] proved that the sequential order of a product of two sequential spaces cannot exceed the sum of sequential orders of factors, provided one factor is regular and countably compact. In particular, the sequential order of product of two compact Fréchet spaces is $\leqslant 2$. Without countable compactness, if X is sequential and Y first-countable, then the sequential order of $X \times Y$ is at most 1 more than the sequential order of X; here, however, the product need not be sequential in general. Under CH, there are two Fréchet spaces such that their product is sequential and its sequential order equals ω_1.

A. Shibakov [14, 15] gave remarkable examples under CH. For any countable ordinal α, the additive group of rationals admits a group topology, which is sequential and its order is α. There is a countable Fréchet topological group, the square of which is sequential, but not Fréchet. And there is a countable Fréchet α_3 topological group, which is not α_2.

Before stating the results concerning $C_p(X)$, let us recall several notions. If X is a topological space and $\mathcal{U}$ is an open cover of X, then $\mathcal{U}$ is called an **ω-cover** if for every finite subset F of X there is some $U \in \mathcal{U}$ with $F \subseteq U$. If $\{A_n: n \in \omega\}$ is a countable family of sets, then $\liminf A_n$ is defined as $\bigcup_{k\in\omega} \bigcap_{n\geqslant k} A_n$, i.e., $\liminf A_n = \{x: (\exists k)(\forall n \geqslant k)(x \in A_n)\}$. A topological space X is a **γ-space**, if each open ω-cover of X contains a countable subfamily $\{U_n: n \in \omega\}$ such that $\liminf U_n = X$. The space of all continuous real-valued functions on X with the topology of pointwise convergence, i.e., a subspace of Tychonov product $\mathbb{R}^X$, is denoted $C_p(X)$. It is well known that the space $C_p([0, 1])$ is not sequential. The following statement is due to J. Gerlits and Zs. Nagy [8]: For a Tychonov space X, the statements (a)–(d) below are equivalent. If X is moreover compact Hausdorff then (a)–(e) are equivalent. (a) $C_p(X)$ is Fréchet, (b) $C_p(X)$ is sequential, (c) $C_p(X)$ is a k-space,

(d) X is a γ-space, (e) X is scattered. D.H. Fremlin [7] showed another remarkable property of $C_p(X)$, namely, the only values of the sequential order of $C_p(X)$ can be 0, 1, or ω_1. Note the subtle distinction between these results: If the sequential order of $C_p(X)$ is ω_1, then $C_p(X)$ is not sequential, and if $X \subseteq \mathbb{R}$ is a Sierpiński set, then $C_p(X)$ is not Fréchet, but its sequential order equals 1.

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d-4 Pseudoradial Spaces

The class of pseudoradial spaces, with the name of **folgenbestimmte Räume** was introduced by H. Herrlich in 1967 (see [13]); they have also been called **chain-net** spaces. They are defined using convergence of transfinite sequences. A (transfinite) sequence is a map whose domain is an ordinal; we employ the usual notation to denote such sequences: $\langle x_\alpha \rangle_{\alpha<\lambda}$ is a sequence with domain λ. We say that such a sequence converges to the point x if for every neighbourhood U of x there is $\beta < \lambda$ such that $x_\alpha \in U$ for all $\alpha \geqslant \beta$; we write $x_\alpha \to x$.

A ***topological space*** X is said to be **pseudoradial** if for any non-closed subset A of X there are a point $x \in \operatorname{cl} A \setminus A$ and a sequence $\langle x_\alpha \rangle_{\alpha<\lambda}$ such that $x_\alpha \in A$ and $x_\alpha \to x$. A space is said to be **radial** (or **Fréchet chain-net** or **stark folgenbestimmt**) if for any $A \subset X$ and any $x \in \operatorname{cl} A$ there is a sequence $\langle x_\alpha \rangle_{\alpha<\lambda}$, with $x_\alpha \in A$, and $x_\alpha \to x$. These classes of spaces are obvious generalizations of the classes of ***sequential*** and ***Fréchet–Urysohn*** spaces respectively. It is easily seen that in these definitions only regular initial ordinals need to be considered, i.e., regular cardinal numbers.

Some subclasses of the class of pseudoradial spaces were introduced successively since they were useful for solving natural problems raised for pseudoradial spaces. First came the class of **almost radial** spaces. We recall that a sequence $\langle x_\alpha \rangle_{\alpha<\lambda}$ is called a **strict** (see [6]) or **thin sequence** (see [HvM, Chapter 17]) if $x_\alpha \to x$ but $x \notin \operatorname{cl}\{x_\gamma\colon \gamma < \beta\}$ for all $\beta < \lambda$. A topological space X is said to be **almost radial** if for any non-closed subset A of X there is a thin sequence with elements in A converging to a point $x \in \operatorname{cl} A \setminus A$. These spaces were initially considered in connection with the problem of characterizing sequential spaces in the class of pseudoradial ***T_1-spaces***. It was proved in [2] that a T_1-space is sequential if and only if it is almost radial and has countable ***tightness***. Herrlich [13] proved that the pseudoradial spaces are quotients of ***linearly ordered topological spaces*** and are quotients of spaces in which every point has a well ordered local base of neighbourhoods. Arkhangel'skiĭ [1] showed that radial spaces are pseudo-open images of ordered spaces. All Fréchet–Urysohn spaces are sequential and radial. All radial and all sequential spaces are almost radial and all almost radial spaces are pseudoradial. The ordinal $\omega_1 + 1$ with the usual order topology shows that there are ***compact Hausdorff*** radial spaces which are not sequential. On the other hand, any compact sequential non-Fréchet–Urysohn space is an example of a compact sequential space which is not radial.

A still unsolved problem concerns the existence of a compact radial space with countable ***cellularity*** which is not Fréchet–Urysohn. Under the assumption that the ***tower number*** $\mathfrak{t}$ equals $\mathfrak{c}$ or that the ***dominating number*** $\mathfrak{d}$ equals ω_1 there is a compact separable radial space that is not Fréchet–Urysohn (see [HvM, Chapter 17]; more information on the 'small' cardinals $\mathfrak{t}$, $\mathfrak{d}$ and others see [KV, Chapter 3] and [vMR, Chapter 8]). However Dow in [7] produced a model in which all compact separable radial spaces are Fréchet–Urysohn.

Several important cardinal inequalities were proved in the class of pseudoradial and radial spaces. For instance, Arkhangel'skiĭ [1] showed that for pseudoradial Hausdorff spaces $|X| \leqslant 2^{\ell(X)\cdot\psi(X)}$ and $|X| \leqslant d(X)^{c(X)}$, while for radial Hausdorff spaces $|\operatorname{cl} A| \leqslant 2^{|A|}$ for any subset $A \subset X$ (and hence $|X| \leqslant 2^{d(X)}$). For compact Hausdorff pseudoradial spaces he showed that $w(X) \leqslant 2^{c(X)}$ and for compact Hausdorff radial spaces $|X| \leqslant 2^{c(X)}$. Given a topological space X, a subset $A \subset X$ is called **radially closed** (**κ-radially closed**) if it contains the limit of all convergent sequences (of size at most κ) consisting of points of A. The **radial character** of the space X, denoted by $R\chi(X)$, is the smallest cardinal number κ such that every κ-radially closed set is radially closed. Obviously, a pseudoradial space X is sequential if and only if $R\chi(X) = \omega$. In any Hausdorff pseudoradial space X we have $t(X) \leqslant s(X)$ and $t(X) \leqslant R\chi(X) \leqslant 2^{t(X)}$; moreover $R\chi(X) \leqslant \psi(X)$ holds for any pseudoradial T_1 space.

One of the first problems considered in connection with the notion of pseudoradialness was to find ZFC examples of non-sequential pseudoradial spaces with countable tightness. It is evident, because of Balogh's Theorem that it is consistent that compact spaces of countable tightness are sequential, that such a space cannot be found in the class of compact spaces. Given a point x in a topological space X, define the **set tightness** of X at x (this name is due to Juhász), as the smallest cardinal number τ having the following property: If $x \in \bar{A} \setminus A$ then there is a family γ of subsets of A with $|\gamma| \leqslant \tau$, such that $x \in \operatorname{cl} \bigcup \gamma$, but $x \notin \bigcup\{\operatorname{cl} C\colon C \in \gamma\}$. This cardinal invariant is denoted by $t_s(x, X)$ and $t_s(X) = \sup\{t_s(x, X)\colon x \in X\}$. In general $t_s(x, X) \leqslant t(x, X)$ for any T_1 space. However, the equality $t_s(X) = t(X)$ holds for any compact Hausdorff space and the equality $t_s(x, X) = t(x, X)$ holds for any point in a radial T_1 space. In [16] and in [12], there are examples of regular pseudoradial spaces for which either $t(X) < R\chi(X)$ or $t_s(X) < t(X)$. It is very simple to find a pseudoradial T_1 space of countable tightness which is not sequential. Let $p \notin \mathbb{R}$ and put $X = \mathbb{R} \cup \{p\}$. Points $x \in \mathbb{R}$ have their usual neighbourhoods, while neighbourhoods of p are sets of the type $U \cup \{p\}$, where U is open in $\mathbb{R}$ and $\mathbb{R} \setminus U$ is countable. Under CH this example can be modified to obtain a non-sequential Hausdorff pseudoradial space with countable tightness. To describe a more sophisticated example in

ZFC, consider the following sketch of a construction. Let κ_0 be any cardinal number and define $\kappa_{n+1} = 2^{\kappa_n}$ and $\kappa = \sup\{\kappa_n : n \in \omega\}$. Endow κ_n with the discrete topology and set $M = \prod_{n\in\omega} \kappa_n$. M is a complete metric, ***zero-dimensional*** space and $|M| = 2^\kappa = \kappa^\omega$. The metric topology on a suitable subset X of M is refined in such a way that $\langle X, \tau\rangle$ is a ***first-countable***, ***locally compact***, ***locally countable*** space and has the property that if $C \subset X$ is closed in X either $|C| \leqslant \kappa$ or $|C| = 2^\kappa$. Let $p \notin X$ and $Y = X \cup \{p\}$, topologized as $\mathbb{R} \cup \{p\}$ above but now with $|X \setminus U| \leqslant \kappa$. Then Y is a pseudoradial Hausdorff space, with $t(Y) = \kappa$. Starting with $\kappa_0 = 2$, the construction gives a space Y which is ***normal*** pseudoradial, has countable tightness but is not sequential. If $\kappa_0 = \omega$, the space Y can be shown to be a ***regular*** zero-dimensional pseudoradial space in which $t_s(Y) = \omega$, while $t(Y) = \kappa > \omega$. Finally, observe that the one-point compactification of ***Ostaszewski's space*** is a Hausdorff compact pseudoradial and not sequential space of countable tightness.

Concerning the possibility to embed a space into a pseudoradial space, not much is known. It was shown in [10] that, under MA, every T_1 space of countable tightness is T_1-subpseudoradial and every countable space with only one accumulation point is T_2 subpseudoradial. Dow and Zhou have shown that CH implies that for any $p \in \beta\mathbb{N} \setminus \mathbb{N}$ the space $\mathbb{N} \cup \{p\}$ can be embedded into a regular pseudoradial space and there is a model in which for any P-point $p \in \beta\mathbb{N} \setminus \mathbb{N}$ the space $\mathbb{N} \cup \{p\}$ cannot be embedded in any regular pseudoradial space.

The class of pseudoradial spaces received new attention after Šapirovskiĭ's theorem in 1990 on the equivalence, under CH, of sequential compactness and pseudoradialness in compact spaces (see [15]). Šapirovskiĭ's result was then improved by Juhász and Szentmiklóssy (see [11]). They proved that if $\mathfrak{c} \leqslant \aleph_2$ then every compact sequentially compact space is pseudoradial and that it is consistent with ZFC + $\mathfrak{c} = \aleph_3$ that there is a compact sequentially compact space which is not pseudoradial. Furthermore, in [8] it is shown that the statement "every compact sequentially compact space is pseudoradial" is consistent with the continuum be arbitrarely large.

The crucial point in the proof given in [11] is the following fact: if X is a compact and sequentially compact space and $H \subset X$ is radially closed but not closed then there exists a closed G_λ-set K such that $K \cap H = \emptyset$ and $K \cap \operatorname{cl} H \neq \emptyset$, where λ has uncountable cofinality and $\lambda^+ < \mathfrak{c}$. As a byproduct of the previous result we have that any compact space which is not pseudoradial has a closed non-pseudoradial space of density less than $\mathfrak{c}$. In fact this was improved in [8], by showing that the continuum $\mathfrak{c}$ can be replaced by the ***splitting number*** $\mathfrak{s}$. This suggests the question as to what extent pseudoradialness is determined by the closed separable subspaces.

It was proved in [5] that if a compact space cannot be mapped onto I^{ω_2} and all of its closed separable subspaces are pseudoradial, then the space is pseudoradial. As a corollary one gets that if we assume that 2^{ω_2} is not pseudoradial then every non-pseudoradial compact space has a closed separable non-pseudoradial subset. This reinforces Šapirovskiĭ's opinion that the structure of the space 2^{ω_2} plays an important role in the study of pseudoradial spaces. Recall that a subset $A \subset X$ is **κ-closed** if for any $B \subset A$, with $|B| \leqslant \kappa$, $\operatorname{cl} B \subset A$. Šapirovskiĭ called a space **$\aleph_0$-pseudoradial** provided that every $\aleph_0$-closed non-closed subset $A \subset X$ contains a sequence converging to a point outside A. He proved that if a compact space cannot be mapped onto I^{ω_1}, then it is $\aleph_0$-pseudoradial and conjectured that the same holds if I^{ω_2} replaces I^{ω_1}. In [5] it was proved that, assuming the **independence number** $\mathfrak{i}$ (the miminum size of a maximal ***independent family***) is larger than ω_1, if a compact space cannot be mapped onto I^{ω_2}, then it is $\aleph_0$-pseudoradial. Dow has recently proved that there is a model of ZFC in which the space 2^{ω_2} is pseudoradial.

It was pointed out by Herrlich [13] that in general the class of pseudoradial spaces is not productive. In fact it has a very bad behaviour with respect to the product operation. To see this, consider the following example due to Gerlits and Nagy. It shows that even the product of two very good radial spaces, one compact metric and the other a ***Lindelöf*** space with only one non-isolated point, may fail to be pseudoradial. Let $X = \omega_1 \cup \{p\}$ be the one point Lindelöfization of the discrete space ω_1 (i.e., neighbourhoods of p are co-countable subsets of X) and let I be the unit segment with the Euclidean topology. Fix a one-to-one map $f : \omega_1 \to I$ and let $A = \{\langle\alpha, f(\alpha)\rangle : \alpha \in \omega_1\} \subset X \times I$. Observe first that no sequence contained in A can converge to a point outside A. Indeed, if $S \subset A$ is a sequence converging to a point outside A then $\pi_X(S)$ must converge to p and hence $|S| = \omega_1$. On the other hand, since I is first-countable, $\pi_I(S)$ cannot converge to any point. But A is not closed. If $x \in I$ is a complete accumulation point of $f[\omega_1]$ then $\langle p, x\rangle \in \operatorname{cl} A \setminus A$. The example just discussed consists of a product of a compact space with a Lindelöf one. This leaves a certain hope to find positive results on the productivity of pseudoradialness in the class of compact spaces. In what follows we shall need two more special classes of pseudoradial spaces: the R-monolithic and the semiradial ones. A space X is called **R-monolithic** if every $|A|$-radially closed subset of $\operatorname{cl} A$ is closed for any $A \subset X$. Equivalently, a R-monolithic space is a pseudoradial space X satisfying the formula $R\chi(\operatorname{cl} A) \leqslant |A|$ for any $A \subseteq X$. A space X is said to be **semiradial** provided that all its κ-radially closed subsets are κ-closed. The class of semiradial spaces contains both radial and R-monolithic spaces and is contained in the class of almost radial spaces. It is not difficult to find a compact almost radial space which is not semiradial. Let a $\{T_\alpha : \alpha < \mathfrak{t}\}$ be a defining family for the tower number, i.e., $T_\alpha \subseteq^* T_\beta$ whenever $\alpha > \beta$ and if A is an infinite subset of ω then there exists some α such that $|A \setminus T_\alpha| = \aleph_0$. Let $\{A_\alpha : \alpha \in \mathfrak{t}\}$ be the sequence of complements and put $A_\mathfrak{t} = \mathbb{N}$. A topology on the set $X = \mathbb{N} \cup \{x_\alpha : \alpha \leqslant \mathfrak{t}\}$ is given such that each point of $\mathbb{N}$ is isolated and neighbourhoods of x_α are the sets $\{x_\gamma : \beta < \gamma \leqslant \alpha\} \cup (A_\alpha \setminus (A_\beta \cup F))$, where F is a finite subset of $\mathbb{N}$ and $\beta < \alpha$. The space X is compact and almost radial, but it is not semiradial: the set $X \setminus \{x_\mathfrak{t}\}$ is non-closed separable and sequentially closed.

The comparison of both radial and semiradial with R-monolithic spaces still deserves study. It is very easy to find an R-monolithic compact space which is not radial: every compact sequential space which is not Fréchet–Urysohn space does the job. For the converse, the consistent examples mentioned before of compact separable radial non-Fréchet–Urysohn spaces are obviously examples of compact radial non-R-monolithic spaces. Another compact semiradial space which is not R-monolithic is $2^{\aleph_1}$ under the assumption $\mathfrak{p} > \aleph_1$.

The possibility to have a ZFC example of a compact radial non-R-monolithic space in the separable case is clearly disproved by the result of Dow that there is a model in which every compact separable radial space is Fréchet–Urysohn. A quite surprising result is that there are no ZFC example even with density $\aleph_1$. Chang's Conjecture implies that a compact radial space of density $\aleph_1$ that is not R-monolithic has a compact separable subspace that is not R-monolithic either, see [4]. **Chang's Conjecture** is the following statement: For any structure of the form $\mathfrak{A} = \langle \omega_2, \omega_1, (R_i)_{i<\omega} \rangle$, in which ω_1 is treated as a distinguished unary relation and each R_i is a finitary relation symbol, there is an elementary substructure $\mathfrak{B}$ of $\mathfrak{A}$ such that its universe B has cardinality $\aleph_1$ and $B \cap \omega_1$ is countable.

It can be shown that the construction of the model in [7] can be modified (assuming the existence of a ***measurable cardinal***) so as to ensure that it is also a model of Chang's Conjecture. Therefore, in the resulting model every compact radial space of density at most ω_1 is R-monolithic. R-monolithic spaces seem very closed to sequential spaces and it is evident that a separable R-monolithic space is always sequential. It is not difficult to find a 'good' ccc R-monolithic space which is not sequential (see below). But things are more interesting in the class of compact spaces and indeed in [4] it is proved that the existence of a compact ccc R-monolithic non-sequential space is independent of ZFC.

Coming back to the product operation, it is clear that, as a consequence of the result of Juhász and Szentmiklóssy, assuming $\mathfrak{c} \leqslant \aleph_2$ the class of compact pseudoradial spaces is countably productive. Taking into account that the assertion $\mathfrak{p} = \mathfrak{c}$ is also called **Booth's Lemma** (BL), we may denote the weaker assumption $\mathfrak{p}^+ \geqslant \mathfrak{c}$ by BL^-. It was proved in [5] that if BL^- holds, then a compact sequentially compact space that cannot be mapped onto I^{ω_2} is pseudoradial. As a consequence on products we have that if BL^- holds and 2^{ω_2} is not pseudoradial, then the product of countably many compact pseudoradial spaces is pseudoradial. Furthermore, under BL^-, the class of compact pseudoradial spaces of weight not exceeding $\aleph_2$ is countably productive.

In ZFC even the finite productivity of the class of compact pseudoradial spaces is still an open problem. The best results known so far are: [6] the product of two compact Hausdorff pseudoradial spaces is pseudoradial if one of them is semiradial; [5] the product of two compact pseudoradial spaces is pseudoradial provided that one of them has radial character not exceeding ω_1; [4] the class of compact R-monolithic spaces is countably productive; [14] the class of compact pseudoradial spaces having size less than $2^{\aleph_2}$ is countably productive. Moreover, D. Malykhin has proved that the product of countably many compact semiradial spaces is almost radial and E. Murtinová has proved that the product of ω_1 many compact semiradial spaces is pseudoradial if and only if it is sequentially compact.

Concerning $C_p(X)$ (the space of continuous real functions on a Tychonoff space X, with the topology of pointwise convergence) investigations about their pseudoradiality were made by Gerlits, Nagy and Szentmiklóssy (see [9]). They proved that $C_p(X)$ is radial if and only if it is Fréchet–Urysohn. For the pseudoradiality the situation is more complex and was completely described only for spaces of the type $C_p(\xi)$, where ξ is an ordinal number with the order topology. $C_p(\xi)$ is pseudoradial if and only if $\operatorname{cf}\xi \leqslant \omega$ (in this case it is Fréchet) or ξ is an ω-inaccessible regular cardinal. Recall that a cardinal κ is said to be **ω-inaccessible** if $\mu^\omega < \kappa$ for ever $\mu < \kappa$. Next, in [4] it was observed that if ξ is a regular ω-inaccessible cardinal then $C_p(\xi)$ is actually R-monolithic. Hence $C_p(\omega_1)$ is not pseudoradial, while $C_p(\mathfrak{c}^+)$ is a ccc R-monolithic space which is not sequential. We do not know any example of a Tychonoff space X for which $C_p(X)$ is pseudoradial but not R-monolithic.

We finish by considering another notion somehow very similar to pseudoradialness. We say that a space X has the **weak Whyburn property** if for any non-closed set $A \subseteq X$ there exists a set $B \subseteq A$ satisfying $|\operatorname{cl} B \setminus A| = 1$. It is evident that here the set B plays the role of a sequence in the definition of pseudoradialness. In [3] it was proved that every semiradial space has the weak Whyburn property and that every compact space with the weak Whyburn property is pseudoradial. It is still unknown whether the class of compact spaces with the weak Whyburn property is finitely productive, but (see [3]) at least the product of a compact space with the weak Whyburn property with a compact semiradial space has the weak Whyburn property. Dow, using $\diamondsuit$, has recently constructed a compact pseudoradial space which has not the weak Whyburn property. The relationship between weak Whyburn property and almost radialness is not very clear. The one-point compactification of the Ostaszewski space is a compact space with the weak Whyburn property which is not almost radial. The question here is if there is a model of ZFC where every compact space with the weak Whyburn property is almost radial. A bit stronger problem, already formulated in [HvM, Chapter 17], asks whether there is a model in which for any compact space X and any non-isolated point $x \in X$ there exists a thin sequence converging to x.

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d-5 Compactness (Local Compactness, Sigma-Compactness etc.)

A ***topological space*** X is called a **compact space** provided that every ***open cover*** of X has a finite ***subcover***. De Morgan laws provide the following equivalent formulation: a space X is compact iff every family, which has the **finite intersection property** (i.e., every finite subfamily has a nonempty intersection) and consists of closed subsets of X, has nonempty intersection. Historically, the concept of compactness (for ***regular spaces***) was introduced by L. Vietoris [19] in 1921. His definition was the following: a regular space X is compact provided that every ***net*** in X has a ***cluster point***. In the class of regular spaces such a definition is equivalent to the usual one; see, e.g., [E, 3.23]. The definition using covers was given by P.S. Alexandroff and P.S. Urysohn [1] in 1923. There it is also announced that a space X is compact iff each infinite subset of X has a ***complete accumulation point***. A very general definition of compactness together with a detailed analysis is contained in Alexandroff and Urysohn's Mémoire [2]: a space X is called **$[\alpha, \beta]$-compact** (where α and β are infinite cardinals) whenever every open cover of X of cardinality at most β has a subcover of cardinality less than α. Moreover, X is called $[\alpha, \infty]$-compact whenever it is $[\alpha, \beta]$-compact for every $\beta \geqslant \alpha$. Compact spaces are precisely those which are both $[\aleph_0, \aleph_0]$-compact (***countably compact***) and $[\aleph_1, \infty]$-compact (***Lindelöf***). By this reason Russian mathematicians use the term 'bicompact' instead of 'compact'.

The classical Heine–Borel–Lebesgue Theorem says that every open cover of a closed interval has a finite subcover. The conclusion of this theorem was converted into the definition of compactness. Consequences are very useful for analysis: every continuous real-valued function defined on a closed interval is uniformly continuous. The same property remains valid for all compact metric spaces: if f is a continuous map of a compact metric space X into a metric space Y then for every positive number ε there exists a positive number δ such that for each two points $x, y \in X$ the distance between $f(x)$ and $f(y)$ is not greater than ε whenever the distance between x and y is not greater than δ, see [E, 4.1.8]. Every open cover of a compact metric space has a so-called Lebesgue number. More precisely: for every open cover $\mathcal{P}$ of a compact metric space X there exists a positive number ε, called the **Lebesgue number** of $\mathcal{P}$, such that each subset of X of diameter less than ε is contained in an element of $\mathcal{P}$; see, e.g., [K, Theorem 5.26]. It is also easy to show that if (X, ρ) is a compact metric space, then for each positive number $\varepsilon > 0$ there is a so-called **ε-net**, i.e., there exists a finite set $A \subseteq X$ such that for each $x \in X$ there exists $a \in A$ such that $\rho(x, a) < \varepsilon$. Consequently, each compact metric space is separable. On the other hand, a metric space is separable iff it has a countable ***base***. Thus, every compact metric space has a countable base. The converse theorem is also true: the topology of a compact Hausdorff space with a countable base is ***metrizable***; see [E, Chapter 4.1])

Several constructions in mathematics lead to compact spaces. One of them is the ***order topology***: for a linearly ordered set $(X, <)$ let the topology on X be generated by the subbase consisting of all sets of the form $\{x \in X \colon x < a\}$ and $\{x \in X \colon b < x\}$, where a and b are points of X. If the order is complete, i.e., if every subset of X has a supremum and an infimum (in sense of $<$), then the order topology is compact. In particular, every closed interval of the reals is compact. Another example comes from the theory of Boolean algebras: the set $\mathrm{Ult}(\mathbb{B})$ of all ultrafilters of a Boolean algebra $\mathbb{B}$, endowed with the topology generated by sets of the form $\{p \in \mathrm{Ult}(\mathbb{B}) \colon u \in p\}$, where $u \in \mathbb{B}$, is compact, Hausdorff and ***zero-dimensional***. The space $\mathrm{Ult}(\mathbb{B})$ is called the ***Stone space*** of $\mathbb{B}$. Every Boolean algebra $\mathbb{B}$ is isomorphic to the field of all closed-open subsets of the Stone space $\mathrm{Ult}(\mathbb{B})$. A similar representation theorem is also true for distributive lattices: each distributive lattice is isomorphic to the lattice of all closed subsets of a compact (not necessary Hausdorff) space. In the ring theory there is considered the so-called **Zariski topology**. It is defined on the set $\mathrm{Spec}(R)$ of all prime ideals of a commutative ring R (with the unity) and generated by sets of the form $\{I \in \mathrm{Spec}(R) \colon A \subseteq I\}$, where $A \subseteq R$. The Zariski topology is compact and it is additionally Hausdorff iff each prime ideal in R is maximal.

Compactness is usually considered together with the Hausdorff separation property; see, e.g., [E, Chapter 3]; a compact Hausdorff space is quite often referred to as a **compactum**. This is because of the following properties: a compact Hausdorff spaces is ***normal*** and a compact subspace of a Hausdorff space is closed. Closed subsets of compact spaces are compact, and continuous maps preserve compactness, i.e., the image of a compact space under a ***continuous map*** is compact. From this property we get the following: every continuous map of a compact space into a Hausdorff one is ***closed***. On the other hand, a subset of a Euclidean space is compact iff it is closed and bounded. This implies another important property frequently used in analysis: every real-valued continuous function defined on a compact space is bounded and attains its bounds. Since continuous maps of compact spaces into Hausdorff spaces are closed we get also the following: a one-to-one continuous map of a compact space onto a Hausdorff one is a homeomorphism. Therefore each compact Hausdorff topology on a set X is maximal in the set of all compact topologies on X and also it is minimal in the set of all Hausdorff topologies on X. Minimal Hausdorff topology can be characterized as an ***H-closed*** topology which has a base consisting of regular open sets. H-closed spaces are strongly related to the compact ones: regular H-closed spaces are compact.

There are several notions extending compactness. Perhaps the most natural ones are local compactness and σ-compactness. A Hausdorff space X is called a **locally compact space** provided that every point of X has an open neighbourhood the closure of which is compact, whereas X is called **σ-compact** provided it is the union of countably many compact subspaces. The space $\mathbb{R}$ of the reals is both locally compact and σ-compact but the subspace of all irrationals is neither locally compact nor σ-compact. Open subspaces as well as closed subspaces of locally compact spaces are locally compact. Locally compact spaces are ***completely regular*** but σ-compact ones do not even have to be regular. Indeed, the topology on the closed interval $[0, 1]$ generated by all open sets in the natural topology together with the set $[0, 1] \setminus \{\frac{1}{n}: n \in \mathbb{N}\}$ is σ-compact but not regular. Regular σ-compact spaces are ***normal*** since they have the ***Lindelöf*** property. For locally compact spaces this is not the case since, for instance, the ***Tychonoff plank*** $(\omega + 1) \times (\omega_1 + 1) \setminus \{(\omega, \omega_1)\}$ is locally compact but not normal; see [E, 3.12.20]. Another simple example of such a space is $\{0, 1\}^{\aleph_1} \setminus \{x\}$, where x is an arbitrary point of the ***Cantor cube*** $\{0, 1\}^{\aleph_1}$.

Basic in the subject of compact spaces and fundamental in all general topology is the **Tychonoff Product Theorem**. It says that the ***product*** of a family of compact spaces is compact. The same is true for compact Hausdorff spaces: product of compact Hausdorff spaces is compact Hausdorff. Both two theorems are non-effective. The first one is equivalent to the Axiom of Choice whereas the second one is equivalent to the Prime Ideal Theorem. It should be noted here that for both locally compact spaces and for σ-compact ones only the finite version of the Tychonoff Product Theorem is true; i.e., product of a finite family of locally compact (σ-compact) spaces is locally compact (σ-compact respectively). One cannot extend, however, this theorem to infinite products. In fact, the countable product of the set of natural numbers, i.e., the space $\mathbb{N}^{\aleph_0}$, is neither locally compact nor σ-compact since it is homeomorphic to the space of irrationals.

Among the consequences of the Tychonoff Product Theorem we counts the fact that for every cardinal κ both the ***Tychonoff cube*** I^κ and the Cantor cube $\{0, 1\}^\kappa$ are compact Hausdorff. On the other hand, by the **Tychonoff Embedding Theorem** (see [E, 2.3.23]) every completely regular space can be embedded into a Tychonoff cube, more precisely a completely regular space of weight κ can be embedded into I^κ. Therefore, every completely regular space X can be embedded as a dense subspace into a compact Hausdorff space cX. The space cX is called a ***compactification*** of X. In the set of all compactifications of a given completely regular space X one may consider a partial order; see, e.g., [E, Chapter 3.5]. The greatest element in this order is called the ***Čech–Stone compactification*** of the space X and denoted βX. The space βX can be characterized (up to a homeomorphism) by the following property: for every continuous map f of the space X into a compact Hausdorff space Y there exists a continuous map $\tilde{f}$ from βX into Y such that $\tilde{f}$ restricted to X equals f.

If the space X is locally compact then it has also the smallest compactification αX known as the ***Alexandroff one-point compactification***. The compactification αX can be obtained by adding a new point to the space X, say $*$. The neighbourhood base at the point $*$ consists of sets of the form $\{*\} \cup U$, where U is an open subset of X such that the complement of U in X is compact, whereas the remaining points have the same neighbourhoods as in X. Clearly, X is an open subspace of αX. Therefore local compactness can be characterized as follows: a space is locally compact iff it is an open subspace of a compact Hausdorff space. For the space $X = \{0, 1\}^{\aleph_1} \setminus \{x\}$ considered above both Čech–Stone and Alexandroff compactification are equal (in the topological sense) to the Cantor cube $\{0, 1\}^{\aleph_1}$.

A special role in the realm of compact spaces belongs to the space $\beta\mathbb{N}$, the Čech–Stone compactification of the set of natural numbers with discrete topology. In particular it is widely used to investigate the structure of compact Hausdorff spaces. It can also be obtained as the Stone space of the Boolean algebra (in fact the field of sets) of all subsets of the set of natural numbers. Topologically, this space can be characterized by the following properties: it is compact Hausdorff, ***extremally disconnected***, and contains a countable and dense set of ***isolated points***. The space $\beta\mathbb{N}$ apears also in real analysis: let us consider the set $N = \{p_n: n \in \mathbb{N}\}$, where p_n is the nth projection from the Cantor set $\{0, 1\}^{\aleph_0}$ onto $\{0, 1\}$, i.e., $p_n(x) = x(n)$ for every $n \in \mathbb{N}$. The set N is discrete in the Cantor cube $\{0, 1\}^{2^{\aleph_0}}$ and each two disjoint subsets of N have disjoint closures. Therefore, by the above characterization, the closure of the set N in the Cantor cube $\{0, 1\}^{2^{\aleph_0}}$ is homeomorphic to $\beta\mathbb{N}$; see also the article "The Čech–Stone compactifications of $\mathbb{N}$ and $\mathbb{R}$" in this Encyclopedia. It was proved in fact by Sierpiński [11] that all functions that are in the closure of the set N but not in N are non-measurable. Actually, he considered non-trivial finitely additive measures on the field of all subsets of $\mathbb{N}$ rather than points of $\beta\mathbb{N} \setminus \mathbb{N}$; see also Z. Semadeni [17, p. 247].

If cX is a compactification of the space X then we can assume that X is contained in cX. The subspace $cX \setminus X$ is called the ***remainder*** of X in cX. In particular, $\beta X \setminus X$ is called the Čech–Stone remainder of X. The Čech–Stone remainder of a completely regular space X is compact Hausdorff iff X is locally compact. Moreover, if X is locally compact and σ-compact then the Čech–Stone remainder of X is a compact ***F-space***; a completely regular space Y is an F-space iff each two disjoint open F_σ-subsets of Y have disjoint closures. Both $\beta\mathbb{R} \setminus \mathbb{R}$ and $\beta\mathbb{N} \setminus \mathbb{N}$ are examples of F-spaces. Every regular extremally disconnected space is an F-space and every F-space that satisfies the ***Souslin property*** (the ***countable chain condition***) is extremally disconnected. Every closed subspace of an F-space is an F-space.

It follows that every infinite compact Hausdorff F-space contains a subspace homeomorphic to $\beta\mathbb{N}$. Since non-empty G_δ-subsets of $\beta\mathbb{N}\setminus\mathbb{N}$ have non-empty interior the space $\beta\mathbb{N}\setminus\mathbb{N}$ cannot to be extremally disconnected; see [KV, Chapter 11].

Each separable compact Hausdorff space X is a continuous image of $\beta\mathbb{N}\setminus\mathbb{N}$. To see this it is enough to take a map from $\mathbb{N}$ onto a dense subset of X that the preimage of each point is infinite. The theorem of I.I. Parovičenko says that every compact Hausdorff space of weight not greater than $\aleph_1$ is the image of $\beta\mathbb{N}\setminus\mathbb{N}$ by a continuous map; see [KV, Chapter 11] for related results. If a compact space X is mapped by a continuous map onto a Hausdorff space Y then the weight of Y is not greater than that of X. On the other hand weight of the space $\beta\mathbb{N}\setminus\mathbb{N}$ equals continuum. Therefore, assuming the Continuum Hypothesis, we obtain the following: each compact Hausdorff space of weight at most continuum is a continuous image of $\beta\mathbb{N}\setminus\mathbb{N}$. This assertion, however, cannot be proved in ZFC. According to a theorem by A. Dow and K.P. Hart [7], under assumption of the ***Open Colouring Axiom*** [OCA] the Stone space of the measure algebra is not a continuous image of $\beta\mathbb{N}\setminus\mathbb{N}$. By the measure algebra we understand the quotient Boolean algebra of Borel subsets of the reals by the ideal of measure zero sets in the Lebesgue sense. The Stone space of the measure algebra is extremally disconnected, has the Souslin property but is not separable, so A. Dow and K.P. Hart ask whether it is consistent with ZFC that each continuous image of $\beta\mathbb{N}\setminus\mathbb{N}$ which is extremally disconnected has to be separable.

Metrizable spaces and $\beta\mathbb{N}$ lie in completely different regions of the class of compact spaces: no metrizable space can contain $\beta\mathbb{N}$ as a subspace and no infinite closed subspace of $\beta\mathbb{N}$ is metrizable. In the class of metric spaces compactness can be described in terms of ***sequences***: a metric space is compact iff each sequence of elements of the space has a convergent subsequence, i.e., in the class of metric spaces compactness and sequential compactness coincide. Compact spaces, however, need not contain convergent sequences. For instance, F-spaces do not contain non-trivial convergent sequences and so, in particular, $\beta\mathbb{N}\setminus\mathbb{N}$ also has this property. This justifies the following Dichotomy Problem posed by B.A. Efimov: is it true that every infinite compact space contains either a non-trivial convergent sequence or a subspace homeomorphic to $\beta\mathbb{N}$? Assuming $2^{\aleph_0}=2^{\aleph_1}$ and $\mathfrak{s}=\aleph_1$ (see [KV, Chapter 3] for definition of $\mathfrak{s}$) V.V. Fedorčuk [8] proved that there exists a compact Hausdorff space of cardinality continuum which has no isolated points and no convergent sequences. Clearly, such a space does not contain $\beta\mathbb{N}$ since $\beta\mathbb{N}$ is of power $2^{2^{\aleph_0}}$. In [9] V.V. Fedorčuk, under the assumption of Continuum Hypothesis, constructed for every natural number $n>0$, a compact Hausdorff space X such that each closed infinite subspace of X is of ***dimension*** n. Since $\beta\mathbb{N}$ is zero-dimensional, this again produces a compact Hausdorff space which contains neither a nontrivial convergent sequence nor a copy of $\beta\mathbb{N}$. However, it seems to be unknown whether the answer to the Dychotomy Problem can be positive in other variants of set theory.

From the Hewitt–Marczewski–Pondiczery Theorem (see, e.g., [E. 2.3.15]), the Tychonoff cube $I^{2^{\aleph_0}}$ is separable and thus it is a continuous image of the space $\beta\mathbb{N}$. Therefore, from Tietze–Urysohn Extension Theorem (see, e.g., [E, 2.1.8.]), a compact Hausdorff space contains a copy of $\beta\mathbb{N}$ iff it has a continuous map onto the Tychonoff cube of weight continuum. This fact leads to a much more general problem: what is the greatest cardinal κ such that a compact Hausdorff space X has a continuous map onto the Tychonoff cube of weight κ? The answer to this question has been given by B.E. Šapirovskiĭ [15]. He has proved that a compact Hausdorff space X has a continuous map onto the Tychonoff cube of weight κ iff X contains a closed subset $A\subseteq X$ such that for every $x\in A$ the ***π-character*** of x in A is at least κ. Another result of B.E. Šapirovskiĭ [16] says that if X is a regular space and κ is an infinite cardinal and the set of all points of X in which the π-character is not greater than κ is dense, then the family of all ***regular closed*** subsets of X is of the power not greater than $\kappa^{c(X)}$. The two Šapirovskiĭ's Theorems together imply the following: every compact Hausdorff space with the Souslin property which is at least of weight $(2^{\aleph_0})^+$ has a continuous map onto the Tychonoff cube of weight $(2^{\aleph_0})^+$. A corresponding theorem for Boolean algebras was proved by S. Shelah [12]. Finally, let us add that if a compact Hausdorff space X is extremally disconnected, then it has a continuous map onto the Cantor cube of weight κ, where κ is the weight of X; see B. Balcar and F. Franěk [3].

Another consequence of the Tychonoff Product Theorem asserts that the ***limit*** of an ***inverse system*** consisting of (non-empty) compact Hausdorff spaces and continuous ***bonding map***s is compact Hausdorff and non-empty; see [E, 3.2.13]. This theorem is frequently used in several constructions of peculiar compact spaces (see, e.g., the construction of V.V. Fedorčuk [8]) but can also be used to describe some classes of compact spaces. The class of Dugundji spaces is one of them. A compact Hausdorff space X is called **Dugundji compact** provided that every continuous map from a closed subspace of a Cantor cube $\{0,1\}^\kappa$, where κ is an arbitrary cardinal, into X has a continuous extension over the whole cube $\{0,1\}^\kappa$. Every compact Hausdorff space is a continuous image of a closed subset of a Cantor cube; see, e.g., [E, 3.2.2]. Therefore Dugundji spaces are ***dyadic***. The converse implication, however, is not true. On the other hand, all compact metric spaces are Dugundji. A very useful characterization of Dugundji spaces in terms of inverse systems was given by R. Haydon (see [HvM, Chapter 18]): a compact Hausdorff space is a Dugundji space iff it is the limit of an inverse system $\{X_\sigma, p^\delta_\sigma;\ \sigma,\delta\in\Sigma\}$ satisfying the following conditions: (1) all the spaces X_σ are compact, (2) all the maps p^δ_σ are ***open*** surjections and have countable weight, (3) the set Σ is well ordered and the system is ***continuous***, i.e., every X_σ is the inverse limit of all X_δ for $\delta<\sigma$ whenever δ is a limit element in Σ, (4) the space X_0 (where 0 is the smallest element of Σ) is a metric space. The notion of **countable weight of a continuous map** f from a compact Hausdorff space X onto a Hausdorff space Y used here means that there exists a countable family $\mathcal{P}$ of ***cozero sets*** of X such that the family $\{f^{-1}(U) \cap V: U$ is open in Y and $V \in \mathcal{P}\}$ is a base in X. The class of κ-metrizable spaces introduced by E.V. Ščepin (see [HvM, Chapter 18]) contains the class of Dugundji spaces. This class also has a characterization in terms of inverse limits. A compact Hausdorff space X is κ-metrizable iff it is the limit of a continuous inverse system $\{X_\sigma, p^\delta_\sigma;\ \sigma, \delta \in \Sigma\}$ in which all the spaces X_σ are compact metric, all the bonding maps p^δ_σ are open surjections and the set Σ is partially ordered with the property that all of its countable chains have the supremum. E.V. Ščepin proved also that all Dugundji spaces are κ-metrizable and all compact κ-metrizable spaces of weight $\aleph_1$ are Dugundji; see [HvM, Chapter 18]. The original definition of a **κ-metrizable space** is similar to that of metric: a non-negative real-valued function ρ defined on the set $X \times RC(X)$, where $RC(X)$ stands for the family of all regular closed subsets of X, is called a **κ-metric** on X provided that the following conditions hold true: (1) $\rho(x, F) = 0$ iff $x \in F$, (2) $\rho(x, F) \geqslant \rho(x, G)$ whenever $F \subseteq G$, (3) for every fixed F the function $\rho(x, F)$ is continuous with respect to x, (4) if $\mathcal{L}$ is a chain in $RC(X)$ then $\rho(x, \mathrm{cl} \bigcup \mathcal{L}) = \inf\{\rho(x, F)$: $F \in \mathcal{L}\}$ for each $x \in X$. Now, a space is κ-metrizable provided it has a κ-metric.

A well-known proof of the Tychonoff Product Theorem, see, e.g., [K, Chapter 5], use the **Alexander Subbase Lemma**: if $\mathcal{S}$ is a subbase of a space X then X is compact iff every cover of X consisting of elements of $\mathcal{S}$ admits a finite subcover. J. de Groot utilized the Alexander Lemma to introduce a new class of compact spaces. A **supercompact space** is a Hausdorff space X with a **binary subbase** $\mathcal{S}$, which means that every cover of X by elements taken from $\mathcal{S}$ has a subcover consisting of two elements. Every compact ordered topology is supercompact. In particular, closed intervals are supercompact. M. Strok and A. Szymański proved that all compact metric spaces are supercompact; see [HvM, Chapter 18]. In contrast to that M. Bell (see [HvM, Chapter 18]) proved that the space $\beta\mathbb{N}$ is not supercompact. The mysterious structure of supercompactness became much more clear when E.K. van Douwen and J. van Mill proved: if A is an infinite countable subset of a continuous image of a supercompact space then all but countably many points of $\mathrm{cl}\, A$ are limits of nontrivial convergent sequences; see [HvM, Chapter 18]. The theorem was recently improved by Zhongqiang Yang [20] who proved that in fact each cluster point of a countable set contained in a continuous image of a supercompact space is a limit of a nontrivial sequence. In particular, no infinite compact F-space can be supercompact. The class of supercompact spaces is closed under products. Therefore, both Tychonoff cubes and Cantor cubes are supercompact. Countable images of supercompact spaces do not have to be supercompact since M. Bell constructed a dyadic space of weight $\aleph_3$ which is not supercompact; see [HvM, Chapter 18]. It appears to be an open question whether all Dugundji spaces are supercompact. In the zero-dimensional case the question was answered affirmatively by L. Heindorf [10]. Assuming $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ L. Shapiro [13] proved that there exists a κ-metrizable compact Hausdorff space without any non-trivial convergent sequences and thus the space is not supercompact. For a κ-metrizable space X the space $\exp(X)$ of all closed subsets of X endowed with the ***Vietoris topology*** is κ-metrizable as well. However, it was proved by M. Bell [4] that $\exp(\{0, 1\}^{\aleph_2})$ is not supercompact. In fact it cannot be even a continuous image of a supercompact space. So, in particular, it is not ***dyadic***.

For locally compact spaces, the ***Baire Category Theorem*** holds true: the union of countably many nowhere dense subsets of a locally compact space has empty interior; see, e.g., [K, 3.9.3]. This assertion is not valid for uncountable families of nowhere dense sets. For instance, if X is the set of all countable ordinals endowed with the (compact Hausdorff) topology generated by all intervals and sets of the form $\{0\} \cup (\alpha, \omega_1)$, where $\alpha < \omega_1$, then the space $X^{\aleph_1}$ is the union of a family of $\aleph_1$ nowhere dense subsets. ***Martin's Axiom***, however, is equivalent to the following extension of the Baire Category Theorem: if X is a compact Hausdorff space with the Souslin property, then the union of fewer than $2^{\aleph_0}$ nowhere dense subsets of X has an empty interior; for other consequences of Martin's Axiom see [Ku, Chapter II.2]. Recently L. Shapiro [14] has shown that under the assumption of Martin's Axiom, for every compact Hausdorff space X of weight less than $2^{\aleph_0}$ and for every infinite cardinal $\kappa < 2^{\aleph_0}$ the following conditions are equivalent: (1) X has a continuous map onto I^κ, (2) X contains a Cantor cube of weight κ, (3) X contains a closed subset Y such that each point of Y has a character in Y not less than κ. Thus, in particular, under the assumption of Martin's Axiom, each infinite compact Hausdorff space of weight less than the continuum contains a non-trivial convergent sequence.

There are several types of compact spaces arising from functional analysis. For every space X let $C_p(X)$ be the set of all continuous real-valued functions on X endowed with the topology inherited from the product $\mathbb{R}^X$. In connection with the Dichotomy Problem discussed above let us mention the following Rosenthal Theorem (see, e.g., S. Todorčević [18]): if X is a compact metric space then every infinite set of continuous real-valued functions on X contains an infinite subset whose closure in $\mathbb{R}^X$ is either a convergent sequence or is homeomorphic to $\beta\mathbb{N}$. In this context let us also mention a theorem of W.F. Eberlein: if a space X is compact Hausdorff and Y is a compact subspace of $C_p(X)$ and A is a subset of Y then every cluster point of A is a limit of a sequence of elements of A; see, e.g., [18].

There is a number of properties of compact spaces which are of combinatorial nature. The Auslander–Ellis Theorem says that if X is a compact Hausdorff space and $(X, +)$ is a **left topological semigroup** (i.e., for ever $x \in X$ the **left translation** $l_x: y \mapsto x + y$ is a continuous map; as apposed to a **topological semigroup**, where $(x, y) \mapsto x + y$ is continuous), then there exists a point $x \in X$ such that $x + x = x$. Since the space $\beta\mathbb{N}$ can be considered as a left topological semigroup this implies the famous Hindman's Theorem: if the set $\mathbb{N}$ is divided into finitely many pieces then one of them contains an infinite set S such that all finite nonrepeating sums of members of S remain in this piece. The classical **Van der Waerden Theorem** (asserting that every finite partition of the set of natural numbers has an element containing arbitrary large arithmetic progresions) can be derived from the following: if X is a compact Hausdorff space and G is a commutative countable group of homeomorphisms of‘X onto itself such that for every $x \in X$ the set $\{h(x)\colon h \in G\}$ is dense in X then for every non-empty set $U \subseteq X$ and every $h_1, h_2, \ldots, h_n \in G$ there exists an integer m such that the family of iterations $\{h_1^m(U), h_2^m(U), \ldots, h_n^m(U)\}$ has non-empty intersection; see [5] and for other results of this type see also [18]. Another type of combinatorics on compact spaces is represented by the following theorem: if X is a zero-dimensional compact Hausdorff space and f is a continuous map of X into itself and f does not have ***fixed points*** then X can be divided into 3 clopen parts such that if U is an element of this partition then $f(U)$ is disjoint with U; see [6].

This short review of notions and results shows that compactness is a wide-spread notion in topology. The general scheme of compact-like properties introduced by Alexandroff and Urysohn is also a source of other notions involving open covers. This, however, is the subject of other autonomic parts of topology.

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d-6 Countable Compactness

1. Definitions

A ***topological space*** X is called **countably compact** provided that for every countable ***open cover*** $\mathcal{U}$ of X there is a finite $\mathcal{V} \subset \mathcal{U}$ such that $\mathcal{V}$ also covers X (we say $\mathcal{V}$ is a finite **subcover** of $\mathcal{U}$). This definition is a special case of a general scheme for compactness properties introduced in 1929 by P. Alexandroff and P. Urysohn [1]. For any pair $a \leqslant b$ of infinite cardinal numbers, they defined a space X to be $[a, b]$**-compact** provided that every open cover $\mathcal{U}$ of X, with the cardinality of $\mathcal{U}$ at most b, has a subcover $\mathcal{V} \subset \mathcal{U}$ with the cardinality of $\mathcal{V}$ strictly less than a. The notation $b = \infty$ means that there is no upper bound on the cardinality of $\mathcal{U}$. Much work has been done on this general property (e.g., see [17]), and on special cases of it: In the terminology of Alexandroff–Urysohn, ***compactness*** is the same as $[\omega, \infty]$-compactness, countable compactness is the same as $[\omega, \omega]$-compactness, and the ***Lindelöf*** property is the same as $[\omega_1, \infty]$-compactness. It should be noted that some authors require the ***Hausdorff*** separation axiom as part of the definition of many open covering properties such as compactness and countable compactness (cf. [E]).

2. Theorems and examples

Countable compactness can be characterized in several useful ways: The following are equivalent for a space X: (a) X is countably compact, (b) Every sequence (x_n) in X has a ***cluster point*** (c) every countably infinite $A \subset X$ has a ***complete accumulation point*** (d) every countable family $\mathcal{F}$ of ***closed*** subsets of X with the property that every finite intersection of members of $\mathcal{F}$ is non-empty, satisfies $\bigcap\{F\colon F \in \mathcal{F}\} \neq \emptyset$.

Countable compactness behaves like compactness in some ways. Among these are: countable compactness is preserved by ***continuous*** images and ***perfect*** preimages; every closed subset of a countably compact space is countably compact; every continuous real-valued function defined on a countably compact space has bounded range; the ***projection*** $\pi : X \times Y \to Y$ is a ***closed map*** whenever X is countably compact and Y is a ***sequential space***; and a countably compact Hausdorff space X whose ***diagonal*** in $X \times X$ is a G_δ***-set***, is ***metrizable*** (all these results can be found in [E, Chapter 3]).

Countable compactness is related to a number of compactness-like properties which evoke countability. Among such properties are sequential compactness (a space is **sequentially compact** if every sequence has a ***convergent subsequence***), and ***pseudocompactness*** (every real valued continuous function is bounded). It is easy to see that sequential compactness is the strongest of the three properties, and pseudocompactness is the weakest. Closely related to pseudocompactness is feeble compactness (every ***locally finite*** family of open sets is finite). In ***completely regular*** spaces, pseudocompactness and feeble compactness are equivalent. Another property that is stronger than countable compactness, but not directly related to sequential compactness, is ω**-boundedness** (every countable set is contained in a compact set).

Another related property arises from an interesting characterization of countable compactness in the class of Hausdorff spaces that does not explicitly mention countability: For every open cover $\mathcal{U}$ of X there is a finite $F \subset X$ such that $\{U \in \mathcal{U}\colon F \cap U \neq \emptyset\}$ covers X (see [E, 3.12.23 (d)]). This characterization leads to a property called **absolute countable compactness** (for every open cover $\mathcal{U}$ and every ***dense*** set D, there exists a finite $F \subset D$ such that $\{U \in \mathcal{U}\colon F \cap U \neq \emptyset\}$ covers the space) (see [11]). We mention that there exists a ***separable*** countably compact space that is not absolutely countably compact [19], and a ***normal***, countably compact space that is not absolutely countably compact [14].

Countable compactness is equivalent to compactness in many important classes of spaces (e.g., in the class of metrizable spaces). Thus, for example, one can check that a metrizable space is compact by checking the formally simpler condition of countable compactness. One way to see that these two properties are equivalent in metrizable spaces is to use the theorem of A.H. Stone that metrizable spaces are ***paracompact*** (i.e., every open cover has a locally finite open refinement) since in countably compact spaces, every locally finite family is finite. Countable compactness and compactness are equivalent in larger classes such as the class of ***meta-Lindelöf spaces***, and the class of ***subparacompact spaces***. It follows that compactness and countable compactness are equivalent in the class of spaces which have a ***point-countable base***, and in the class of ***Moore spaces***. A list of such properties is given in [KV, Chapter 12].

The class of countably compact non-compact spaces is a deep and difficult class with interesting relations to many topics in general topology and set theory. The study of this class has generated many interesting theorems and examples. A fundamental example in this class is the set of countable ordinals with the ***order topology***: ω_1. This is a non-compact space (since it has no largest element) which is countably compact, and has a number of interesting properties such as ***normality*** and ***first-countability*** (hence ω_1 is sequentially compact).

In contrast to the Tychonoff product theorem (every product of compact spaces is compact) J. Novák and H. Terasaka constructed two countably compact subsets $X, Y \subset \beta\omega$ such that $X \times Y$ is not countably compact (see [E, Chapter 3]). Indeed, many interesting countably compact, non-compact spaces have been constructed as subsets of $\beta\omega$ (the ***Čech–Stone compactification*** of the natural numbers).

Compact Hausdorff spaces are normal (e.g., satisfy the Tietze–Urysohn extension theorem [E, 2.1.8]) but countably compact Hausdorff spaces need not be normal [E, 3.12.20]. There has been significant effort devoted to understanding this difference. A classical example of a countably compact Hausdorff space which is not normal is the square version of the Tychonoff plank $(\omega_1 + 1) \times (\omega_1 + 1) \setminus \{(\omega_1, \omega_1)\}$ [GJ, 8L]. Moreover, there is a countably compact, first-countable non-normal Hausdorff space (such a space is necessarily absolutely countably compact) (see [18] and [11]).

Sequential compactness clearly implies countable compactness, but the converse is false since $\beta\omega$ is compact and has no non-trivial convergent sequences. A product of countably many sequentially compact spaces is sequentially compact, hence countably compact [E, 3.10.35]. A.H. Stone and C.T. Scarborough proved that in fact, every product of ω_1 sequentially compact spaces is countably compact [16], and raised the Scarborough–Stone problem: Is every product of sequentially compact spaces, countably compact? This problem, which is still not completely solved, has generated much work. P. Nyikos and J. Vaughan [13] constructed families of sequentially compact Hausdorff spaces (one for each ***ultrafilter*** $u \in \beta\omega \setminus \omega$) whose product is not countably compact, but these spaces are not ***regular*** and the problem is still open in the class of regular, completely regular, and ***normal*** spaces. We mention this problem twice more.

A. Dow [5] proved the following interesting reflection theorem: If X is countably compact, and is a space in which every subset of cardinality at most ω_1 is metrizable, then X is metrizable.

A recent idea of I. Juhász, J. Klimó and Z. Szentimiklóssy [10] is to apply a topological partition theorem to the study of countable compactness, and more generally to **initial κ-compactness**, which is the same as $[\omega, \kappa]$-compactness. For example, the following is a simple corollary to their partition theorem: If X is a countably compact T_1-space and $\mathcal{U}$ is an open cover of X such that every uncountable subset of X has a finite subset which is contained in at most countable many elements of $\mathcal{U}$ then $\mathcal{U}$ has a finite subcover. This strengthens Aquaro's well-known theorem [E, 3.12.23] that says a point-countable open cover of a countably compact T_1-space has a finite subcover.

All results mentioned above are provable in ZFC.

3. Results using set theory

Returning to the Scarborough–Stone problem, Nyikos and Vaughan gave several models in which there are sequentially compact completely regular spaces $\{X_\alpha : \alpha < \kappa\}$, where $\kappa < \mathfrak{c} = 2^\omega$ (the cardinality of the continuum) such that $\prod_{\alpha<\kappa} X_\alpha$ is not countably compact. Of course, $\omega_2 \leqslant \kappa$ by the result of Scarborough–Stone above.

Next we consider a question which we know cannot be settled in ZFC. Is every countably compact regular space in which every closed set is a G_δ-set, compact? W. Weiss proved that the answer is "yes" assuming $\mathfrak{p} > \omega_1$ (see [KV, Chapter 3]), and A. Ostaszewski proved the answer is "no" assuming the combinatorial principle $\diamondsuit$ (see [Ku, Chapter 2]) by constructing a remarkable space now called Ostaszewski's space. Among other interesting properties, Ostaszewski's space is countably compact, ***perfectly normal*** (hence first-countable and every closed set is G_δ), ***locally compact***, ***locally countable***, every closed set is either countable or co-countable, ***hereditarily separable*** and not compact. From these properties, it is clear that Ostaszewski's space is an example of a first-countable, countably compact, non-compact space that does not contain a copy of ω_1. It is consistent, however, that every first-countable, countably compact, non-compact space contains a copy of ω_1, and this holds in certain iterated forcing models and under the proper forcing axiom which we discuss next (see [3]).

The ***Proper Forcing Axiom*** (PFA) is a strong set-theoretic principle which implies $2^\omega = \omega_2$ (see [4]). We mention some results that hold under the assumption of PFA. The result (2) concerns the Scarborough–Stone problem again, and solves the problem in the opposite direction to the ZFC solution mentioned above for Hausdorff spaces.

Assuming PFA:

(1) in a countably compact ***hereditarily normal*** space, every countable subset has a compact ***Fréchet–Urysohn closure***, hence, every hereditarily normal countably compact Hausdorff space is sequentially compact [12],
(2) every product of countably compact, hereditarily normal spaces is countably compact [12],
(3) a separable hereditarily normal countably compact space does not contain a copy of ω_1 [12],
(4) a countably compact ***manifold*** is metrizable if and only if it does not contain a copy of ω_1 (see [2]),
(5) every first-countable, countably compact space is either compact or contains a copy of ω_1 [3], and
(6) in a regular, countably compact space X of ***character*** $\leqslant \omega_1$ (e.g., $[0,1]^{\omega_1}$) the closure of any set A can be obtained in two steps: if we let A_1 be the set of all limits of a convergent sequences from A, and let A_2 denote the set of all limits of convergent ω_1-sequences from A_1; then A_2 is the closure of A in X [3],
(7) all countably compact hereditarily normal topological groups are metrizable [6].

One of the most widely used set theoretic assumptions in topology is the ***Continuum Hypothesis*** (CH): "$2^\omega = \omega_1$". Nevertheless, many set-theoretic assumption used in topology contradict CH (such as PFA or $\mathfrak{p} > \omega_1$). Statements proved using such assumptions, raise the question as to how the statement is related to CH. Such questions often go deeper into set theory. For example, concerning the statement (5) above, T. Eisworth and P. Nyikos proved, among other things, the strong result that it is consistent with CH that $(*)$ every first-countable countably compact space is either compact or contains a copy of ω_1 [8]. Using $(*)$ and CH,

G. Gruenhage proved that every countably compact space with a ***small diagonal*** is metrizable (Gruenhage also proved the consistency of the statement that there exist a countably compact space with a ***small diagonal*** that is not metrizable [9]). Moreover, Oleg Pavlov [15] has announced that $\diamondsuit^+$ implies there exists a countably compact space with a ***small diagonal*** that is not metrizable. Since $\diamondsuit^+ \Rightarrow \diamondsuit \Rightarrow \mathsf{CH}$, the statement "every countably compact space with a small diagonal is metrizable" is consistent with and independent of the Continuum Hypothesis (i.e., of $\mathsf{ZFC} + \mathsf{CH}$). Concerning the theorem of Weiss mentioned above using $\mathfrak{p} > \omega_1$, Eisworth proved that it is consistent with the Continuum Hypothesis that every perfectly normal countably compact space is compact [7]. Thus, $\diamondsuit$ implies, but CH does not imply, the existence of a perfectly normal, countably compact, non-compact space.

The Handbook article [KV, Chapter 12] surveys material on countable compactness up to 1984.

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d-7 Pseudocompact Spaces

For ***topological*** spaces X and Y, $C(X,Y)$ will denote the family of continuous functions from X into Y, $C(X)$ will denote $C(X,\mathbb{R})$, and $C^*(X)$ will denote the family of bounded functions in $C(X)$. A space X is called **pseudocompact** provided that $C(X)=C^*(X)$. This definition was first given for ***Tychonoff space***s, i.e., completely regular T_1-spaces, by Hewitt [11].

Researchers have found useful a number of conditions equivalent to pseudocompactness. Before stating them we give related definitions. For $f\in C(X)$, the **zero set** of f, $f^{-1}(0)$, will be denoted $Z(f)$, and by the **cozero set** of f is meant $X\setminus Z(f)$. One defines $\mathcal{Z}(X)=\{Z(f)\colon f\in C(X)\}$ and $\mathcal{C}(X)=\{X\setminus Z\colon Z\in\mathcal{Z}(X)\}$. For a collection $\mathcal{B}$ of sets, $\mathcal{B}^\wedge$ denotes the family of all finite intersections of members of $\mathcal{B}$, and $\mathcal{B}$ is called **fixed** if $\bigcap\mathcal{B}\neq\emptyset$. The **cluster set** of a filter base $\mathcal{F}$ on a space X is defined to be the intersection of the closures of the members of $\mathcal{F}$. By a **cluster point** of a sequence of subsets $\{S_n\colon n\in\mathbb{N}\}$ of a space X is meant a point $p\in X$ such that for every neighbourhood V of p, $V\cap S_n\neq\emptyset$ for infinitely many integers n. An open filter base $\mathcal{F}$ on X (open cover $\mathcal{F}$ of X) is called **completely regular** (**cocompletely regular**) provided that for each $F\in\mathcal{F}$ there exists $G\in\mathcal{F}$ such that G and $X\setminus F$ (F and $X\setminus G$) are completely separated. A subset Y of X is said to be ***C*-embedded** in X provided that every function in $C(Y)$ can be extended to a function in $C(X)$. A sequence $\{f_n\}$ of real valued functions is said to **converge to a function** f **uniformly at** x provided that for every $\varepsilon>0$ there are a neighbourhood U_x of x and number N such that $|f_n(y)-f(y)|<\varepsilon$ whenever $y\in U_x$ and $n\geqslant N$. The terminology used in this chapter generally agrees with that in [Ke] – except where noted otherwise, no separation axioms are required. For definitions not given here or in [Ke], see [E].

For any space X the following are equivalent.

(a) X is pseudocompact.
(b) For every space Y and $f\in C(X,Y)$, the image $f(X)$ is pseudocompact.
(c) If $f\in C^*(X)$ ($f\in C(X)$), then $f(X)$ is a compact subset of $\mathbb{R}$.
(d) If $f\in C^*(X)$ ($f\in C(X)$), then $f(X)$ is a closed subset of $\mathbb{R}$.
(e) For every $f\in C^*(X)$ there exists $x_0\in X$ such that $f(x_0)=\sup_{x\in X}f(x)$.
(f) If $\mathcal{B}$ is any countable family in $\mathcal{Z}(X)$ such that $\emptyset\notin\mathcal{B}^\wedge$, then $\mathcal{B}$ is fixed.
(g) If $\mathcal{U}\subset\mathcal{C}(X)$ is any countable cover of X, then $\mathcal{U}$ has a finite subcover.
(h) Every locally finite subfamily of $\mathcal{C}(X)$ is finite.
(i) Every countable completely regular filter base on X is fixed.
(j) Every countable cocompletely regular cover of X has a finite subcover.
(k) X contains no C-embedded copy of $\mathbb{N}$.
(l) Every sequence in $C(X)$ that converges uniformly at each point of X converges uniformly on X.
(m) Dini's theorem holds for $C^*(X)$, that is, if $\{f_n\}$ is any sequence in $C^*(X)$, $f\in C^*(X)$, each $f_i\leqslant f_{i+1}$, and $f_n(x)\to f(x)$ for all $x\in X$, then $f_n\to f$ uniformly.
(n) Ascoli's theorem holds for $C^*(X)$: Every bounded equicontinuous family in the Banach space $C^*(X)$ has compact closure.
(o) For every nonnegative functional J on $C^*(X)$ there is a measure m with respect to which every element of $C^*(X)$ is measurable for which $J(f)=\int f(x)\,dm$ for each $f\in C^*(X)$.

Hewitt [11] proved that (a), (c), (e), and (f) are equivalent for a Tychonoff space X, Glicksberg [9] proved that (a), (e), (m), (n), and (o) are equivalent for any space X, and Bagley, Connell and McKnight [2] proved (a) and (l) are equivalent. Proofs and discussion concerning some of the other equivalences can be found in [E], [7] and [15].

1. Between countably compact and pseudocompact

Every ***countably compact*** space is pseudocompact. This implication can be divided into a number of subimplications.

For a topological space X, each of the following conditions implies the next, and in each of the groups, (C_1)–(C_6), (D_1)–(D_2), (E_1)–(E_2), (F_1)–(F_2), and (G_1)–(G_2), the conditions in that group are equivalent. No other implications hold among (A)–(H) for spaces at least T_1. If the space X is regular, then (C)–(G) are equivalent. If X is completely regular, then (C)–(H) are equivalent, and if X is a weakly normal (respectively, normal) T_1-space, then (A)–(G) (respectively, (A)–(H)) are equivalent.

(A) The space X is countably compact.
(B) There is a dense subset D of X such that every countable filter base on D has an adherent point in X.
(C_1) Every locally finite family of open sets of X is finite.
(C_2) Every pairwise disjoint locally finite family of open sets of X is finite.
(C_3) If $\mathcal{U}=\{U_n\colon n\in\mathbb{N}\}$ is a sequence of nonempty open subsets of X such that $U_i\cap U_j=\emptyset$ whenever $i\neq j$, then $\mathcal{U}$ has a cluster point in X.
(C_4) Every countable open filter base on X has an adherent point.
(C_5) Every countable open cover of X has a finite subcollection whose union is dense in X.

(C_6) If $\mathcal{U}$ is a countable open cover of X and A is an infinite subset of X, then the closure of some member of $\mathcal{U}$ contains infinitely many points of A.
(D_1) Every locally finite open cover of X has a finite subcover.
(D_2) Every countably infinite locally finite open cover of X has a finite subcover.
(E_1) Every countably infinite locally finite open cover of X has a finite subcollection whose union is dense in X.
(E_2) Every countably infinite locally finite open cover of X has a proper subcollection whose union is dense in X.
(F_1) Every star-finite open cover of X is finite.
(F_2) Every star-finite open cover of X has a finite subcover.
(F_3) Every star-finite open cover of X has a finite subcollection whose union is dense in X.
(F_4) Every countably infinite open cover of X by sets each of which meets at most two others has a proper subcollection whose union is dense in X.
(G_1) Every countable regular filter base on X is fixed.
(G_2) Every countable coregular cover of X has a finite subcover.
(H) The space X is pseudocompact.

For the above, we recall that a space X is said to be **weakly normal** provided that whenever F and G are disjoint closed subsets of X, at least one of which is countable, then there exist disjoint open subsets U and V of X such that $F \subset U$ and $G \subset V$. An open filter base $\mathcal{F}$ on a space X (open cover $\mathcal{F}$ of X) is called a **regular filter base** (a **coregular cover**) provided that for each $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that $\overline{G} \subset F$ ($\overline{F} \subset G$). An open cover $\mathcal{U}$ of a space is called **star-finite** provided that each member of $\mathcal{U}$ meets at most finitely many other members of $\mathcal{U}$.

The conditions (A)–(H) have been studied by a number of authors, especially (C_2), which has been referred to in [18] as **feebly compact** and attributed to S. Mardešić and P. Papić, and (C_1), which was called **lightly compact** in [2]. A space satisfying (B) is called **e-countably compact** (or **e-countably compact with respect to** D). Discussions and proofs concerning the implications holding among the conditions (A)–(H) can be found in [2, 7, 12, 15, 18], and in the articles by J. Colmez, K. Iseki, S. Kasahara, S. Mardešić, P. Papić and S. Watson cited in the preceding.

2. Star-covering and other conditions

Some other similar conditions studied are listed next. Two of them involve stars. For an open cover $\mathcal{U}$ of a space X and a subset A of X, the **star of** A **with respect to** $\mathcal{U}$ is defined by

$$\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon U \cap A \neq \emptyset\};$$

we set $\mathrm{St}^1(A,\mathcal{U}) = \mathrm{St}(A,\mathcal{U})$ and $\mathrm{St}^{n+1}(A,\mathcal{U}) = \mathrm{St}(\mathrm{St}^n(A,\mathcal{U}),\mathcal{U})$. Stars with respect to open covers provide a nice way of characterizing some topological properties such as those being considered here. For example, a Hausdorff space X is countably compact if and only if for every open cover $\mathcal{U}$ of X there is a finite subset F of X such that $\mathrm{St}(F,\mathcal{U}) = X$. The latter may be compared with the result that: (1) a regular T_1-space X is countably compact if and only if every point-finite open cover of X has a finite subcover; or (2) a T_1-space X is countably compact if and only if every infinite open cover of X has a proper subcover – see [E], [Ke] and [12].

Let X be a topological space. Then the condition (C_1) implies each condition below, (I_1) and (I_2) are equivalent, (I_2) implies (J), (J) implies (K), (K) implies X is pseudocompact, (I_1) implies (L), (L) implies (M), and (M) implies (E_2). If X is a regular space, then each of (I)–(M) is equivalent with (C_1).

(I_1) If $\mathcal{U} = \{U_n\colon n \in \mathbb{N}\}$ is a sequence of nonempty open subsets of X such that $\overline{U_i} \cap \overline{U_j} = \emptyset$ whenever $i \neq j$, then $\mathcal{U}$ has a cluster point in X.
(I_2) Every discrete family of open sets in X is finite.
(J) For every open cover $\mathcal{U}$ of X there is a finite subset $\mathcal{A}$ of $\mathcal{U}$ such that $\mathrm{St}^2(\bigcup\mathcal{A},\mathcal{U}) = X$.
(K) For every open cover $\mathcal{U}$ of X there is a finite subset F of X such that $\mathrm{St}^3(F,\mathcal{U}) = X$.
(L) Every infinite open cover of X has a proper subcollection whose union is dense in X.
(M) Every point-finite countably infinite open cover of X has a proper subcollection whose union is dense in X.

The condition (I_1) is due to Glicksberg [9], where he proved that in any space X, (I_1) implies X is pseudocompact, and for a Tychonoff space X, the converse holds. Then in [10] Glicksberg noted that a similar statement could be made about the condition (C_2) or (C_3). Condition (I_2) is called the **discrete finite chain condition (DFCC)** and has been used by J.W. Green and many other authors. For proofs and further examples concerning DFCC and the other statements in the previous paragraph, see the articles by Matveev [12] and van Douwen, Reed, Roscoe and Tree [5].

In case a space X is Tychonoff, the list above can be expanded, and one obtains a very nice characterization theorem. For any Tychonoff space X, the following are equivalent: X is pseudocompact; every nonempty zero set in the Stone–Čech compactification βX of X meets X; no nonempty closed G_δ-subset of βX is contained in $\beta X \setminus X$ (i.e., X is G_δ**-dense** in βX); and the ***Hewitt realcompactification*** υX of X satisfies $\upsilon X = \beta X$. This theorem is due to Hewitt – see [E], [7] or [11].

When combined with any of a variety of other conditions, pseudocompactness implies or nearly implies compactness. Some examples: Every regular T_1-, locally feebly compact space is a Baire space. Every pseudocompact space that is realcompact or metrizable is compact. By a proof of S. Watson obtained in 1980, every regular feebly compact T_1-space (e.g., every pseudocompact Tychonoff space) that is weakly paracompact is compact. For definitions of these concepts and proofs or references to proofs, see [E] or [17].

Here are a few examples of pseudocompact spaces that are not compact, some of which illustrate that a number of the conditions above are different. See also [2], [E] and [18].

Example: Let $T = (\omega_1 + 1) \times (\omega_0 + 1) \setminus \{(\omega_1, \omega_0)\}$, with its usual topology. As noted by Hewitt [11], T is a pseudocompact Tychonoff space that is not countably compact.

Example: Let $\mathcal{M}$ be a maximal infinite family of infinite subsets of $\mathbb{N}$ such that the intersection of any two members of $\mathcal{M}$ is finite, and let $\Psi = \mathbb{N} \cup \mathcal{M}$, where a subset U of Ψ is defined to be open provided that for any set $M \in \mathcal{M}$, if $M \in U$ then there is a finite subset F of M such that $\{M\} \cup M \setminus F \subset U$. The space Ψ is then a first-countable pseudocompact Tychonoff space that is not countably compact. The space Ψ is due independently to J. Isbell and S. Mrówka – see [E] or [7]. Both of the spaces Ψ and T are e-countably compact and locally compact.

Example: Let A be an uncountable set, T be the circle group, and define $G = \{g \in T^A$: for all but countably many $a \in A$, $g_a = e^{2\pi i q}$ for some $q \in \mathbb{Q}\}$ and $H = \{g \in T^A$: for all but countably many $a \in A$, $g_a = 1\}$. Then G and H are topological groups. Since H is a dense subspace of G which is countably compact, G is pseudocompact, and since G is a proper dense subspace of T^A, neither H nor G is compact. G also fails to be countably compact. Similar examples are in [E].

Example: Let $X = \mathbb{Q} \times \mathbb{Q}^+$, where a subset U of X is defined to be open if and only if: for each point $(p, 0) \in U$ there exists $\varepsilon > 0$ so that $\{(x, 0) \in X: |x - p| < \varepsilon\} \subset U$; and for each point $(p, q) \in U$, where $q > 0$, there exists $\varepsilon > 0$ so that $\{(x, 0) \in X: |x - (p - q/\sqrt{3})| < \varepsilon$ or $|x - (p + q/\sqrt{3}) < \varepsilon\}$ $\subset U$. The space X was given in 1953 by R. H. Bing as an example of a countable connected Hausdorff space. As noted in [15], X satisfies the discrete finite chain condition. Hence by the result stated earlier, it also satisfies condition (E_1). It can be shown that X fails to satisfy condition (D_1).

Example: Let $X = [0, 1]$, $\mathcal{T}$ be the usual topology on X, in $(X, \mathcal{T})$, choose dense subsets $\{D_n: n \in \mathbb{N}\}$ such that whenever $i \neq j$, $D_i \cap D_j = \emptyset$, and let $\mathcal{U}$ be the topology on X which has as a base $\{D_{2n-1}: n \in \mathbb{N}\} \cup \{D_{2n-1} \cup D_{2n} \cup D_{2n+1}: n \in \mathbb{N}\}$. Then $C((X, \mathcal{U})) = C((X, \mathcal{T}))$, and hence $(X, \mathcal{U})$ is pseudocompact [15], but $(X, \mathcal{U})$ satisfies none of the other conditions among (A)–(M).

3. Subspaces, images and extensions

We consider next some of the results known about whether these properties are or can be inherited by subspaces, preserved or reflected by continuous maps, or possessed by extension spaces.

While countable compactness is inherited by closed subsets, the preceding examples illustrate that in general this is not the case with any of the other conditions considered above. Some inheritance conclusions that do hold are stated next.

Let X be a space. If X is e-countably compact with respect to D and $A \subset D$, then $\bar{A}$ is e-countably compact. If X is feebly compact (respectively, satisfies DFCC) and A is an open subset of X, then $\bar{A}$ is feebly compact (respectively, satisfies DFCC). If X and $\bar{A} \setminus A$ are pseudocompact and A is an open subset of X, then $\bar{A}$ is pseudocompact. If every countably infinite closed subset of X is pseudocompact (and X is a T_1-space), then every infinite subset of X has a limit point in X (and hence X is countably compact).

Before stating some mapping theorems, we recall that a map $f: X \to Y$ is called **Z-closed** (**closed**) provided that for every zero set F (closed subset F) of X, $f(F)$ is a closed subset of Y. In case a map $f: X \to Y$ satisfies $f(F)$ is a proper closed subset of Y for every proper closed subset F of X, then f is called **irreducible**.

Let X and Y be topological spaces and $f \in C(X, Y)$ be onto Y. If $\mathcal{P}$ denotes one of the conditions (A)–(M) and X satisfies $\mathcal{P}$, then so does Y. If f is an open and Z-closed map onto Y, and Y and each fiber $f^{-1}(y)$, $y \in Y$, are pseudocompact, then X is pseudocompact. If f is closed, and Y and each $f^{-1}(y)$, $y \in Y$, are countably compact, then X is countably compact. If f is irreducible, each $f^{-1}(y)$, $y \in Y$, is countably compact, and Y is feebly compact (e-countably compact), then X is feebly compact (e-countably compact).

For proofs or references to proofs of some of these map and subspace theorems, see [E] or [15].

Example: Here is an example which illustrates that the condition "irreducible" cannot be removed from the preceding two map theorems. Let Ψ be the Isbell–Mrówka space described earlier, let $\mathbb{N}^-$ be the set of the negative integers, with the discrete topology, and let X be the discrete union of Ψ and $\mathbb{N}^-$. List in a 1–1 manner as $\{M_n: n \in \mathbb{N}\}$ the members of an infinite subset of $\mathcal{M}$, and define $f: X \to \Psi$ by the rule: $f(x) = x$ if $x \in \Psi$, and $f(x) = M_{-x}$ if $x \in \mathbb{N}^-$. Then X is not pseudocompact, Ψ is e-countably compact, and $f: X \to \Psi$ is a closed, continuous map of X onto Ψ, each of whose fibers is finite.

An area of topology in which there has been much interest is concerned with the question, for a given topological property $\mathcal{P}$, of which spaces are minimal $\mathcal{P}$, maximal $\mathcal{P}$, or $\mathcal{P}$-closed. By a **minimal $\mathcal{P}$-space** (respectively **maximal $\mathcal{P}$-space**) is meant a $\mathcal{P}$-space $(X, \mathcal{T})$ such that for every $\mathcal{P}$-topology $\mathcal{U}$ on X, if $\mathcal{U} \subset \mathcal{T}$ (respectively, $\mathcal{U} \supset \mathcal{T}$) then $\mathcal{U} = \mathcal{T}$. A $\mathcal{P}$-space X is a **$\mathcal{P}$-closed space** provided that X is a closed subspace of any $\mathcal{P}$-space in which it can be embedded. Here are some answers related to pseudocompact and feebly compact spaces.

First we give some definitions. A space $(X, \mathcal{T})$ is called **semiregular** provided that each of its closed subsets is an intersection of **regular closed** sets, i.e., sets having the form $\overline{T}$ (note that when used in this way, the words "regular closed" contain no hyphen), $T \in \mathcal{T}$. A space is called an **E_1-space** provided that each point of the space is a countable intersection of regular closed neighbourhoods of that point. A subset F of a space X is said to be **relatively pseudocompact** provided that for every $f \in C(X)$, $f(F)$ is a bounded subset of $\mathbb{R}$.

Let $(X, \mathcal{T})$ be a space.

(a) $(X, \mathcal{T})$ is maximal feebly compact if and only if it is feebly compact, every feebly compact subset of $(X, \mathcal{T})$ is closed, and every dense subset of $(X, \mathcal{T})$ is open in $(X, \mathcal{T})$.

(b) $(X, \mathcal{T})$ is maximal pseudocompact if and only if $(X, \mathcal{T})$ is pseudocompact and for every subset F of X, if F is relatively pseudocompact in the space whose topology is generated by $\mathcal{T} \cup \{X \setminus F\}$, then F is closed in $(X, \mathcal{T})$.
(c) If $\mathcal{P}$ denotes the property first-countable and Tychonoff, then X is minimal $\mathcal{P}$ if and only if it is a pseudocompact $\mathcal{P}$-space.
(d) If $\mathcal{P}$ denotes the property first-countable and regular T_1, then X is minimal $\mathcal{P}$ if and only if it is a feebly compact $\mathcal{P}$-space.
(e) Suppose $\mathcal{P}$ denotes one of the properties: first-countable and Hausdorff; or E_1. Then X is minimal $\mathcal{P}$ if and only if it is a feebly compact semiregular $\mathcal{P}$-space.
(f) If $(X, \mathcal{T})$ is a feebly compact semiregular space and $\mathcal{U}$ is any E_1-topology on X, then $\mathcal{U} \subset \mathcal{T}$ if and only if $\mathcal{U} = \mathcal{T}$.
(g) Suppose $\mathcal{P}$ denotes one of the properties: E_1; first-countable and Hausdorff; first-countable and regular T_1; first-countable and Tychonoff; Moore; perfectly normal T_1; or metrizable. Then X is $\mathcal{P}$-closed if and only if it is a feebly compact $\mathcal{P}$-space.

Proofs of the above can be found in articles of C.E. Aull, D. Cameron, J.W. Green, J.R. Porter, G.M. Reed, R.M. Stephenson Jr and R.G. Woods, and some use the result noted by Glicksberg in [9] that every G_δ-point in a feebly compact, regular space has a countable neighbourhood base. The space Ψ described earlier is a maximal feebly compact space, as well as a maximal pseudocompact space [13]. It is not known whether every maximal pseudocompact space is also maximal feebly compact. An example of a maximal feebly compact space that is not maximal pseudocompact is given in [13]. We observe that statement (f) in the preceding list implies that *every one-to-one continuous map of a pseudocompact completely regular space, in fact, of any feebly compact semiregular space, onto an E_1-space is a homeomorphism.*

A space X is called an **extension space** of a space Y if Y is a dense subspace of X. For properties $\mathcal{P}$ like those defined above in (e) or (g), authors such as M. Bell, E.K. van Douwen, B. Fitzpatrick, P. Nyikos, J. Porter, T.C. Przmusiński, G.M. Reed, P. Simon, R.M. Stephenson Jr, T. Terada, J. Terasawa, G. Tironi, S. Watson and R.G. Woods have investigated which spaces have $\mathcal{P}$-closed extension spaces – for example, see [17]. We state a few results obtained in this area. If $\mathcal{P}$ = first-countable and Hausdorff, every $\mathcal{P}$-space has a $\mathcal{P}$-closed extension space. Every locally pseudocompact (locally compact), first-countable Tychonoff space Y has a pseudocompact, first countable Tychonoff extension space X such that X is locally compact if Y is. There is a first-countable zero-dimensional Čech-complete T_1-space which has no first-countable, feebly compact, regular T_1-extension space. Every locally feebly compact regular T_1-space Y has a feebly compact regular T_1-extension space X such that Y is an open subset of X, and X is first countable at each point of $X \setminus Y$. Every locally feebly compact (locally compact) separable Moore space has a (locally compact) Moore-closed extension space.

4. Product spaces

We consider next some results concerning when a product of pseudocompact (feebly compact) spaces has that property. Let us first note that spaces such as the following show that the product of just two feebly compact spaces need not be pseudocompact.

Example: Let $\mathbb{N}$ have the discrete topology. For each infinite subset I of $\mathbb{N}$ choose one limit point P_I of I in $\beta\mathbb{N}$. Let $X = \mathbb{N} \cup \{P_I : I \text{ infinite}, I \subset \mathbb{N}\}$ and $Y = \mathbb{N} \cup (\beta\mathbb{N} \setminus X)$. Then X has 2^{ω_0} points, and since every infinite subset of $\beta\mathbb{N}$ has $2^{2^{\omega_0}}$ limit points [E], it follows that every infinite subset of $\mathbb{N}$ has limit points in X and limit points in Y. Thus, X and Y are pseudocompact, but the diagonal of $X \times Y$, $\{(n, n): n \in \mathbb{N}\}$, is an infinite, open-and-closed discrete subspace of $X \times Y$, and so $X \times Y$ is not pseudocompact.

More complicated examples, e.g., see [E] or [7], show that there is a countably compact Tychonoff space T such that $T \times T$ is not pseudocompact, and [3] there is a family of pseudocompact Tychonoff spaces whose product is not pseudocompact but whose finite subproducts are pseudocompact.

In checking whether some product space is pseudocompact, the following result [10, 14, 15] enables one to focus on subproducts: Let $\mathcal{F} = \{X_a : a \in A\}$ be a family of spaces. If $\prod \mathcal{F}$ is pseudocompact (respectively, feebly compact), then so is $\prod X_{b\in B}$ for every nonempty subset B of A. If $\prod_{c\in C} X_c$ is pseudocompact (respectively, feebly compact) for every countable subset $C \subset A$, then $\prod \mathcal{F}$ is pseudocompact (respectively, feebly compact).

If suitable restrictions are satisfied, pseudocompactness is productive. Some known results are the following. The product of a family of topological groups is pseudocompact if and only if each group in the family is pseudocompact. Every product of pseudocompact (respectively, feebly compact) spaces, of which all but one (at most) are sequentially compact, is pseudocompact (feebly compact). Every product of feebly compact spaces, of which all but one (at most) are locally compact, is feebly compact. If $X = \prod X_a$, where each X_a is feebly compact, and if (except perhaps for one value of a) each non-P point of X_a has a countable neighbourhood base in X_a, then X is feebly compact. The product of a pseudocompact k-space and a pseudocompact Tychonoff space is pseudocompact. Finally, suppose $\mathcal{F} = \{X_a : a \in A\}$ is a family of Tychonoff spaces. If $\prod \mathcal{F}$ is pseudocompact, then $\beta \prod \mathcal{F} = \prod_{a\in A} \beta X_a$. If $\beta \prod \mathcal{F} = \prod_{a\in A} \beta X_a$ and if for each $a_0 \in A$, $\prod_{a\neq a_0} X_a$ is infinite, then $\prod \mathcal{F}$ is pseudocompact.

Proofs of these product theorems can be found in [4, 14, 15]. Several of these theorems are generalizations of results which were obtained originally only for Tychonoff spaces.

Some researchers (whose interests lie particularly with properties of rings of continuous functions) consider $C(X)$ only for Tychonoff spaces X. One reason, as noted in [9], is that associated with every topological space (X, τ) is a unique completely regular space $(X, w\tau)$ such that $C((X, \tau)) = C((X, w\tau))$. The topology $w\tau$ is defined to be the weak topology induced on X by $C((X,\tau))$, i.e., the topology generated by all sets having the form $\{f^{-1}(T)\colon f \in C((X,\tau))$ and $T \in \tau\}$. The space $(X, w\tau)$ is a Tychonoff space if and only if $C((X,\tau))$ separates the points of X. (This notation is sometimes simplified to X and wX.) An example shows, however, that despite the nice properties of Tychonoff spaces, the "operator" w does not always behave well.

Example [16] There exists a space X such that wX is compact Hausdorff (and hence X is pseudocompact), but $X \times X$ fails to be pseudocompact, and consequently the compact Hausdorff space $wX \times wX$ and the nonpseudocompact Tychonoff space $w(X \times X)$ are not even homeomorphic.

In [8] J. Ginsberg and V. Saks extended results of Z. Frolík and studied a number of conditions which, when imposed on factor spaces, produce pseudocompact products. One such condition is given next. For any free ultrafilter $\mathcal{D}$ on $\mathbb{N}$, let us call a space X **$\mathcal{D}$-feebly compact** provided that for every sequence $\{U_n\colon n \in \mathbb{N}\}$ of nonempty open subsets of X there is a point $p \in X$ such that for every neighbourhood V of p, $\{n\colon V \cap U_n \neq \emptyset\} \in \mathcal{D}$. A Tychonoff space X having this property is called **$\mathcal{D}$-pseudocompact** [8].

The following holds. Let $\mathcal{D}$ be a free ultrafilter on $\mathbb{N}$. Every $\mathcal{D}$-feebly compact space (respectively, $\mathcal{D}$-pseudocompact space) is feebly compact (respectively, pseudocompact). Every product of $\mathcal{D}$-feebly compact spaces (respectively, $\mathcal{D}$-pseudocompact spaces) is $\mathcal{D}$-feebly compact (respectively, $\mathcal{D}$-pseudocompact).

Example: Let $\mathbb{N}$ have the discrete topology. For each point $p \in \beta\mathbb{N} \setminus \mathbb{N}$, let $K_p = \beta\mathbb{N} \setminus \{p\}$, and form $X = \prod\{K_p\colon p \in \beta\mathbb{N} \setminus \mathbb{N}\}$. Then [8] X is not $\mathcal{D}$-pseudocompact for any free ultrafilter $\mathcal{D}$ on $\mathbb{N}$, but all powers of X are pseudocompact.

5. τ-pseudocompact spaces

We conclude by briefly considering some other variations on pseudocompactness, defined and studied by J.F. Kennison in 1962 and A.V. Arhangel'skiĭ in 1998: For an infinite cardinal number τ, a Tychonoff space X is said to be **τ-pseudocompact** (**strongly τ-pseudocompact**) provided that for every function $f \in C(X, \mathbb{R}^\tau)$, $f(X)$ is compact ($f(A)$ is a closed subset of $\mathbb{R}^\tau$ for every closed subset A of X). In [1] and [6] one can find recent results concerning these concepts, such as the following: As τ becomes larger, τ-pseudocompactness becomes stronger and stronger, and ω-pseudocompactness is the same as pseudocompactness for Tychonoff spaces. Every normal τ-pseudocompact space is strongly τ-pseudocompact, and every strongly τ-pseudocompact space is countably compact. A 1969 result of N. Noble that every Tychonoff space can be embedded as a closed subset of a pseudocompact Tychonoff space is generalized to: For any infinite cardinal τ, every Tychonoff space is homeomorphic to a closed C^*-embedded subspace of a τ-pseudocompact space.

As noted in [1] and elsewhere, many interesting problems remain to be solved concerning spaces which are strongly τ-pseudocompact, τ-pseudocompact, pseudocompact, feebly compact or DFCC. For some very good lists of references including many significant contributions to this area, see [2, 5], [E], [8, 12] and [18].

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d-8 The Lindelöf Property

A topological space is said to be a **Lindelöf space**, or have the Lindelöf property, if every ***open cover*** of X has a countable ***subcover***. The Lindelöf property was introduced by Alexandroff and Urysohn in 1929, the term 'Lindelöf' referring back to Lindelöf's result that any family of open subsets of Euclidean space has a countable sub-family with the same union. Clearly, a space is ***compact*** if and only if it is both Lindelöf and ***countably compact***, though weaker properties, for example ***pseudocompactness***, imply compactness in the presence of the Lindelöf property. The real line is a Lindelöf space that is not compact and the space of all countable ordinals ω_1 with the order topology is a countably compact space that is not Lindelöf. It should be noted that some authors require the ***Hausdorff*** or ***regular*** (which we take to include T_1) separation axioms as part of the definition of many open covering properties (cf. [E]). For any unreferenced results in this article we refer the reader to [E].

There are a number of equivalent formulations of the Lindelöf property: (a) the space X is Lindelöf; (b) X is $[\omega_1, \infty]$***-compact*** (see the article by Vaughan in this volume); (c) every open cover has a countable ***refinement***; (d) every family of closed subspaces with the **countable intersection property**[1] has non-empty intersection; (e) (for regular spaces) every open cover of X has a countable subcover $\mathcal{V}$ such that $\{\overline{V}: V \in \mathcal{V}\}$ covers X (where $\bar{A}$ denotes the closure of A in X). In the class of ***locally compact*** spaces, a space is Lindelöf if and only if it is σ**-compact** (i.e., is a countable union of compact spaces) if and only if it can be written as an increasing union of countably many open sets each of which has compact closure.

It is an important result that regular Lindelöf spaces are ***paracompact***, from which it follows that they are ***collectionwise normal***. Conversely, every paracompact space with a ***dense*** Lindelöf subspace is Lindelöf (in particular, every separable paracompact space is Lindelöf) and every locally compact, paracompact space is a disjoint sum of ***clopen*** Lindelöf subspaces. A related result is that any ***locally finite*** family of subsets of a Lindelöf space is countable.

Closed subspaces and countable unions of Lindelöf spaces are Lindelöf. Continuous images of Lindelöf spaces are Lindelöf and inverse images of Lindelöf spaces under ***perfect maps***, or even ***closed maps*** with Lindelöf fibres, are again Lindelöf. In general, the Lindelöf property is badly behaved on taking either ***Tychonoff products*** or ***inverse limits***.

The ***Tychonoff product*** of two Lindelöf spaces need not be Lindelöf or even normal, although any product of a Lindelöf space and a compact space is Lindelöf and countable products of Lindelöf ***scattered*** spaces are Lindelöf [HvM, Chapter 18, Theorem 9.33]. It is also true that both the class of ***Čech-complete*** Lindelöf and Lindelöf Σ-spaces are closed under countable products. The **Sorgenfrey line**, which one obtains from the real line by declaring every interval of the form $(a, b]$ to be open, is a simple example of a Lindelöf space with non-normal square. Even more pathological examples are possible: Michael constructs a Lindelöf space, similar to the Michael line, which has non-normal product with a subset of the real line and, assuming the Continuum Hypothesis, constructs a Lindelöf space whose product with the irrationals is non-normal. Details and further results may be found in [KV, Chapter 18] and Section 9 of [HvM, Chapter 18]. A space is said to be ***realcompact*** if it is homeomorphic to a closed subspace of the Tychonoff product $\mathbb{R}^\kappa$ for some κ. Every regular Lindelöf space is realcompact and, whilst the ***inverse limit*** of a sequence of Lindelöf spaces need not be normal, both inverse limits and arbitrary products of realcompact spaces are realcompact. Hence arbitrary products and inverse limits of regular Lindelöf spaces are realcompact. In fact a space is realcompact if and only if it is the inverse limit of a family of regular Lindelöf spaces.

All **second-countable** spaces (i.e., spaces with a countable base to the topology) are both Lindelöf and ***separable***. The Sorgenfrey line is an example of a separable, Lindelöf space that is not second-countable. On the other hand, if X is ***metrizable*** (or even ***pseudometrizable***), then X is second-countable if and only if it is separable if and only if it has the ***countable chain condition*** if and only if it is Lindelöf. By ***Urysohn's Metrization Theorem***, a space is second-countable and regular if and only if it is a Lindelöf metrizable space if and only if it can be embedded as subspace of the ***Hilbert cube***.

A space X is said to be **hereditarily Lindelöf** if every subspace of X is Lindelöf. Since any space can be embedded as a dense subspace of a (not necessarily Hausdorff) compact space, not every Lindelöf space is hereditarily Lindelöf. However, a space is hereditarily Lindelöf if and only if every open subspace is Lindelöf if and only if every uncountable subspace Y of X contains a point y whose every neighbourhood contains uncountably many points of Y. A regular Lindelöf space is hereditarily Lindelöf if and only if it is ***perfect*** and hereditarily Lindelöf spaces have the ***countable chain condition*** but need not be separable.

In fact, for regular spaces there is a complex and subtle relationship between the hereditary Lindelöf property and **hereditary separability**[2] (both of which follow from second countability). An hereditarily Lindelöf regular space that is not (hereditarily) separable is called an L-space; an hereditarily separable regular space that is not (hereditarily) Lindelöf is called an S-space. The existence of S- and L-spaces

[1] A family of sets has the *countable intersection property* if every countable sub-family has non-empty intersection.

[2] A space is hereditarily separable if each of its subspaces is separable.

is, to a certain extent, dual and depends strongly on the model of set theory. For example, the existence of a ***Souslin line*** implies the existence of both S- and L-spaces, $\mathsf{MA} + \neg\mathsf{CH}$ is consistent with the existence of S- and L-spaces but implies that neither compact S- nor compact L-spaces exist. However, the duality is not total: Todorčević [11] has shown that it is consistent with MA that there are no S-spaces but that there exists an L-space, i.e., that every regular hereditarily separable space is hereditarily Lindelöf but that there is a non-separable, hereditarily Lindelöf regular space. It is currently an open question whether it is consistent that there are no L-spaces. For further details about S and L see Roitman's article [KV, Chapter 7], or indeed [11]. It is fair to say that the S/L pathology, along with Souslin's Hypothesis and the Normal Moore Space Conjecture, has been one of the key motivating questions of set-theoretic topology and it crops up frequently in relation to other problems in general topology, such as: the metrizability of ***perfectly normal manifolds*** [10]; Ostaszewski's construction of a countably compact, perfectly normal non-compact space [9]; and the existence of a counter-example to Katětov's problem "if X is compact and X^2 is hereditarily normal, is X metrizable?" [5].

The **Lindelöf degree** or **Lindelöf number**, $L(X)$, of a space X is the smallest infinite cardinal κ for which every open cover has a subcover of cardinality at most κ. The **hereditary Lindelöf degree**, $hL(X)$, of X is the supremum of the cardinals $L(Y)$ ranging over subspaces Y of X. The Lindelöf degree of a space is one of a number of cardinal invariants or ***cardinal function***s one might assign to a space. Cardinal functions are discussed in the article by Tamano in this volume, however, one result due to Arkhangel'skiĭ [1] is worth particular mention here. The **character** $\chi(x, X)$ of a point x in the space X is smallest cardinality of a ***local base*** at x and the **character** $\chi(X)$ of the space X is the supremum $\sup\{\chi(x, X)\colon x \in X\}$. A space with countable character is said to be **first-countable**. Arkhangel'skiĭ's result says that the cardinality of a Hausdorff space X is at most $2^{L(X)\cdot\chi(X)}$. In the countable case this theorem tells us that the cardinality of a first-countable, Lindelöf Hausdorff space is at most the continuum, $2^{\aleph_0}$, and that, in particular, the cardinality of a first-countable, compact Hausdorff space is at most the continuum.[3] This impressive result solved a problem posed thirty years earlier by Alexandroff and Urysohn (whether a first-countable compact space could have cardinality greater than that of the continuum), but was, moreover, a model for many other results in the field. The theorem does not remain true if we weaken first-countability, since it is consistent that the cardinality of a regular, (***zero-dimensional*** even) Lindelöf Hausdorff space with countable ***pseudocharacter*** can be greater than that of the continuum [12], and Lindelöf spaces can have arbitrary cardinality. However, de Groot has shown that the cardinality of a Hausdorff space X is at most $2^{hL(X)}$ [KV, Chapter 1, Corollary 4.10]. For a much more modern proof of Arkhangel'skiĭ's theorem than the ones given in [1] or [KV, Chapter 1], we refer the reader to Theorem 4.1.8 of the article by Watson in [HvM].

[3]In fact it turns out that first-countable, compact Hausdorff spaces are either countable of have cardinality exactly $2^{\aleph_0}$.

A space is compact if and only if every infinite subset has a ***complete accumulation point*** if and only if every increasing open cover has a finite subcover and a space is countably compact if and only if every countably infinite subset has a complete accumulation point. However, the requirement that every uncountable subset has a complete accumulation point is implied by, but does not characterize the Lindelöf property. Spaces satisfying this property are called **linearly Lindelöf**, since they turn out to be precisely those spaces in which every open cover that is linearly ordered by inclusion has a countable subcover. Surprisingly little is known about such spaces. There are (somewhat complex) examples of regular linearly Lindelöf, non-Lindelöf space in ZFC, but there is, at present, no known example of a normal linearly Lindelöf, non-Lindelöf spaces under any set theory. Such a space would be highly pathological: the problem intrinsically involves singular cardinals and any example is a **Dowker space**, that is, a normal space which has non-normal product with the closed unit interval $[0, 1]$. Nevertheless one can prove some interesting results about linearly Lindelöf spaces, for example every first-countable, linearly Lindelöf Tychonoff space has cardinality at most that of the continuum, generalizing Arhangel'skiĭ's result mentioned above. For more on linearly Lindelöf spaces see the paper by Arkhangel'skiĭ and Buzyakova [2].

One important sub-class of Lindelöf spaces, the **Lindelöf Σ-space**s, deserves mention. The notion of a Σ-space was introduced by Nagami [8], primarily to provide a class of spaces in which covering properties behave well on taking products. It turns out that there are a number of characterizations of Lindelöf Σ-spaces, two of which we mention here. A Tychonoff space is Lindelöf Σ if it is the continuous image of the preimage of a separable metric space under a perfect map. An equivalent (categorical) definition is that the class of Lindelöf Σ-spaces is the smallest class containing all compact spaces and all separable metrizable spaces that is closed under countable products, closed subspaces and continuous images. So, as mentioned above, countable products of Lindelöf Σ-spaces are Lindelöf Σ. Every σ-compact space, and hence every locally compact Lindelöf space, is a Lindelöf Σ-space. Lindelöf Σ-spaces play an important rôle in the study of function spaces (with the topology of pointwise convergence). For details, see the article on *C_p-theory* by Arkhangel'skiĭ [HvM, Chapter 1].

There are several strengthenings and weakenings of the Lindelöf property in the literature, for example: almost Lindelöf, n-starLindelöf, totally Lindelöf, strongly Lindelöf, Hurewicz, subbase Lindelöf. We mention one in passing. A space is **weakly Lindelöf** if any open cover has a countable subfamily $\mathcal{V}$ such that $\bigcup\{V\colon V \in \mathcal{V}\}$ is ***dense*** in X. Weakly Lindelöf spaces are of some interest in Banach space theory [HvM, Chapter 16] and, assuming CH, the weakly Lindelöf subspaces of $\beta\mathbb{N}$ are precisely those which are ***C^*-embedded*** into $\beta\mathbb{N}$ (1.5.3 of [KV, Chapter 11]). Covering properties such as ***para-Lindelöf*** or ***meta-Lindelöf*** are discussed in the article on generalizations of paracompactness.

Finally, we list a number of interesting results concerning the Axiom of Choice and the Lindelöf property. The Countable Axiom of Choice is strictly stronger than either of the statements 'Lindelöf metric spaces are second-countable' or 'Lindelöf metric spaces are separable' [7]. In Zermelo–Fraenkel set theory (without choice) the following conditions are equivalent: (a) $\mathbb{N}$ is Lindelöf; (b) $\mathbb{R}$ is Lindelöf; (c) every second-countable space is Lindelöf; (d) $\mathbb{R}$ is hereditarily separable; (e) $f:\mathbb{R}\to\mathbb{R}$ is continuous at x iff it is sequentially continuous at x; and (f) the axiom of countable choice holds for subsets of $\mathbb{R}$ [6]. There are models of ZF in which every Lindelöf T_1-space is compact [3] and models in which the space ω_1 is Lindelöf but not countably compact [4].

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d-9 Realcompactness

The study of realcompact spaces began in the latter half of the 1940s with independent contributions by both Edwin Hewitt and Leopoldo Nachbin. Over the course of the years these spaces have been studied under a variety of different names: **e-complete**, **functionally closed**, **Hewitt–Nachbin**, **realcomplete**, **replete**, and **saturated**. Hewitt, who introduced these spaces to the literature, originally called them **Q-space**s [4]. Though the term Q-space is occasionally still used to refer to realcompactness, as a rule the term Q-space now refers to a special class of metrizable spaces (cf. [KV, p. 718]). The term realcompact, popularized by the enormous success of Gillman and Jerison's work [GJ], is now prevalent in the literature.

The definition of realcompactness which requires perhaps the least upfront machinery is the following: a space X is said to be **realcompact** if and only if it is homeomorphic to a closed subspace of a product of real lines. $\mathbb{R}$ itself is thus probably the easiest example of a realcompact space. Since a ***compact Hausdorff*** space is one which is homeomorphic to a closed subspace of a product of unit intervals, it is easy to see that realcompactness is a generalization of compactness.

We would do well at this point to note that our definition requires that realcompact spaces be at least ***Tychonoff***, that is to say ***completely regular*** and Hausdorff. This is so because the real line is Tychonoff, and because the property of being Tychonoff is both productive and hereditary. It's also worthy of note that although compact Hausdorff spaces are normal, realcompact spaces need not be. The ***Moore Plane*** Γ (also called the ***Niemytzki Plane***) is realcompact, but not normal. Precisely because realcompact spaces are necessarily Tychonoff, the literature on realcompactness almost always begins with the explicit assumption that all the otherwise unidentified spaces under consideration will be Tychonoff. We too will make that assumption from this point onwards.

Gillman and Jerison [GJ] devote an entire chapter to a discussion of the importance of complete regularity, and the Tychonoff assumption will afford us a variety of useful ways to characterize realcompactness. For our purposes, suffice it to say that the complete regularity of a space guarantees three principal things: (1) the ***z-ultrafilters*** of the space will converge to at most one point; (2) the space will be ***uniformizable***, i.e., there will be a finest ***uniformity*** that generates the topology of the space; and (3) the space $C(X)$ of continuous real-valued functions on X, endowed with the ***compact-open topology***, will be a ***locally convex*** Hausdorf topological vector space.

Our first application of the complete regularity assumption results in a very useful characterization of realcompactness: X is realcompact iff every z-ultrafilter on X with the ***countable intersection property*** has non-empty intersection. Given that a space is compact iff each of its z-ultrafilters with the *finite* intersection property has non-empty intersection, one can see again how neatly the concept of realcompactness generalizes compactness.

Realcompactness is inherited by a variety of sets. ***F_σ-sets*** inherit realcompactness, and consequently both closed sets and ***cozero*** sets (or ***functionally open*** sets) inherit realcompactness. ***Baire sets*** and weak Baire sets also inherit realcompactness. If we define a subpace F of X to be **G_δ-closed** (or **real-closed** or **Q-closed**) in X iff every point of X not in F is contained in a G_δ-subset of X which misses F, then it turns out that every G_δ-closed subspace of a realcompact space is realcompact. Since every point in a ***first-countable*** space is a G_δ, it follows that every subspace of a first-countable realcompact space is realcompact. From this it follows that $\mathbb{N}$ and $\mathbb{Q}$ are realcompact.

Realcompactness is arbitrarily productive with respect to the usual product topology. Moreover an arbitrary intersection of realcompact subspaces of a given space is again realcompact. Realcompactness is not necessarily preserved by unions however. Though the union of a compact space and a realcompact space will always be realcompact, there are a number of examples of realcompact spaces whose unions fail to be realcompact. In addition to those spaces identified by Engelking in his historical notes [E, p. 219], Kato has identified the Dieudonné Plank as a non-realcompact space which is the union of two realcompact subspaces. This problem with unions can be cured in the countable case with the additional stipulation that the union be either a cb-space or normal, or provided that the union's members are individually ***z-embedded*** in the ambient space.

As for mapping theorems, if $f: X \to Y$ is a ***perfect map*** from a Tychonoff space X onto a realcompact space Y, then X is realcompact. Realcompactness is not however preserved by perfect maps. If though, in addition to being perfect, f is an ***open map*** from a realcompact space X onto a Tychonoff space Y, then we may claim Y to be realcompact. Many other types of maps preserve and pull back realcompactness, but they can be quite technical. Weir's book has a nice chapter devoted to these maps.

Lindelöf spaces are known to be realcompact, and consequently so is every ***σ-compact***, ***second-countable*** and ***separable metric space***. But not every realcompact space is Lindelöf. The Sorgenfrey plane is realcompact but not Lindelöf. The converse does not appear to be easy to recover without strong additional hypothesis. If we define a space to be **hereditarily realcompact** iff each of its subspaces is realcompact, and if we define a space to be **linearly Lindelöf** iff every open cover that is linearly ordered by inclusion has a countable subcover, then Arkhangel'skiĭ and Buzyakova have shown the existence of an example consistent with ZFC that is hereditarily realcompact and linearly Lindelöf, and yet fails to be Lindelöf.

Yet another way to gauge the distance between Lindelöfness and realcompactness affords us the opportunity to discuss another characterization of realcompactness via measure theory. Nagata has a nice exposition of the interplay between topology and measure in [N, VIII §4]. For the sake of expedience though we follow [KV, p. 1031] and point out that X is realcompact iff every two-valued **Baire measure** (i.e., defined on the ***Baire σ-algebra***) on X is ***τ-additive***. If we define X to be **measure-compact** iff every Baire measure on X is τ-additive, then it follows that Lindelöf $\Rightarrow$ measure-compact $\Rightarrow$ realcompact. These implications are not reversible. Gale has provided rather strong examples based on the Cantor Tree: a locally compact, realcompact space which is not measure-compact; a locally compact, measure-compact space which is not Lindelöf; and a normal measure-compact space which is not Lindelöf. None of Gale's examples requires more than ZFC.

Discrete spaces of cardinality $\mathfrak{c}$ are easily shown to be realcompact, and the question comes to mind whether discrete spaces of *any* cardinality might be realcompact. The answer amounts to whether or not there exist such things as ***measurable cardinal***s. It is in fact consistent with ZFC that no such cardinals exist. We know that a discrete space X is realcompact iff $|X|$ is nonmeasurable.

We have already noted that realcompactness is productive with the usual product topology. It is natural to wonder whether or not realcompactness remains productive in the ***box product***. The answer is yes, provided that the number of factors is nonmeasurable.

Since regular Lindelöf spaces are known to be ***paracompact***, it is also natural to wonder at the relationship between paracompactness and realcompactness. In general, realcompact spaces need not be paracompact. The Sorgenfrey plane serves as an example (it is not even ***countably paracompact***). On the other hand Katětov has shown that a paracompact space is realcompact iff every closed discrete subspace of it is realcompact. In particular, this will be true when its cardinality is nonmeasurable.

It is known that a realcompact ***pseudocompact*** space is compact. This proves to be a convenient tool for identifying spaces which cannot be realcompact. The standard examples $[0, \omega_1)$ and the ***Tychonoff plank*** are known to be pseudocompact and non-compact, so neither is realcompact.

We have so far mentioned only three characterizations of realcompactness, despite the fact that we've alluded to many such formulations in the opening paragraphs of this article. To begin gathering alternative formulations we must first explore the concepts of compactification and realcompactification.

Let us first recall that a space Y is a called a ***compactification*** of a space X if Y is compact and X is dense in Y. By analogy, a space Y is called a **realcompactification** of a space X if Y is realcompact and X is dense in Y. In this article we will limit ourselves to a discussion of the most important realcompactification of X, the **Hewitt realcompactification** υX.

We know that for any Tychonoff space X, there is a compact space βX called the ***Čech–Stone compactification*** of X which contains X as a dense subspace, and for which every continuous function $f : X \to I$ can be extended to a continuous function $\beta f : \beta X \to I$. Likewise, for any Tychonoff space X, the Hewitt realcompactification υX contains X as a dense subspace, and enjoys the property that every continuous function $f : X \to \mathbb{R}$ can be extended to a continuous function $\upsilon f : \upsilon X \to \mathbb{R}$. Furthermore, just as βX is maximal in the collection of all compactifications of X, υX is maximal in the collection of all realcompactifications of X. Note that $X \subseteq \upsilon X \subseteq \beta X$ and that υX is the smallest realcompact subspace between X and βX. In particular, if X is realcompact, then $X = \upsilon X$.

In light of our characterization of realcompactness in terms of z-ultrafilters, it is often useful to think of the points of υX as the z-ultrafilters on X which have the countable intersection property. If the intersection of one such z-ultrafilter $\mathcal{F}$ on X is non-empty, i.e., if $\mathcal{F}$ is 'fixed', we identify $\mathcal{F}$ with the point of X to which it converges. If the intersection of $\mathcal{F}$ on X is empty, i.e., if $\mathcal{F}$ is 'free', then we think of $\mathcal{F}$ as a new point in the extension $\upsilon X - X$.

With the concept of compactification at our disposal, we now have access to a number of equivalences to realcompactness:

THEOREM 1 [11, 8.8]. *If X is Tychonoff, then the following are equivalent*:

(1) *X is realcompact.*
(2) *If Y is a Tychonoff space in which X is dense and C-embedded, then $X = Y$.*
(3) *For each point $p_0 \in \beta X - X$, there exists a continuous function $f \in C(\beta X)$ such that $f(p_0) = 0$ and $f(p) > 0$ for all points $p \in X$.*
(4) *The space X is G_δ-closed in βX.*
(5) *The space X is an intersection of F_σ-sets containing X and contained in βX.*
(6) *The space X is an intersection of cozero sets in βX.*
(7) *The space X is an intersection of σ-compact subspaces of βX.*

Before pressing on to discuss further characterizations of realcompactness, we might look briefly at an interesting application of realcompactification. We have noted that realcompactness is productive, so it is natural to wonder about the conditions under which $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$. Glicksburg has shown that $\beta(X \times Y) = \beta X \times \beta Y$ iff $X \times Y$ is pseudocompact. It's not too hard to show that if $X \times Y$ is pseudocompact, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$. Hušek has shown, however, that the equation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ implies no topological property of $X \times Y$. Biconditional statements involving the equation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ are nonetheless available. Building on the work of Comfort, Hušek and McArthur, Ohta has proved that X is locally compact, realcompact and has nonmeasurable cardinality iff $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ for any space Y.

Another approach to studying realcompactness is from the perspective of ***uniform spaces***. It is here that we make use of the second principle consequence of restricting our attention to Tychonoff spaces, their uniformizability. Let the ***universal uniformity*** for a topological space be the uniformity generated by the collection of all ***normal covers***, and let the **e-uniformity** for a space be the uniformity generated by all covers with countable normal refinement. We say that a uniform space is ***complete*** if every ***Cauchy filter*** converges with respect to the given uniformity. The ***completion*** of a uniform space X is a uniform space which contains X homeomorphically as a dense subspace and is complete with respect to the given uniformity. Spaces which are complete with respect to their universal uniformities are called ***completely uniformizable*** (or ***topologically complete*** or ***Dieudonné complete***).

Shirota proved that a space X is realcompact iff X is e-complete, and that the **e-completion** of a space is homeomorphic to its Hewitt-realcompactification, i.e., $eX = \upsilon X$. It may be readily inferred from Hewitt's original paper that every realcompact space is completely uniformizable (cf. [E, 8.5.13]). A much deeper result, commonly called Shirota's Theorem, adds a 'partial' converse: If $|X|$ is nonmeasureable, then X is realcompact iff X is completely uniformizable. [GJ, Chapter 15] contains a classic discussion of these results; Howes provides a rather nice synthesis of the work of Hewitt and Shirota in [5].

Our next characterization of realcompactness utilizes our third consequence of the complete regularity assumption. Let $C(X)$ be endowed with the compact-open topology. We say that $C(X)$ is **bornological** iff each seminorm that is bounded on the bounded sets of $C(X)$ is continuous, and that $C(X)$ is **ultrabornological** iff each seminorm that is bounded on the convex compact sets of $C(X)$ is continuous. Then we have the following:

THEOREM 2 [11, p. 155]. *If X is Tychonoff, then the following are equivalent*:

(1) *X is realcompact.*
(2) *$C(X)$ is bornological.*
(3) *$C(X)$ is ultrabornological.*

If, instead of the compact-open topology, we endow $C(X)$ with the ***topology of pointwise convergence***, denoted $C_p(X)$, Uspenskiĭ has shown that X is realcompact iff $C_p(C_p(X))$ is realcompact. Moreover, if X is realcompact and $C_p(X) \cong C_p(Y)$, then Y is realcompact.

This would seem to be the moment to point out that Hewitt's original motivation to study realcompact spaces was sparked by an *algebraic* question. In particular, Hewitt wanted to identify the class of spaces for which the ring structure of $C(X)$ determined the topology on X. After establishing that X is realcompact iff every real maximal ideal in $C(X)$ is fixed, he showed that two realcompact spaces X and Y are homeomorphic iff $C(X)$ and $C(Y)$ are isomorphic. Hewitt's result has since been strengthened by the work of Araujo, Beckenstein and Narici. If we define an additive map $T : C(X) \to C(Y)$ to be **separating** iff $fg = 0$ implies $(Tf)(Tg) = 0$, and say that T is **biseparating** iff T and T^{-1} are separating, then two realcompact spaces X and Y are homeomorphic iff there is a biseparating map T from $C(X)$ onto $C(Y)$.

We would probably be remiss if we did not at least mention that there is an approach to studying realcompactness via nonstandard analysis. The nonstandard window is a particularly difficult one to open in this brief survey article, but the interested reader should consult [9]. A number of more recent nonstandard characterizations of concepts mentioned in this article may be found in [7].

One of the problems in realcompactness that has remained open for nearly forty years is the following: Is every ***perfectly normal*** T_1 space of nonmeasurable power realcompact? Robert Blair first posed the question in 1962 and conjectured that it was consistent with ZFC that the answer was affirmative. Blair's conjecture has resisted proof. The answer is known to be negative if, in addition to ZFC, one accepts as axiomatic Jensen's ***Diamond Principle*** $\diamondsuit$. For references, more open questions, and a discussion of the current state of the problem see [10].

Every characterization of realcompactness invites a generalization of one kind or another. The most well-known of these generalizations is that of almost realcompactness. A space is said to be **almost realcompact** iff every ultrafilter of regular closed subsets with the countable intersection property is fixed. There are dozens of such generalizations however, making any systematic discussion of them in this brief article impossible. The interested reader might find [8] a helpful introduction to the study of generalized realcompactness.

A different type of generalization is that of E-compactness. One takes a fixed space E and defines a space X to be **E-compact** if it is homeomorphic to a closed subset of some topological power of E. Thus $\mathbb{R}$-compactness is realcompactness and $[0, 1]$-compactness is just compact Hausdorff. An interesting special case occurs when $E = \mathbb{N}$, the space of natural numbers. It turns out that **$\mathbb{N}$-compactness** is equivalent to $\mathbb{R}$-compactness plus ***strong zero-dimensionality*** and that $\mathbb{N}$-compactness is characterized by the property that every ultrafilter of clopen sets with the countable intersection property has a non-empty intersection. ***Roy's Example*** Δ is realcompact and zero-dimensional but not $\mathbb{N}$-compact.

The exercises of Engelking's text [E] contain yet a few more characterizations of realcompactness which we have failed to mention. Comfort points to still others in his review [1] of Weir's book. Indeed, Comfort's review is highly recommended if for no other reason than his placement of [11] in historical context. More than twenty-five years after its publication, [11] remains the premier reference for results on realcompact spaces.

When we began this article we noted that work on realcompactness started with indepedent contributions by both Hewitt and Nachbin. Though evidence for Hewitt's priority seems clear, the issue is not altogether uncontroverted. The interested reader might find Comfort's article [2] helpful in this regard. [E] and [GJ], as always, contain wonderful historical notes. The articles of Henriksen [3] and Nagata [6] also provide interesting historical contributions.

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d-10 k-Spaces

The notion of k-space has turned out to be important in various applications, and the analysis of this use brings out some interesting questions on the role of general topology. On the one hand, k-spaces are a particular kind of space, and so the analysis of their properties such as subspaces, products, and so on is of interest. On the other hand, the notion of topological space arose partly to give a useful setting to the notion of continuity, when it was found that metric spaces were inadequate for the applications. Thus from this point of view, we may not care what is the definition of topological space as long as it plays an adequate role for the purposes in mind. So then we have to ask how we define these purposes.

Since we need a notion of continuity, which defines a relation of a topological space to all other spaces, we are led to a categorical viewpoint, that is, the study of the category Top of all topological spaces and all continuous maps. This global viewpoint suggests asking if this category has all the properties we would desire, and if not, is there a 'better' candidate? It is not clear if the 'final answer' has yet been obtained, but the notion of k-space has played a key role in this investigation. An influential step was the remark in R. Brown's 1963 paper [3]: "It may be that the category of Hausdorff k-spaces is adequate and convenient for all purposes of topology". Of course 'adequate' means that the category contains the basic spaces with which one wishes to deal, for example metric spaces. A key list of desirable properties for a **convenient category** was given in [4], which amounted to the property of being what is now called ***Cartesian closed*** [17]. This term 'convenient category' was adopted in Steenrod's widely cited 1967 paper [20], and the scope of the idea was extended by other writers. For example, the 1998 book by Kriegl and Michor [16] provides a 'convenient setting for global analysis'. It is interesting to see from their account that workers in that field started by taking up the advantages of k-spaces, but eventually found a different setting was needed, and that is the main topic of [16]. But in certain applications, for example algebraic topology, it is often sufficient to have and to use a convenient category without knowing the specific details of its construction. There are also a number of purely topological questions of interest in k-spaces and these we will come to later.

We first give the definition in the Hausdorff case. A Hausdorff space is said to be a k**-space** if it has the ***final topology*** with respect to all inclusions $C \to X$ of compact subspaces C of X, so that a set A in X is closed in X if and only if $A \cap C$ is closed in C for all compact subspaces C of X. Examples of k-spaces are ***Hausdorff*** spaces which are ***locally compact***, or satisfy the ***first axiom of countability***. Hence all metric spaces are k-spaces. Also all ***CW-complex***es are k-spaces. A closed subspace of a k-space is again a k-space, but this is not true for arbitrary subspaces [E]. A space is ***Fréchet–Urysohn*** if, whenever a point x is in the closure of a subset A, there is a sequence from A converging to x; it is proved in [1] that a space is hereditarily k, i.e., every subspace is a k-space, if and only if it is Fréchet–Urysohn.

The product of k-spaces need not be a k-space. Let W_A be the wedge of copies of the unit interval $[0, 1]$ indexed by a set A, where $[0, 1]$ is taken to have base point 0, say. (The **wedge** $\bigvee_{a\in A} X_a$ of a family $\{X_a\}_{a\in A}$ of pointed spaces is the space obtained from the disjoint union of all the X_a by shrinking the disjoint union of the set of base points to a point.) Consider the product $X = W_A \times W_B$ where A is an uncountable set and B is countably infinite. It is proved in [9] that X is not a CW-complex, although W_A and W_B are CW-complexes. Kelley in [15] states as an Exercise that the product of uncountably many copies of the real line $\mathbb{R}$ is not a k-space. A solution is in effect given in [6], since the example is used in showing that various topologies on $X \times Y$ are in general distinct.

One place where k-spaces arise is with the **Ascoli Theorem** (see [E]).

THEOREM 1. *Let X be a k-space, let $\mathcal{B}$ be the family of compact subsets of X, and let $(Y, \mathcal{U})$ denote a uniform space. Then a closed subspace F of the space* Top(X, Y) *of continuous functions $X \to Y$ with the compact-open topology is compact if and only if the following conditions hold*:

(a) *$F|Z$ is equicontinuous for all $Z \in \mathcal{B}$,*
(b) *for all $x \in X$ the set $F(x) = \{f(x)\colon f \in F\}$ is a compact subset of Y.*

However the most widespread applications of k-spaces are in algebraic topology, for dealing with identification spaces and function spaces.

A problem with identification maps is that the product of identification maps need not be an identification map. One example derives from Dowker's example mentioned above – another is $f \times 1 : \mathbb{Q} \times \mathbb{Q} \to (\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}$, where $\mathbb{Q}/\mathbb{Z}$ denotes here the space $\mathbb{Q}$ of rational numbers with the subspace $\mathbb{Z}$ of integers shrunk to a point. (A proof is given on p. 105 of [3].)

Identification spaces present another problem in that an identification of a Hausdorff space need not be Hausdorff, and so there grew pressure, in view of the important applications of identification spaces, to extend the definition of k-space to non Hausdorff spaces. A useful definition was found by a number of writers as follows.

We now say that a space X is a k-space (also called **compactly generated**) if X has the final topology with respect to all maps $C \to X$ for all compact Hausdorff spaces C. We view maps t of this form as **test map**s. A consequence of this definition is that a necessary and sufficient condition for X

to be a k-space is that for all spaces Y a function $f : X \to Y$ is continuous if and only if $f \circ t : C \to Y$ is continuous for all test maps $t : C \to X$. If a space satisfies this characterization of continuity only for real-valued functions (i.e., in the special case $Y = \mathbb{R}$) then X is called a k_R**-space**.

It may seem against common sense to have to test properties of a space X by considering all compact Hausdorff spaces, but in fact since X has only a set of closed subspaces, it is easy to show we can choose a *set* of test maps to determine if X is a k-space. This allows one to show that X is a k-space if and only if X is an identification space of a space which is a ***sum*** (disjoint union) of compact Hausdorff spaces.

Let kTop denote the subcategory of Top of k-spaces and continuous maps. The inclusion $i : \mathsf{kTop} \to \mathsf{Top}$ has a left adjoint $\mathsf{k} : \mathsf{Top} \to \mathsf{kTop}$ which assigns to any space X the space with the same underlying set but with the final topology with respect to all test maps $t : C \to X$. The adjointness condition means that there is a natural bijection

$$\mathsf{Top}(iX, Y) \to \mathsf{kTop}(X, \mathsf{k}Y)$$

for all k-spaces X and topological spaces Y. Consequently k preserves limits, and in particular the product $X \times_{\mathsf{k}} Y$ in the category kTop is the functor k applied to the usual product $X \times Y$. Further, i preserves colimits.

The importance for function spaces of the maps $\mathsf{k}X \to Y$ was first emphasised by Kelley [15] in the context of Hausdorff spaces. If we now work in the category kTop, it is natural to define the function space for k-spaces Y and Z to be the set of continuous maps $Y \to Z$ with a modification of the compact open topology to the **test open topology**. This has a subbase all sets

$$W(t, U) = \{ f \in \mathsf{Top}(Y, Z) \colon ft(C) \subseteq U \}$$

for all U open in Z and all test maps $t : C \to Y$. Finally we apply k to this topology to get the function space $\mathsf{kTOP}(Y, Z)$. The major result is that the exponential correspondence gives a bijection

$$\mathsf{kTop}(X \times_{\mathsf{k}} Y, Z) \cong \mathsf{kTop}\big(X, \mathsf{kTOP}(Y, Z)\big)$$

for all k-spaces X, Y and Z. For a detailed proof, see, for example, [2] or the references given there. Thus this result says that the category kTop is Cartesian closed. A consequence is that in this category the product of identification maps is an identification map.

As an example of what can be proved formally from the Cartesian closed property, we note that the composition map

$$\mathsf{kTOP}(X, Y) \times_{\mathsf{k}} \mathsf{kTOP}(Y, Z) \to \mathsf{kTOP}(X, Z)$$

is continuous. Consequently, $\mathrm{END}(X) = \mathsf{kTOP}(X, X)$ is a monoid in the category kTop (using of course the product $- \times_{\mathsf{k}} -$).

Another important use of this law in algebraic topology is to be able to regard a homotopy as either a map $I \times X \to Y$ or as a path $I \to \mathsf{kTOP}(X, Y)$ in a space of maps. It is awkward for applications to have to restrict X to be for example locally compact, or Hausdorff.

In using the category kTop it is convenient to replace the usual Hausdorff condition by **weakly Hausdorff** which means that the diagonal is closed in $X \times_{\mathsf{k}} X$, which is equivalent to any map from a compact Hausdorff space to X is a closed map. This condition is between T_1 and Hausdorff and is more stable than Hausdorff with respect to many categorical constructions. For work in this area, using fibred mapping spaces, see [11].

As an example of the use of the category kTop we consider the construction of free products $G * H$ of k**-group**s G and H (by which is meant groups in the category kTop so that we require the difference map $(g', g) \mapsto g'g^{-1}$ on G to be continuous as a function $G \times_{\mathsf{k}} G \to G$). The group $G * H$ is constructed as a quotient set $p : W \to G * H$ where W is a monoid of words in $G \sqcup H$, and it is easy to show the multiplication $W \times_{\mathsf{k}} W \to W$ is continuous. Because $p \times_{\mathsf{k}} p$ is an identification map in kTop, it follows that the difference map on $G * H$ (which is given the identification topology) is also continuous. This type of application is pursued in [10].

There is another way of making this type of colimit construction, which also applies much more generally, for example to topological groupoids and categories, see [8]. This method, based on what is known as the Adjoint Functor Theorem, see [17], supplies an object with the appropriate universal property but the method gives no easy information on the open sets of the constructed space. It can be argued that the universal property, since it defines the object uniquely, is all that is required and any more detailed information should be deduced from this property.

A more direct way has been found as follows. A topological space X is a k_ω**-space** if it has the final topology with respect to some countable increasing family $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots$ of compact subspaces whose union is X. The following are some main results on these spaces, proved in [18].

THEOREM 2. *Let X be a Hausdorff k_ω-space, and let $p : X \to Y$ be an identification map. Then the following are equivalent*:

(i) *The graph of the equivalence relation associated with p is closed in $X \times X$.*

(ii) *Y is a Hausdorff k_ω-space.*

THEOREM 3. *If $p : X \to Y$ and $q : Z \to W$ are identification maps of Hausdorff k_ω-spaces X, Y, Z and W then $p \times q : X \times Z \to Y \times W$ is also an identification map of Hausdorff k_ω-spaces.*

These results are used in [7] for the construction of free products of Hausdorff k_ω-groups (and more generally for constructions on topological groupoids).

There are other solutions to the problem of the inconvenient nature of the category Top. An early solution involved what are called quasi-topologies [19]. This was gradually thought to be unacceptable because the class of quasi-topologies on even a 2-point set did not form a set. However, this objection can be questioned.

Another solution involves the space $\mathbb{N}^\wedge$ which is the ***one-point compactification*** of the discrete space of positive integers, that is, it involves a sequential approach. For any topological space X one defines the ***s*-test maps** to be the continuous maps $\mathbb{N}^\wedge \to X$. The space X is said to be a ***sequential space*** (which we abbreviate here to s-space) if X has the final topology with respect to all s-test maps to X. The study of such spaces was initiated in [12]. By working in a manner analogous to that for k-spaces one finds the category sTop of s-spaces has a product $X \times_{\mathsf{s}} Y$ and a function space sTOP satisfying an exponential law

$$\mathsf{sTop}(X \times_{\mathsf{s}} Y, Z) \cong \mathsf{sTop}\big(X, \mathsf{sTOP}(Y, Z)\big)$$

for all s-spaces X, Y and Z [21]. In fact the k-space and s-space exponential laws are special cases of a general exponential law defined by a chosen class of compact Hausdorff spaces satisfying a number of properties [2].

For sequential spaces, the property corresponding to Hausdorff is having unique sequential limits, which we abbreviate to 'has unique limits'. A space X has unique limits if and only if the diagonal is sequentially closed in $X \times X$.

Another advantage of sequential spaces is with regard to ***proper map*s**. The **one-point sequential compactification** $X^\wedge$ of a sequential space X is defined to be the space X with an additional point ω, say, and with a topology such that X is open in $X^\wedge$ and any sequence in X which has no convergent subsequence converges to the additional point ω. Any function $f : X \to Y$ has an extension $f^\wedge : X^\wedge \to Y^\wedge$ in which the additional point of X is mapped to the additional point of Y. Consider the following conditions for an s-map $f : X \to Y$: (a) $f^\wedge$ is an s-map; (b) $f \times 1 : X \times \mathbb{N}^\wedge \to Y \times \mathbb{N}^\wedge$ is sequentially closed; (c) $f \times 1 : X \times Z \to Y \times Z$ is sequentially closed for any s-space Z; (d) if s is a sequence in X with no subsequence convergent in X, then fs has no subsequence convergent in Y; (e) if B is a sequentially compact subset of Y, then $f^{-1}(B)$ is a sequentially compact subset of X; (f) if s is a convergent sequence in Y then $f^{-1}(\bar{s})$ is sequentially compact.

It is proved in [5] that (d)$\Rightarrow$(e)$\Rightarrow$(f)$\Rightarrow$(c)$\Rightarrow$(b), and that if X is T_1 then (b)$\Rightarrow$(d). Further, if X and Y have unique limits, then (a)$\Leftrightarrow$(c). It is reasonable therefore to call a map satisfying (a) **sequentially proper**.

A further advantage of the sequential theory as shown in [14] is that the category sTop can be embedded in a **topos**, that is a category which is not only Cartesian closed but also has finite limits and a 'sub-object classifier'. This has a number of implications, including fibred exponential laws and spaces of partial maps, which are developed in books on topos theory and pursued further in this case in the thesis of Harasani [13]. We cannot go into these implications here, but they do suggest that extensions of the notion of topological space may be very important for future applications.

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d-11 Dyadic Compacta

A ***compact Hausdorff*** topological space is called **dyadic** if it is a continuous image of a ***Cantor cube*** D^m, where m is some infinite cardinal number. This notion was inspired by the following theorem of P.S. Alexandroff, announced in 1925: every compact metric space is a continuous image of the ***Cantor set*** D^{ω_0}. E. Marczewski proved that any Cantor cube D^m satisfies the ***countable chain condition*** [E, 2.3.18]. This condition is preserved by continuous maps, so not all compact spaces are dyadic. The first example of a non-dyadic space is the ***one-point compactification*** of an uncountable discrete space.

The above definition of the class of dyadic compacta was given in Shanin's paper [15] with the following statement: "in particular, the origin of the term 'dyadic compactum' was suggested by P. Alexandroff". In [16] N.A. Shanin introduced the following notion: an uncountable cardinal number n is a **caliber** of a topological space, if for any family of cardinality n of nonempty open sets there exists a subfamily of the same cardinality with nonempty intersection. In the same paper, he proved that any uncountable regular cardinal number is a caliber of a dyadic compactum. The importance of the notion caliber was mentioned by W.W. Comfort and S. Negrepontis in their monograph *Chain conditions in topology*: "... Calibers are the creation of the Soviet mathematician Shanin, without whose highly original papers, the present monograph could not have been written, or ever ..."

A.S. Esenin–Volpin confirmed the following natural conjecture posed by P.S. Alexandroff: a dyadic compactum satisfying the ***first axiom of countability*** is metrizable. He proved that the ***weight*** of a dyadic compactum is equal to the supremum of ***character***s of all points of the original compactum. Later B.A. Efimov generalized Esenin–Volpin's theorem in the following way: the weight of a dyadic compactum is equal to the supremum of characters of all points of a fixed dense subspace of the original compactum [E, 3.12.12]. In the fundamental paper [5] B.A. Efimov gives new proofs of well-known results and their generalizations using strong sequences, which form the primary notion of that work. In addition he obtained the following important results:

(1) Every closed G_δ-subset of dyadic compactum is dyadic;
(2) Every ***regular closed*** subspace of dyadic compactum is dyadic;
(3) For any family $\mathcal{F}$ of G_δ-subsets of a Cantor cube there exists a countable subfamily $\mathcal{F}'$ such that $\mathrm{cl}(\bigcup\mathcal{F}) = \mathrm{cl}(\bigcup\mathcal{F}')$ and moreover the set $\mathrm{cl}(\bigcup\mathcal{F})$ is a G_δ subset of the Cantor cube.

Generalizations of 3 were given by E.V. Ščepin [13] (for the class of ***κ-metrizable space***s) and by I. Bandlow [2] (for arbitrary compacta).

A.V. Arkhangel'skiĭ and V.I. Ponomarev obtained in some sense the definitive result in the theory of cardinal invariants of dyadic compacta: the weight of a dyadic compactum is equal to the supremum of tightnesses of all points of the original compactum [E, 3.12.12]. A.V. Arkhangel'skiĭ presented a very useful approach of the cardinal invariant theory of dyadic compacta [1].

L. Ivanovski and V.I. Kuz'minov confirmed another natural conjecture posed by P.S. Alexandroff: any compact ***topological group*** is dyadic, see [HvM, Chapter 2, 3.6C].

A great impulse to the theory of dyadic compacta was given by the book of A. Pełczynski [11] in which was introduced the class of Dugundji spaces – an important subclass of dyadic spaces. Let us remark that any compact topological group is a Dugundji space. The original definition is the following: a compactum X is called **Dugundji compact** if and only if for any embedding $e: X \to Y$ of X into a compact space Y there exists a linear extension operator $\mathcal{U}: C(X) \to C(Y)$ (the ***Banach space***s of real-valued continuous functions) of norm 1, i.e., $\mathcal{U}$ is linear, $\mathcal{U}(f) \restriction X \equiv f$ for all f and $\|\mathcal{U}\| = 1$. In this book a series of problems were formulated whose solutions had great influence on the theory of dyadic compacta. R. Haydon [6] solved problem 15 of Pełczynski, at the same time proving that every Dugundji space is dyadic (problem 14).

Another important problem from the above mentioned list (problem 22) was solved by J. Gerlits and B.A. Efimov: every dyadic compactum X contains a topological copy of the Cantor cube D^m for every regular cardinal number $m \leqslant w(X)$ (see [E, 3.12.12]). E.V. Ščepin [14] gave a different proof using the method of inverse spectra, which helped him to obtain the same result for Tychonoff cubes: a continuous image of Tychonoff cube X contains topological copy I^m for every regular cardinal number $m \leqslant w(X)$. In his fundamental paper [12] E.V. Ščepin, using the technique of inverse spectra, solved problem 15 of A. Pełczynski by showing that the classes of Dugundji and Miljutin spaces are not identical (a compact space is called a **Miljutin space** if and only if there exists a surjection $\varphi: D^m \to X$ admitting a regular linear **average operator** $\mathcal{U}: C(D^m) \to C(X)$, the average condition means that for any function $f \in C(X)$ we have $\mathcal{U}(\varphi \circ f) = f$, regularity means that $\|\mathcal{U}\| = 1$). We point out that the first part of the problem was solved in the above mentioned paper of R. Haydon: every Dugundji space is a Miljutin space. E.V. Ščepin for his example used the functor $\exp_2$ – second hyper-symmetric power; $\exp_2 X$ is the set of one or two point subsets of X endowed with the ***Vietoris topology***.

The connection between dyadic compacta and functors was first mentioned by S. Sirota [22]: the hyperspace of Cantor cube of weight ω_1 is homeomorphic to this cube itself, i.e., $\exp D^{\omega_1} \approx D^{\omega_1}$ ($\exp X$ is the space of closed subsets of X endowed with the Vietoris topology). The same result for the functor of probability measures P ($P(X)$ is the set of linear functionals on $C(X)$ endowed with the ***weak* topology***) was obtained by E.V. Ščepin [14]: if X is Dugundji space such that $\chi(x, X) = w(X) = \omega_1$ for any $x \in X$ then $P(X) \approx I^{\omega_1}$. For spaces of weight larger than ω_1 it is not possible, more precisely, in the paper of L.B. Shapiro [17] the following results were obtained: if $\exp X$ is a dyadic compactum then $w(X) \leqslant \omega_1$; if $\exp X$ is a continuous image of Tychonoff cube then $w(X) \leqslant \omega_0$. As a corollary of the last result, modulo the Mazurkiewicz–Hahn–Sierpiński theorem, one obtains the final statement: $\exp X$ is a continuous image of a Tychonoff cube if and only if X is a Peano continuum. In [18] the following result was obtained: if $P(X)$ is a dyadic compactum then $w(X) \leqslant \omega_1$. The previous results were used for proving the non-homogeneity of the spaces $\exp X$ and $P(X)$ for dyadic X. More accurately [19]: if X is a dyadic and $\exp X$ is homogeneous then X is either finite, or a finite disjoint union of non-degenerate Peano continuum, or a Cantor cube D^m, where $m = \omega_0, \omega_1$. For the functor P the following is proved [20]: if X is a dyadic and $P(X)$ is homogeneous then X is either an infinite metrizable compactum ($P(X) \approx I^{\omega_0}$) or X is a Dugundji space such that $\chi(x, X) = w(X) = \omega_1$ ($P(X) \approx I^{\omega_1}$). The homogeneity can be used for characterization of Cantor cubes of small weights. In answering a question of B.A. Efimov [5], M.G. Bell [3] and L.B. Shapiro [20] proved that: if X is a zero-dimensional homogeneous dyadic compactum of weight ω_1 then $X \approx D^{\omega_1}$. Previously V.V. Pašenkov [10] constructed examples of dyadic homogeneous compacta which are not homeomorphic to Cantor cubes (the weights of his examples are greater than 2^{ω_0}).

The previous results show that without additional conditions dyadic compacta by their properties are very different from Cantor cubes. But the ***Boolean algebra*** $\mathrm{RO}(X)$ of ***regular open*** sets of a dyadic space X is similar to the corresponding algebras of Cantor cubes. More precisely the following result is true [7, Theorem 5.2.2]: if X is a dyadic compactum then $\mathrm{RO}(X)$ is isomorphic to $\mathrm{RO}(\bigoplus\{D^{m_i}\colon i \leqslant k\})$ or $\mathrm{RO}(\alpha(\bigoplus\{D^{m_i}\colon i < \omega_0\}))$ (the ***one-point compactification*** of a countable sum of Cantor cubes). Using the notion of completion of Boolean algebras it is possible to translate the last result into Boolean language: if B is a subalgebra of a free Boolean algebra, then there exists a countable family $\{m_i\colon i < \omega_0\}$ of cardinal numbers such that the completion B^{cm} of B is isomorphic to the completion of the product of the free Boolean algebras $\prod\{\mathrm{Fr}(m_i)\colon i < \omega_0\}$ ($\mathrm{Fr}(m)$ denotes the free Boolean algebra with m generators).

In topology isomorphism of the algebras $\mathrm{RO}(X)$ and $\mathrm{RO}(Y)$ means that the compacta X and Y are **coabsolute** (i.e., their ***absolute***s are homeomorphic). It is clear that for coabsoluteness the existence in X and Y of homeomorphic dense subspaces is sufficient. In case of metrizable compacta that property is also necessary. In fact it is true also for dyadic compacta of weight ω_1: dyadic compacta X of weight ω_1 are coabsolute iff they contain homeomorphic dense subspaces [21]. This is also another example when dyadic compacta of weight ω_1 have a property analogous to one in the metrizable case. Another important class of spaces connected with dyadic compacta is that of the so-called the κ-metrizable compacta which was introduced by E.V. Ščepin [12]. Let us remark that any Dugundji space is κ-metrizable. But the continuous images of κ-metrizable compacta (also called κ-adic) are already considered as a generalisation of dyadic compacta. L.V. Shirokov gave a fine characterization of Dugundji space and κ-metrizable compacta, using the notion of extension operator of open sets introduced by K. Kuratowski: if X is a subspace of Y then an extension operator of open sets $e\colon \mathcal{T}(X) \to \mathcal{T}(Y)$ satisfies the condition $e(U) \cap X = U$, for any $U \in \mathcal{T}(X)$. Now we can give a formulation of L.V. Shirokov's characterization: a compactum X is a Dugundji space (κ-metrizable) iff for every (some) embedding of X into I^m there exists an extension operator of open sets $e\colon \mathcal{T}(X) \to \mathcal{T}(I^m)$ that preserves intersections, i.e., $e(U_1 \cap U_2) = e(U_1) \cap e(U_2)$ (preserves disjointness: if $U_1 \cap U_2 = \emptyset$ then $e(U_1) \cap e(U_2) = \emptyset$). A remarkable connection between the described classes was founded by A.V. Ivanov [8]: if X is a κ-metrizable compactum then the ***superextension*** λX in the sense of de Groot is a Dugundji space. A zero-dimensional Dugundji space is the Stone space of a ***projective*** Boolean algebra, the studies of which was begun by P. Halmos. In this sense we must mention the following result of E.V. Ščepin: every zero-dimensional Dugundji space which is homogeneous with respect by character is homeomorphic to a Cantor cube. A very good introduction to the theory of projective Boolean algebras is the paper of S. Koppelberg [9].

Another line of generalization of dyadic compacta takes its roots from the work of S. Mrówka. A continuous image of a product $(\alpha m)^k$ is said to be a **polyadic space**. Many of the theorems that hold for the class of dyadic spaces are also true for the class of polyadic spaces. Generalizing the concept of polyadic space, M. Bell introduced the class of centered spaces, for which also many facts of the theory of dyadic spaces are true. Some new results on the cardinal invariants of centered spaces can be found in [4]. That class but with a different name was studied by M. Talagrand in the theory of Banach spaces of continuous functions and ***Corson compact*** spaces. W. Kulpa and M. Turzański introduced a so-called weakly dyadic spaces, which generalizes the class of centered spaces. A very good background of different generalizations of the concept of dyadic spaces can be found in the manuscript [23].

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d-12 Paracompact Spaces

1. Introduction

As defined by J. Dieudonné in 1944, a ***topological space*** X is **paracompact** if it is ***Hausdorff*** and every ***open cover*** $\mathcal{U}$ of X has a locally finite, open refinement $\mathcal{V}$ (see [E, p. 299]).[1] Here a **cover** $\mathcal{V}$ of X is called a **refinement** of a cover $\mathcal{U}$ of X if every $V \in \mathcal{V}$ is a subset of some $U \in \mathcal{U}$, and a collection $\mathcal{V}$ of subsets of X is called ***locally finite*** if every $x \in X$ has a ***neighbourhood*** W which intersects at most finitely many $V \in \mathcal{V}$.

The refinement $\mathcal{V}$ of $\mathcal{U}$ in the definition of paracompactness can always be chosen to be of the form $\mathcal{V} = \{V_U: U \in \mathcal{U}\}$, with $V_U \subset U$ (or even $\overline{V}_U \subset U$) for every $U \in \mathcal{U}$ [E, 5.1.7]. It cannot, however, always be chosen so that $\mathcal{V} \subset \mathcal{U}$; indeed, that can be done if and only if X is ***compact*** [E, 5.1.A(d)].

There are numerous characterizations of paracompact spaces, many of them in terms of different kinds of refinements of open covers. Some of these characterizations can be found in this article; for some others, see, for example, [E, 5.1.11 and 5.1.12].

Some topics related to paracompact spaces are discussed elsewhere in this encyclopedia, such as in H.J.K. Junnila's and D. Burke's articles on Generalizations of Paracompact Spaces and in P. Gartside's article on Generalized Metric Spaces.

2. Relationships to other properties

Clearly all compact Hausdorff spaces are paracompact; more generally, so are all ***regular Lindelöf*** spaces [E, 5.1.2]. In the opposite direction, all paracompact spaces are ***normal*** [E, 5.1.5] (in fact, even ***collectionwise normal*** [E, 5.1.18]). A much deeper result is the following fundamental theorem obtained by A.H. Stone in 1948 (see [E, 4.4.1]).

THEOREM 1 (Stone's Theorem). *Every **metrizable** space is paracompact.*

Stone proved this theorem by showing that paracompactness is equivalent to another property – ***full normality*** – that had been introduced, and shown to follow easily from metrizability, by J.W. Tukey in 1940. (See [E, p. 313]; full normality is defined in [E, 5.1.12(iii)].)

An important consequence of Stone's theorem is the "only if" part of the following result, obtained independently by J. Nagata and Yu. Smirnov in 1951 (see [E, 4.4.7]), where a collection $\mathcal{U}$ of subsets of X is called **σ-locally finite** if $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ with each $\mathcal{U}_n$ locally finite.

THEOREM 2 (Nagata–Smirnov Metrization Theorem). *A regular space is metrizable if and only if it has a σ-locally finite **base**.*

Analogously to the Nagata–Smirnov theorem, we have the following characterization of paracompact spaces (see [E, 5.1.11]).

THEOREM 3. *A regular space X is paracompact if and only if every open cover of X has a σ-locally finite, open refinement.*

Theorems 2 and 3 together clarify why metrizable spaces are paracompact, and Theorem 3 also clarifies why regular Lindelöf spaces are paracompact.

It should be noted that the characterizations in Theorems 2 and 3 both remain valid if "σ-locally finite" is replaced by the stronger condition "σ-discrete". (A collection $\mathcal{V}$ of subsets of X is called a **discrete collection** if every $x \in X$ has a neighbourhood W intersecting at most *one* $V \in \mathcal{V}$, and $\mathcal{V}$ is called **σ-discrete** if $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ with each $\mathcal{V}_n$ discrete.) The resulting modification of Theorem 2 was obtained, also in 1951, by R.H. Bing (see [E, 4.4.8 and 5.1.12]).

3. Preservation under various operations

The behavior of paracompact spaces under various operations is generally similar to that of normal spaces, with a significant exception: Paracompact spaces are preserved by ***products*** with compact Hausdorff spaces, whereas normal spaces are not.

First, consider subsets: Like normal spaces, paracompact spaces are preserved by closed subsets, and even by F_σ-subsets [E, 5.1.28], but not, in general, by all subsets (since a subset of a compact Hausdorff – thus paracompact – space need not even be normal).

Next, consider products: Just as for normal spaces, the product of two paracompact spaces need not be paracompact. For example, there exists a paracompact space X such that $X \times \mathbb{P}$ (where $\mathbb{P}$ denotes the irrationals) is not even normal (see [E, 5.1.36]; see [E, 5.5.4] for related examples). However, the product $X \times Y$ of a paracompact space X and a *compact Hausdorff* space Y *is* always paracompact [E, 5.1.36]. (More generally, so is any Hausdorff space which admits a ***perfect map*** onto a paracompact space.) This is false for normal spaces; indeed, we have the following

[1] Whenever possible, all our references will be to Engelking's book [E], where more detailed information can be found in the historical and bibliographic notes at the ends of the relevant sections.

striking result of H. Tamano (see [E, 5.1.38]), which characterizes paracompactness in terms of normality:

THEOREM 4 (Tamano's Theorem). *The following are equivalent for a space X:*

(a) *X is paracompact.*
(b) *$X \times Y$ is paracompact for every compact Hausdorff space Y.*
(c) *$X \times Y$ is normal for some compact Hausdorff space $Y \supset X$.*

Finally, consider images under ***continuous closed maps***: Just as for normal spaces, we have the following result for paracompact spaces.

THEOREM 5. *If $f : X \to Y$ is a continuous, closed map onto Y, and if X is paracompact, then so is Y.*

For a direct proof of Theorem 5, see [E, 5.1.33]. Alternatively, Theorem 5 is an immediate consequence of the following characterization (see [3, Theorem 1 and Footnote 7]), where a collection $\mathcal{A}$ of subsets of X is called **closure-preserving** if $(\bigcup \mathcal{B})^- = \bigcup\{\bar{A}\colon A \in \mathcal{B}\}$ for every $\mathcal{B} \subset \mathcal{A}$. Note that every locally finite collection of subsets of X is closure-preserving.

THEOREM 6. *A T_1-space X is paracompact if and only if every open cover of X has closure-preserving, **closed refinement**.*

It should be noted that, just as Theorem 5 follows from Theorem 6, the obvious analogue of Theorem 5 for normal spaces follows from the following simple analogue of Theorem 6: A T_1-space X is normal if and only if every binary open cover $\{U_1, U_2\}$ of X has a binary closed refinement $\{A_1, A_2\}$.

For some other characterizations of paracompact spaces related to Theorem 6, see [3, Theorem 2] and [4].

4. Partitions of unity and their applications

A **partition of unity** on a space X is a family $(f_\alpha)_{\alpha \in A}$ of continuous functions from X to $[0, 1]$ such that, for all $x \in X$, $\sum_{\alpha \in A} f_\alpha(x) = 1$.[2] It is a **locally finite partition of unity** on X if the open cover $(\{x \in X\colon f_\alpha(x) > 0\})_{\alpha \in A}$ of X is locally finite, and it is **subordinated to** a cover $\mathcal{U}$ of X if each f_α is 0 outside some $U_\alpha \in \mathcal{U}$.

THEOREM 7. *The following are equivalent for a T_1**-space** X:*

(a) *X is paracompact.*
(b) *Every open cover of X has a locally finite partition of unity subordinated to it.*
(c) *Every open cover of X has a partition of unity subordinated to it.*

[2] This equation implies, in particular, that $f_\alpha(x) \neq 0$ for at most countably many α.

We now give two applications of Theorem 7. Our first application is another characterization of paracompact spaces (see [2, Theorem 3.2″]), where $\mathcal{F}(Y)$ denotes $\{E \subset Y\colon E \neq \emptyset$, E closed in $Y\}$, a map $\varphi : X \to \mathcal{F}(Y)$ is called **lower semi-continuous** (= **l.s.c.**) if $\{x \in X\colon \varphi(x) \cap V \neq \emptyset\}$ is open in X for every open V in Y, and a **selection** for $\varphi : X \to \mathcal{F}(Y)$ is a continuous $f : X \to Y$ such that $f(x) \in \varphi(x)$ for all $x \in X$.

THEOREM 8. *A T_1-space X is paracompact if and only if, for every **Banach space** Y, every l.s.c. map $\varphi : X \to \mathcal{F}(Y)$ with $\varphi(x)$ **convex** for all $x \in X$ has a selection.*

Our next result is the following theorem of J. Dugundji (see [E, 4.5.20(a)]), where $C(X)$ denotes the linear space of continuous functions $f : X \to \mathbb{R}$.

THEOREM 9 (Dugundji's Extension Theorem). *Let X be metrizable and $A \subset X$ closed. Then there exists a linear map $u : C(A) \to C(X)$ such that, if $f \in C(A)$, then $u(f)$ is an **extension** of f whose range lies in the **convex hull** of the range of f.*

Although paracompactness plays an important role in the proof of Theorem 9 (which depends on Theorems 1 and 7), the theorem does not remain valid for *all* paracompact X. Nevertheless, by a result of C.R. Borges (see [1, 4.3] or [KV, 5.2.3 on p. 464]), it suffices to assume that X is ***stratifiable***, an interesting property lying between metrizability and paracompactness which shares many of the best features of both. See Paul Gartside's article on Generalized Metric Spaces in this encyclopedia.

5. Further results

The following is a small sample of some other result in which paracompact spaces play a role.

(1) Every paracompact and ***locally metrizable*** space is metrizable. (See [E, 5.4.A] or by Theorem 2.)
(2) Every paracompact and ***locally compact*** space has a ***disjoint*** open covering by Lindelöf and locally compact subspaces; see [E, 5.1.27].
(3) Every paracompact and ***countably compact*** space is compact; see [E, 5.1.20]. (Here "paracompact" can be weakened to "***weakly paracompact***"; see [E, 5.3.2].)
(4) Every paracompact space with a ***dense*** Lindelöf subspace is Lindelöf; see [E, 5.1.25].
(5) If $f : X \to Y$ is a closed map from a paracompact space X onto a Hausdorff space Y, and if $C \subset Y$ is compact, then $C = f(K)$ for some compact $K \subset X$. (See [E, 5.5.11(c)].)
(6) Let $f : X \to Y$ be an ***open map*** from a completely metrizable (more generally: ***Čech-complete***) space X onto a paracompact space Y. Then there exists a closed

$A \subset X$ such that $f(A) = Y$ and $f \upharpoonright A : A \to Y$ is a perfect map. (See [E, 5.5.8].)

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d-13 Generalizations of Paracompactness

Compactness and metrizability are the heart and soul of general topology. Also for applications, these two concepts are the most important: metric notions are used almost everywhere in mathematical analysis, while compactness is used in many parts of analysis and also in mathematical logic. Already a study of the Euclidean spaces $\mathbb{R}^n$ shows that the classes $\mathcal{K}$ and $\mathcal{M}$ of compact (Hausdorff) spaces and of metrizable spaces have a very interesting intersection $\mathcal{K} \cap \mathcal{M}$. It is not equally obvious that also the union $\mathcal{K} \cup \mathcal{M}$ of these two classes is interesting, even though it was proved already in the 1920s that every member of $\mathcal{K} \cup \mathcal{M}$ satisfies the important separation property known as ***normality***. The picture became more interesting in 1940 when J.W. Tukey introduced the class of fully normal spaces: a space X is **fully normal** provided that each open cover $\mathcal{U}$ of X has an open **star refinement** (i.e., an open cover $\mathcal{V}$ such that, for every $x \in X$, the set $\bigcup\{V \in \mathcal{V}\colon x \in V\}$ is contained in some set of $\mathcal{U}$). Tukey proved that every member of $\mathcal{K} \cup \mathcal{M}$ is fully normal. A few years later, J. Dieudonné defined ***paracompactness*** as a natural generalization of compactness. In 1948, the period of 'modern general topology' was started by A.H. Stone's landmark paper in which full normality and paracompactness where shown to be equivalent properties. Thus we have $\mathcal{K} \cup \mathcal{M} \subset \mathcal{P}$, where $\mathcal{P}$ is the class of paracompact (Hausdorff) spaces.

Besides compactness and metrizability, there are a few other concepts which are fundamental in general topology: connectedness, countability conditions (separability, first and second countability, the Lindelöf property etc.) and separation (Hausdorffness, regularity, normality, etc.). The importance of paracompactness is manifested in the rich interplay it has with many of these fundamental notions: already Dieudonné showed that regular Lindelöf spaces are paracompact, paracompact Hausdorff spaces are normal and paracompact separable spaces are Lindelöf. Paracompactness gained even much more importance when Stone's result on paracompactness of metrizable spaces was used by J.-I. Nagata, Yu.M. Smirnov and R.H. Bing to give a satisfactory solution, around 1950, for the fundamental 'metrization problem', which asked for purely topological characterizations of metrizability (somewhat less satisfactory solutions for this problem had already been obtained earlier, the first having been the Alexandroff–Urysohn metrization theorem from 1923). In the 1950s, E.A. Michael published a series of papers containing successively weaker conditions characterizing paracompactness, and he used these conditions to derive important properties of paracompact spaces; for example, he proved that the continuous image of a paracompact space under a closed map is paracompact; see Michael's contribution *Paracompact spaces* to this volume for further information.

In general topology, as in many other parts of mathematics, successful notions tend to get generalized. One motivation for such generalizations is the attempt to 'push results to their limits'. Thus paracompactness has been generalized to *meta-Lindelöfness*, a property satisfying the formula 'P + separable ⇒ Lindelöf' and it has also received a long sequence of successively weaker generalizations which satisfy the formula 'P + countably compact ⇒ compact' (see [KV, Chapters 9 and 12]). Another motivation is to 'imitate a successful theory'. Thus paracompactness has been generalized to metacompactness, subparacompactness and submetacompactness, three different concepts, each allowing a characterization theory very similar to that obtained by Michael for paracompactness. If these were the only motivations, it is not likely that we would be led to very interesting concepts. A third guiding principle can sometimes correct this, because it requires that there should exist some well-established class of spaces whose members are 'typical' representatives for the new property. Thus metrizable spaces are typical paracompact spaces, even though Dieudonné did not know it; on the other hand, subparacompactness first appeared as a property of ***developable*** spaces, and only later was it considered as an independent concept. Like in these two instances, the typical representatives often come from some class of ***generalized metric spaces***.

Starting from the early 1950s, a great number of various generalizations of paracompactness have been considered. I shall mention below a only a few of these, and I will try to concentrate on the ones which seem most interesting.

One of the earliest and most straightforward generalizations of paracompactness was defined in 1951 by C.H. Dowker and M. Katětov (see [E, Section 5.2]). A space X is **countably paracompact** provided that each countable open cover has a locally finite open refinement (below we consider 'countable' versions of other covering properties, and the definitions are analogous to that of countable paracompactness). Dowker proved that a normal space X is countably paracompact if, and only if, the product $X \times \mathbb{I}$ of X with the closed unit interval is normal, and he raised an influential problem, known later as the ***Dowker space*** problem, asking about the existence of a normal space which is not countably paracompact; the problem was solved in 1971 by M.E. Rudin with a ZFC example (see [KV, Chapter 17]), but work has continued up to the present day (see, e.g., [2]) with attempts to produce 'smaller and nicer' examples; some difficult problems remain, for example: Is there a Dowker space with a σ-disjoint base? Is there a nonmetrizable normal space which is the union of countably many open metrizable subsets? Even though countably paracompact regular spaces need not be normal, there are many instances of results on normal spaces which remain valid for regular countably paracompact spaces. A striking example of this is D.K. Burke's result that, under the ***Product Measure Extension Axiom***, countably paracompact regular developable spaces are metrizable [3], which extends the earlier 'provisional solution' of the Normal Moore Space Problem by P. Nyikos (see [HvM, Chapter 7]). Dowker's result on $X \times \mathbb{I}$ also pointed towards another important development culminating in the remarkable theorem of H. Tamano (see [E, Chapter 5.1]): a completely regular space X is paracompact if, and only if, the product $X \times \beta X$ of X with its ***Čech–Stone compactification*** is normal.

The archetypal covering property, compactness, is preserved in Cartesian products, by the famous Tychonoff theorem, but most other covering properties do not behave well with products. However, almost all covering properties P satisfy the condition that $X \times K$ has P whenever X has P and K is compact. Tamano's theorem shows that whenever P satisfies the above condition and P implies normality, then P implies paracompactness. As a consequence, most covering properties generalizing paracompactness do not imply normality. One of the few exceptions is the shrinking property. A **shrinking** of an open cover $\mathcal{U}$ of a space X is a cover $\{L_U\colon U \in \mathcal{U}\}$ of X such that $\overline{L_U} \subset U$ for every $U \in \mathcal{U}$. A space X has the **shrinking property** if every open cover of X has an open shrinking. S. Lefschetz had proved in 1942 (see [E, Section 1.5]) that every point-finite open cover of a normal space has an open shrinking, and it is a consequence of this result that every paracompact Hausdorff space is shrinking. On the other hand, it is easy to see that every shrinking space is countably paracompact. These and more results on shrinking spaces can be found in [MN, Chapter 5].

Turning to generalizations of paracompactness not implying normality, we first mention the oldest of these, defined by R. Arens and J. Dugundji in 1950 and by Bing in 1951 (see [E, Section 5.3]): a space is **metacompact** if every open cover has a point-finite open refinement (**weakly paracompact** in [E]; Bing used the term 'pointwise paracompact' for these spaces). It follows from Lefschetz's theorem above that every metacompact and normal space is shrinking and hence countably paracompact. Normal metacompact spaces do not have to be paracompact, as an example given by Michael in 1955 shows; see [E, Problem 5.5.3]. The relation between normal metacompact spaces and paracompact spaces has given rise to difficult problems: A.V. Arkhangel'skiĭ proved in 1971 [1] that every metacompact, locally compact and perfectly normal space is paracompact, and he asked whether 'perfectly' can be omitted from the assumptions. This problem turned out to be dependent of set-theoretical axioms: S. Watson showed in 1982 [9] that the answer is 'yes' under the Axiom of Constructibility ($V = L$), and G. Gruenhage and P. Koszmider [6] completed the solution of Arkhangel'skiĭ's problem by giving a consistent example of a metacompact and locally compact normal space which fails to be paracompact. Actually the covering properties are only added ingredients here, and the real difficulty is in the relation between normality and ***collectionwise normal***ity, because Michael and K. Nagami showed, independently, in 1955 that every metacompact collectionwise normal space is paracompact (see [E, Section 5.3]).

The characterization theory for metacompactness is almost as rich as that initiated by Stone and Michael for paracompactness. In 1966, J.M. Worrell Jr characterized metacompactness by a condition whose relation to the defining condition is similar to the one that full normality has to paracompactness, see; see [KV, Chapter 9, Theorem 3.5]. Dieudonné had observed that every locally finite family $\mathcal{L}$ is **closure-preserving** (i.e., $\overline{\bigcup \mathcal{L}'} = \bigcup\{\overline{L}\colon L \in \mathcal{L}'\}$ for every $\mathcal{L}' \subset \mathcal{L}$) and Michael had proved that a regular space is paracompact if every open cover of the space has a closure-preserving closed refinement; see [KV, Chapter 9]. Metacompactness allows a similar characterization: X is metacompact if, and only if, every directed open cover has a closure-preserving closed refinement; this theorem gives as a corollary the result of Worrell (see [E, Chapter 5.3]) that the continuous image of a metacompact space under a closed map is metacompact. There are even versions of Dowker's and Tamano's theorems for metacompactness, but the role of normality is now taken by a covering property. An open family $\mathcal{U}$ of X is **interior-preserving** if $\bigcap(\mathcal{U})_x$, where $(\mathcal{U})_x = \{U \in \mathcal{U}\colon x \in U\}$, is open for every $x \in X$, and X is **orthocompact** if every open cover of X has an interior-preserving open refinement (with different terminology, these notions were introduced by W.M. Fleischman [5]). B.M. Scott [8] showed that a regular space X is countably metacompact if, and only if, $X \times \mathbb{I}$ is countably orthocompact, and Junnila [7] showed that a completely regular space X is metacompact if, and only if, $X \times \beta X$ is orthocompact.

Typical representatives of metacompact spaces are spaces with a uniform base in the sense of P.S. Alexandroff; a base $\mathcal{B}$ of X is a **uniform base** provided that for every $x \in X$, every infinite subfamily of $(\mathcal{B})_x$ is a base at x; if the conclusion is required only for each infinite *descending* subfamily of $(\mathcal{B})_x$, we have the weaker notion of a **base of countable order**. This class demonstrates well the close relation that covering properties have with generalized metric spaces, since a result of R.W. Heath shows that a T_1-space X has a uniform base if, and only if, X is metacompact and developable; see [E, Lemma 5.4.7]. (This is just one example of many similar results: for Hausdorff spaces we have that developable + paracompact $\Leftrightarrow$ metrizable; developable + orthocompact $\Rightarrow$ quasi-metrizable; developable + meta-Lindelöf $\Rightarrow$ point-countable base; developable $\Leftrightarrow$ submetacompact + base of countable order, etc.; see [KV, Chapter 10].) Another interesting class of metacompact spaces is formed by spaces with a closure-preserving cover by compact closed sets; see [MN, Chapter 13].

There are two other covering properties having a nice characterization theory, subparacompactness and submetacompactness. A space is **subparacompact** provided every open cover of the space has a σ-discrete closed refinement; this property first appeared in Bing's 1951 paper as a property of developable spaces. Explicitly, these spaces were defined by L.F. McAuley in 1958, under the name 'F_σ-screenable' (Bing had defined **screenable** spaces by the condition that every open cover has a σ-disjoint open refinement). In 1966, Arkhangel'skiĭ considered a generalization of full normality which he called 'σ-paracompactness'. In 1969, D.K. Burke showed that the properties of F_σ-screenability and σ-paracompactness are equivalent with each other and also with the property that every open cover has a σ-closure-preserving closed refinement, and he introduced the term 'subparacompact' for spaces having these properties; see [KV, Chapter 9]. Developable spaces are typical subparacompact spaces, and an even larger class of subparacompact spaces if formed by the ***semistratifiable*** spaces.

A space X is **submetacompact** provided that every open cover $\mathcal{U}$ of X has a sequence $\langle \mathcal{V}_n \rangle_{n\in\mathbb{N}}$ of open refinements such that, for every $x \in X$, there exists $n \in \mathbb{N}$ such that $\mathcal{V}_n$ is point-finite at x. These spaces, which contain all metacompact spaces and all subparacompact spaces, were introduced, as 'θ-refinable spaces', by Worrell and H.H. Wicke in 1965. In 1967, Worrell obtained a 'full normality-type' characterization for these spaces (see [KV, Chapter 9]). The name 'submetacompact' was introduced in 1978 by Junnila, who characterized these spaces by the condition that every directed open cover has a σ-closure-preserving closed refinement. One class of representatives for submetacompact spaces is formed by those spaces which have a σ-closure-preserving cover by compact closed sets. This class has been of some interest in the product theory of covering properties since R. Telgarsky showed in 1975 that if a paracompact Hausdorff space X has a σ-closure-preserving cover by compact sets, then $X \times Y$ is paracompact for every paracompact Hausdorff space Y; similar theorems have subsequently been proved for other covering properties by Y. Yajima and others; see [MN, Chapter 13]. An important theorem of S. Jiang from 1984 gives another interesting class of representatives for submetacompact spaces: the ***strict*** ***p-space***s of Arkhangel'skiĭ; see [HvM, Chapter 14].

The three covering properties mentioned above, metacompactness, subparacompactness and submetacompactness, each has a rather interesting theory. An analogous theory could also be developed for **mesocompact space**s, i.e., spaces whose open covers have *compact-finite* open refinements; however, mesocompactness seems to be too close to paracompactness to deserve an independent study. The theories for the other generalizations of paracompactness that have been considered in the literature seem to consist of only a few isolated theorems together with some limiting examples. To illustrate, we mention **para-Lindelöf** spaces, which are defined by the condition that every open cover has a locally countable open refinement. In the 1970s, many topologists worked on paraLindelöf spaces, mainly trying to solve the problem whether every regular paraLindelöf space is paracompact. The solution, in the form of a counterexample, was given by C. Navy in 1981 (see [KV, Chapter 16]), and after that, paraLindelöf spaces lost much of their interest.

Perhaps the most interesting of the generalizations of paracompactness not yet considered here are meta-Lindelöfness and weak submetacompactness. The 'internal' theories of these two properties are quite uninvolved, but they both have many interactions with other topological properties, and they are even important in applications of general topology to other parts of mathematics, notably to functional analysis, measure theory and descriptive set theory.

A space is **meta-Lindelöf** if each open cover has a point-countable open refinement. Typical meta-Lindelöf spaces are spaces with a point-countable base and spaces with a closure-preserving cover by closed Lindelöf subspaces. The relationship between normal meta-Lindelöf spaces and paracompact spaces is even more problematic than that between normal metacompact spaces and paracompact spaces, because the Michael–Nagami theorem does not extend to meta-Lindelöf spaces (Worrell and Wicke showed that it does extend to submetacompact spaces). Nevertheless, Z. Balogh managed to extend Watson's result mentioned above by proving that, under the Axiom of Constructibility ($V = L$), normal, locally compact meta-Lindelöf spaces are paracompact; see [HvM, Chapter 14]. A space is **weakly submetacompact** (**weakly θ-refinable**) if each open cover has a **σ-isolated** refinement (i.e., a refinement $\bigcup_{n\in\mathbb{N}} \mathcal{L}_n$ such that, for every n, every $x \in \bigcup \mathcal{L}_n$ has a neighbourhood meeting only one member of $\mathcal{L}_n$; such families could also be called *σ-relatively discrete*). With a different definition, these spaces were introduced by H.R. Bennett and D.J. Lutzer in 1972; see [KV, Chapter 9]. Typical weakly submetacompact spaces are spaces with a σ-isolated network.

Both meta-Lindelöf spaces and weakly submetacompact spaces have appeared in functional-analytic considerations. R.W. Hansell considered a class of spaces with a σ-isolated network which he called **descriptive space**s (see [HvM, Chapter 8]), and he proved that a ***σ-fragmentable*** Banach space, in its weak topology, is descriptive if, and only if, it is hereditarily weakly submetacompact. A. Dow, Junnila and J. Pelant showed in [4] that not all B_w-spaces, i.e., Banach spaces with weak topology, are weakly submetacompact, but they left open the problem of whether all σ-fragmentable ones are. Meta-Lindelöf spaces are important in the theory of **Corson compact** spaces, i.e., spaces homeomorphic with compact subsets of ***Σ-product***s $\{x \in \mathbb{I}^A\colon x_a \neq 0$ for only countably many $a \in A\}$ (here A is any set and $\mathbb{I}^A$ has usual product topology). N.N. Yakovlev proved that every Corson compact space is hereditarily meta-Lindelöf and Gruenhage obtained the following characterization: a compact Hausdorff space X is Corson compact if, and only if, the space $X \times X$ is hereditarily meta-Lindelöf. S.P. Gul'ko showed that $C_p(K)$, the set of continuous functions on K equipped with the topology of pointwise convergence, is hereditarily meta-Lindelöf for each Corson compact space; see [HvM, Chapter 14]. Even though $C_p(K)$-spaces and B_w-spaces may fail to be meta-Lindelöf (see [4]), there are other non-trivial covering properties satisfied hereditarily by all $C_p(K)$-spaces and B_w-spaces. In 1959, Michael (see [KV, Chapter 9]) characterized paracompactness of a regular space by the existence of **cushioned refinement**s for all open covers: a cover $\mathcal{L}$ of X has a cushioned refinement if, and only if, one can assign $L_x \in \mathcal{L}$, for every $x \in X$, so that $\bar{A} \subset \bigcup\{L_x\colon x \in A\}$ for every $A \subset X$. Generalizing this condition, we say that a cover $\mathcal{L}$ of X is **thick** if one can assign a finite union U_H of sets from $\mathcal{L}$ to each finite subset H of X so that we have $\bar{A} \subset \bigcup\{U_H\colon H \subset A$ and H is finite$\}$ for every $A \subset X$. The space X is **thickly covered** if each open cover of X is thick. The class of thickly covered spaces is quite large as it contains all meta-Lindelöf T_1-spaces and all subspaces of $C_p(K)$-spaces and of B_w-spaces; nevertheless, the class is non-trivial, because every separable thickly covered space is Lindelöf and every countably compact thickly covered space is compact.

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d-14 Countable Paracompactness, Countable Metacompactness, and Related Concepts

A space is **countably paracompact** (respectively **countably metacompact**) if every countable open cover has a locally finite (respectively, point-finite) open refinement. Despite the superficial similarity in both their names and in some of their respective equivalents, these classes of spaces are very different as far as current-day interests of topologists are concerned. Countable paracompactness generally goes hand in hand with normality – so much so that spaces that are ***normal*** but not countably paracompact are singled out by the term "Dowker spaces" while spaces that are countably paracompact but not normal are widely termed **anti-Dowker** spaces. On the other hand, a quarter of a century ago, so few ***regular*** spaces were known *not* to be countably metacompact that Brian M. Scott [10] referred to the few then-known examples as "almost Dowker spaces". Since then, a considerable variety of regular spaces have been found not to be countably metacompact, including some manifolds [7], even smooth ones such as tangent bundles obtained from smoothings of the long line [9]. Nevertheless, while the term "almost Dowker" would be an overstatement, these spaces are still encountered rather infrequently.

There are many concepts with definitions similar to that of these two, such as that of countable subparacompactness, and that of cb-spaces and weak cb-spaces, but these have attracted relatively little attention and their properties are not so well understood. Since ***subparacompactness*** is treated at some length elsewhere in this encyclopedia, it may be worth pointing out that the various alternative definitions of that concept carry over to **countable subparacompactness**. Thus it makes no difference whether one says "every countable open cover has a $\{\sigma$-locally finite, σ-discrete, σ-closure-preserving, σ-cushioned$\}$ closed refinement", and the only novelty is that it is also equivalent to having a countable closed refinement. Proofs of these equivalences can be found in [3], which is also the seminal paper on the subject of countable metacompactness. It also shows that what were called "countably θ-refinable spaces" are actually the same as countably metacompact spaces, and hence that every countably subparacompact space is countably metacompact. An awkward feature of countable subparacompactness is that it is not implied by countable compactness: the product space $(\omega_1 + 1) \times \omega_1$ is not countably subparacompact.

One can require, in the definitions of countable paracompactness and countable metacompactness, that the refinements be countable as well, and obtain an equivalent condition in either case. The really useful equivalent conditions, however, are the ones that begin with a countable descending sequence of closed sets F_n whose intersection is empty, and expand each F_n to an open set U_n, requiring $\bigcap_{n=0}^{\infty} U_n = \emptyset$ in the case of countable metacompactness and $\bigcap_{n=0}^{\infty} \overline{U_n} = \emptyset$ in the case of countable paracompactness. This makes it obvious that the two properties coincide for normal spaces. Thus one can define a **Dowker space** either as a normal space which is not countably paracompact, or as one that is not countably metacompact. Of course, the property of Dowker spaces that makes them so popular is that they are the normal spaces whose product with $[0, 1]$ fails to be normal. Brian Scott [10] found a similar product theorem for ***orthocompactness***: an orthocompact space has orthocompact product with $[0, 1]$ iff it is countably metacompact. Countable metacompactness figures in another interesting equivalence due to Norman Howes [4]: a regular linearly Lindelöf space is Lindelöf iff it is countably metacompact. [Recall that a space is called **linearly Lindelöf** if every ascending open cover has a countable subcover.]

Countable paracompactness has many affinities with normality, including a curious set of parallels involving the axioms $V = L$ and PMEA. If $V = L$, then every normal space and also every countably paracompact space of character $\leqslant \aleph_1$ is ***collectionwise Hausdorff*** [2, 11]. If PMEA, then every normal space of character $< 2^{\aleph_0}$ is ***collectionwise normal***, while every countably paracompact space of ***character*** $< 2^{\aleph_0}$ is collectionwise Hausdorff and expandable [6, 1]. Both of these axioms imply that a subspace of ω_1^2 is normal iff it is countably paracompact [5]. On the other hand, although it is a theorem of ZFC that every normal subspace of ω_1^2 is countably paracompact, the reverse implication is an open problem with interesting set-theoretic equivalents. There are also $V = L$ and PMEA theorems for countable metacompactness: $V = L$ implies closed discrete subsets of locally countable T_1 countably metacompact spaces are G_δ [8], while PMEA implies closed discrete subsets of T_1 countably metacompact spaces of character $< 2^{\aleph_0}$ are G_δ if every one of their points is a G_δ [1].

Morita P-spaces are an important special class of countably metacompact spaces. These spaces are often classed as "generalized metric spaces" because the normal ones are precisely those normal spaces whose product with every metric space is normal. But they can also be looked at as an interesting example in the theory of topological games. They can be defined as those topological spaces in which the second player has a winning strategy in what might be called the countable metacompactness game. This is a topological game with infinitely many moves indexed by the natural numbers, in which two players take turns playing closed sets and open sets in a topological space, and each one has knowledge of past plays but not of future ones. On the nth turn, Player 1 plays a closed set F_n which is a subset of the previously chosen sets F_k, and then Player 2 plays an open set G_n containing F_n. At the end of the game, Player 1 wins iff the set of all F_n have empty intersection but the set of all G_n has nonempty intersection. If either condition fails to obtain, then Player 2 wins.

Morita's original definition was more technical but also lends itself more readily to modification. Given any cardinal number κ and any choice of open sets $G(\alpha_1, \ldots, \alpha_n) \subset X$ for each finite sequence of $\alpha_i \in \kappa$, there are closed sets $F(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n)$ such that for each infinite sequence $\langle \alpha_n : n \in \omega \rangle$ of elements of κ, either

(a) $\bigcup_{n=1}^{\infty} G(\alpha_1, \ldots, \alpha_n) \neq X$, or

(b) $\bigcup_{n=1}^{\infty} F(\alpha_1, \ldots, \alpha_n) = X$.

One variation is to fix $\kappa = \omega$; this gives the class of $P_{\aleph_0}$**-spaces**. If, in addition, one requires that $\bigcup_{n=1}^{\infty} G(\alpha_1, \ldots, \alpha_n) = X$ for each infinite sequence $\langle \alpha_n : n \in \omega \rangle$ of elements of ω, then one defines the class of **weak** $P_{\aleph_0}$**-spaces**. For $X \times Y$ to be normal for all separable ***metrizable*** Y (respectively, for all separable ***completely metrizable*** Y), it is necessary and sufficient that X be a normal (weak) $P_{\aleph_0}$-space. There are also characterizations along similar lines for all spaces X whose product with a single given metrizable space Y is normal (see [KV, Chapter 18]).

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d-15 Extensions of Topological Spaces

Finding a topological space containing a given one is part of many constructions in topology and it has attracted a lot of attention, already in the topological past. Quite a few entries in this book are related to extending spaces, e.g., *Topological embeddings*, *Hyperspaces*, *Free topological groups* and many others. Unless stated otherwise, all spaces in this article will be assumed to be ***completely regular***. Except for Section 5 where extensions will resemble hyperspaces, we are going to deal with extensions which contain the starting space as a dense subspace.

1. Completion of metric spaces

Already Hausdorff [11] proved that each metric space (X, d) has a unique **metric completion** $(\widetilde{X}, \tilde{d})$, i.e., $(\widetilde{X}, \tilde{d})$ is ***complete*** and X is ***isometric*** to a dense subset of $(\widetilde{X}, \tilde{d})$. It holds true that ***weight*** of $(\widetilde{X}, \tilde{d})$ is equal to weight of (X, d) and $\tilde{d}$ is ***totally bounded*** iff d is [E, 4.3.19 and 4.5.6]. Completeness is described by ***Cauchy sequence***s which depend on the metric. As there are more metrics inducing a topology of a metrizable space, we recall that completions $(\widetilde{X}, \tilde{\rho})$ and $(\widetilde{X}, \tilde{\sigma})$ are **uniformly homeomorphic** if the identity map $\mathrm{id} : (X, \rho) \to (X, \sigma)$ is a uniformly continuous homeomorphism [E, 4.3.17], so two (topologically) equivalent metrics on X can induce very different completions. So one can consider completions of a metrizable space without specifying a metric. There are examples of dimension-preserving completions.

For each metrizable space X there is a ***completely metrizable*** space $\widetilde{X}$ such that X is isomorphic to a dense subset of $\widetilde{X}$, $w(X) = w(\widetilde{X})$ and, in addition, $\Phi X = \Phi \widetilde{X}$, where Φ denotes a dimension function, e.g., $\Phi = \mathrm{Ind}$ (see [9, 4.1.20]) or $\Phi = \mathrm{tr\,Ind}$ (Luxemburg 1982, see [9, 7.2.19]).

Mrówka [16, 17] constructed under the assumption $S(\aleph_0)$ an example of a zero-dimensional metric space (i.e., it has a base for topology consisting of ***clopen*** sets) that does not have any zero-dimensional completion; Dougherty showed in 1997 that $S(\aleph_0)$ is relatively consistent with the existence of a certain large cardinal.

A metrizable space is ***separable*** (***compact***) iff X is metrizable by a totally bounded (and complete) metric [E, 4.3.5 and 4.3.29]. So the completion of a separable totally bounded metric space (X, ρ) is also a (metrizable) **compactification** of X. Notice that metrizable compactifications exist only for separable metrizable spaces.

2. Compactifications

For a completely regular Hausdorff space Z, a **compactification** of Z is a compact space K such that Z is homeomorphic to a dense subspace of K. There are again results on dimension-preserving compactifications: For each separable completely metrizable X with a property Ψ, there is a metrizable compactification with this property Ψ, where Ψ can be n-dimensional (for $n = 0$, Sierpiński (1921) [9, 1.3.16], for $n > 0$, Hurewicz (1927) [9, 1.7.2]); ***countable-dimensional*** (Hurewicz (1928), Lelek (1965), [9, 5.3.2]); ***strongly countable-dimensional*** (Schurle (1959) [9, 5.3.5]), or ***weakly countable-dimensional*** (Lelek (1965) [9, 6.1.19]).

Dropping metrizability of a compactification, Engelking and R. Pol proved in 1988 that every countable-dimensional completely metrizable space has a countable-dimensional compactification. Sklyarenko (1958) proved that if X is normal then there is a compactification $\widetilde{X}$ of X such that $\dim \widetilde{X} \leqslant \dim X$ and $w(\widetilde{X}) \leqslant w(X)$ [9, 3.4.2].

Partially ordered set of compactifications
For any completely regular space X, a partial order $\leqslant$ on the set of compactifications of X is defined as follows: let $c_1 X$ and $c_2 X$ be compactifications of X. So $X \subseteq c_i X$, $i = 1, 2$. We define $c_2 X \leqslant c_1 X$ if the identity map $\mathrm{id} : X \to X$ extends to a continuous map $c_1 X \to c_2 X$. Two compactifications $c_1 X$ and $c_2 X$ of X are said to be equivalent if $c_2 X \leqslant c_1 X$ and $c_1 X \leqslant c_2 X$. So for **equivalent compactifications** $c_1 X$ and $c_2 X$, the identity map $\mathrm{id} : X \to X$ extends to a homeomorphism from $c_1 X$ onto $c_2 X$ [E, 3.5.4]. Denote the set of compactifications of X by $\mathcal{K}(X)$. The partially ordered set $(\mathcal{K}(X), \leqslant)$ turns out to be **complete upper lattice**, i.e., any non-empty family $\mathcal{C} \subseteq \mathcal{K}(X)$ has a least upper bound, denoted as $\bigvee \mathcal{C}$, see [E, 3.5.9] or more generally [18, 4.1.6]. The largest element in $(\mathcal{K}(X), \leqslant)$ is called the ***Čech–Stone compactification*** and it is denoted as βX. This compactification is extremely important and a lot of information on it can be found in corresponding sections of this book. The existence of a smallest element in $(\mathcal{K}(X), \leqslant)$ characterizes ***local compactness*** of X. This smallest compactification of a locally compact non-compact space X is called the **Alexandroff compactification** and it is often denoted as αX. As the ***remainder*** $\alpha X \setminus X$ is a singleton, αX is called also the **one-point compactification** [E, 3.5.11, 3.5.12]. Another related characterization of local compactness says that X is locally compact iff $(\mathcal{K}(X), \leqslant)$ is a complete lattice (Lubben (1941), [18, 2.1.e and 4.3.e]). Magill (1968) proved that for any two locally compact spaces X and Y, the remainders $\beta X \setminus X$ and $\beta Y \setminus Y$ are homeomorphic iff the two lattices of compactifications of X and Y are isomorphic. This interesting fact was generalized in many other papers, see [18, p. 768].

For every compactification K of X one has $\mathrm{card}\, K \leqslant 2^{2^{d(X)}}$ and $w(K) \leqslant 2^{d(X)}$ [E, 3.5.3]. For each space X, there is a compactification K with $w(X) = w(K)$ [E, 3.5.2]. For any X, all compactifications from $\mathcal{K}(X)$ have the same density, which may be smaller than the density of X itself (van Douwen (1975), Levy and McDowell (1975), [E, 3.5.J]).

Construction of compactifications
There are several ways how to describe (or generate) compactifications of X. We mention the approach via ***proximities***: Smirnov's theorem (1952, see [E,8.4.13]) establishes a one-to-one correspondence between $\mathcal{K}(X)$ and all proximities on X generating the topology of X. Another approach uses subrings of the ring $C^*(X)$ of bounded continuous functions on X, equipped with the **sup-norm**, i.e., the norm of $f \in C^*(X)$ is $\|f\| = \sup_{x\in X} |f(x)|$. If a subring S of $C^*(X)$ is a **regular subring** (i.e., S with the sup-norm metric is complete, S contains all constant maps and the collection of ***zero set***s of all maps from S form a closed base of X) then the set of all maximal ideals of S – denoted as $m_S X$, equipped with the Stone topology (see [10, 4.9, 7.10-11 and 7M]) is a compactification of X. Moreover, denote by $\mathcal{R}$ the poset of regular subrings of $C^*(X)$, partially ordered by inclusion. Then $\psi(S) = m_S X$ defines an order isomorphism from $\mathcal{R}$ onto $\mathcal{K}(X)$ (Gelfand (1941), [18, 4.5.m and 4.5.q]).

Preserving properties
A lot of effort was concentrated on the question whethner every completely regular ***scattered*** space has a scattered compactification. Kannan and Rajagopalan (1974) proved that a first-countable zero-dimensional Lindelöf Hausdorff space is scattered iff it admits a scattered compactification. The first-counterexample was given by P. Nyikos in 1974. Interesting is the investigation of subspaces of $\beta\mathbb{N}$ of the form $\mathbb{N} \cup \{p\}$, where p is a free ultrafilter on $\mathbb{N}$. Ryll-Nardzewski and Telgarsky (1970) proved that if p has a well-ordered neighbourhood base of type ω_1 in $\beta\mathbb{N} \setminus \mathbb{N}$, i.e., p is a ***P-point*** of character ω_1, then $\mathbb{N} \cup \{p\}$ does have a scattered compactification. Solomon (1976) (Telgarsky (1977) and Malyhin (1978) respectively) proved that $\mathbb{N} \cup \{p\}$, where $p \in F \subseteq \mathbb{N}^*$, has no scattered compactification provided F is the support of a non-atomic Borel measure in $\mathbb{N}^*$ (F is ccc perfect, F is extremely disconnected dense-in-itself subspace of $\mathbb{N}^*$, respectively). In 1987, Malyhin announced the consistency of the fact that for every $p \in \mathbb{N}^*$, each compactification of $\mathbb{N} \cup \{p\}$ contains a copy of $\beta\mathbb{N}$, so no compactification of $\mathbb{N} \cup \{p\}$ is scattered. Finally, Dow and Zhou (1999) proved: if one takes a continuous map f from $\beta\mathbb{N}$ onto $2^{\mathfrak{c}}$ and a closed subset A of $\beta\mathbb{N}$ such that $f \restriction A$ is ***irreducible*** then for every $p \in A$ every compactification of $\mathbb{N} \cup \{p\}$ contains a copy of $\beta\mathbb{N}$.

This is related to examples from [23] where a space Δ is contructed which is first-countable and Lindelöf and all compactifications of which contain a copy of $\beta\mathbb{N}$. We are going to recall Δ at least twice in the sequel.

Taking a ***linearly ordered topological space*** (or ***LOTS***) $(X, \leqslant)$, one is interested in finding ordered compactifications of X (see also [10, 3.O]). The standard process is filling each **gap** or **pseudogap** with one or two ideal points. A gap is a pair (A, B) of open sets such that $a < b$ whenever $a \in A$ and $b \in B$ and A has no maximum and B no minimum; it is a pseudo gap if A has a maximum or B a minimum, these occur only in ***generalized ordered spaces***. The family (in fact, a complete lattice) of all ordered compactifications of $(X, \leqslant)$ has both a largest element and a smallest one, described as the lattice completion of $(X, \leqslant)$ or equivalently as the Dedekind completion of $(X, \leqslant)$ with endpoints added, see once more [10]. This smallest compactification is an at most 2-1 continuous image of the largest compactification (see [13] for a systematic treatment). With minimal effort, one finds out that $bX \setminus X$ is zero-dimensional for each orderable compactification bX of an orderable space X. So each ordered compactification bX is bounded by the ***Freudenthal compactification***, which is constructed precisely for ***rim-compact*** spaces. Sklyarenko (1958) introduced the notion of a **perfect compactification**. A compactification bX of X is perfect iff $\beta\,\mathrm{id} : \beta X \to bX$ has all fibres (preimages of singletons) connected. This perfectness characterizes the Freudenthal compactification among the compactifications with a zero-dimensional remainder (for a precise formulation, see [1, 3.9]). Diamond [7] gave a characterization of a perfect compactification of X: A subring S of $C^*(X)$ is **algebraic** if S contains the constant functions and $f^2 \in S$ implies $f \in S$. Let bX be a compactification of X. For $S \subseteq C^*(X)$, let $S^\sharp$ denote the set $f \in C(\beta X)$ which satisfy: $f \restriction X \in S$ and $f \restriction X$ extends continuously to bX. Then bX is a perfect compactification iff $S^\sharp$ is an algebraic subring of $C(\beta X)$ for each algebraic subring S of $C^*(X)$ iff there is an algebraic subring of $C^*(X)$ that determines bX.

Maximal extensions of first-countable spaces
Terada and Terasawa [20] called a ***first-countable*** space X maximal with respect to first-countability if it has no proper first-countable extension. If a first-countable space X has a first-countable compactification Y then Y is a maximal extension of X. They proved that all metrizable (or locally compact first-countable) spaces have a maximal extension, so also first-countable spaces without any first-countable compactification, can have a maximal extension. The space Δ from [23] is proven to have no maximal extension.

3. Dieudonné completion

A topological space X is said to be **Dieudonné complete** if it is homeomorphic to a closed subset of a Cartesian product $\prod_{i\in I} M_i$ of metrizable spaces M_i, see [E, 8.5.13] also for most facts listed in this section without any reference. So each ***paracompact*** space is Dieudonné complete. Restricting M_i in the product to separable metrizable spaces, we arrive to the definition of Hewitt-complete or ***realcompact*** spaces. Shirota (1952) [E, 8.5.13.h] proved that a Dieudonné complete space X is realcompact iff each discrete closed subspace of X has cardinality smaller than the first ***Ulam***

measurable cardinal. To describe the Dieudonné completion μX, which is a categorical ***epireflection*** on category of ***Tychonoff*** spaces, one can take the ***completion*** of the finest ***uniformity*** inducing the topology of X. It is very close to taking all continuous functions $\mathcal{F}$ from X to metrizable spaces and then the diagonal product of $\mathcal{F}$; this way X is embedded into $\prod_{f\in\mathcal{F}}\mathrm{Range}(f)$ and μX is the closure of X in $\prod_{f\in\mathcal{F}}\mathrm{Range}(f)$. It is worth mentioning that this construction using a diagonal product of continuous maps from X to spaces belonging to some ***productive*** and ***closed-hereditary*** class $\mathcal{P}$ is a standard way how to get an extension of X, see [18], also for other properties like $\mathcal{P}$-regularity. Another description of μ uses the Čech–Stone compactification βX: μX consists of those points p of βX for which *every* continuous map $f: X \to M$ with metrizable codomain can be extended to the subspace $X \cup \{p\}$ of βX [6].

The Dieudonné completion is a very useful and popular technical tool. For example: a ***suborderable*** space is Dieudonné complete iff it is paracompact (Ishii (1959), Engelking, Lutzer (1977), [E, 8.5.13(j)]). For their selection theorem, Nedev and Choban used a restatement of this fact: the Dieudonné completion of a suborderable space is suborderable and paracompact, [5]. Finally, we recall a result from [19]: the Dieudonné completion of a product of an arbitrary family of locally ***pseudocompact topological groups*** is homeomorphic to the product of Dieudonné completions of the factors.

4. Less compact extensions

Morita [15] defined a space defined a space S to be a **countable-compactification** of X if S is ***countably compact***, X is dense in S and every countably compact closed subset of X is also closed in S. Morita showed that such an S may be always taken as a subspace of βX. Moreover, an ***M-space*** has a countable-compactification iff it can be embedded as a closed subset in a product of a countably compact space and a metrizable space. Burke and van Douwen [4] gave an example of a normal locally compact M-space without any countable-compactification. Whitaker (1984) got a result, nicely complementing this counterexample: for each rim-compact space X there is a countably compact extension Y which is 'minimal' in the Freudenthal compactification $F(X)$, i.e., Y is countably compact, $X \subseteq Y \subseteq F(X)$ and if $X \subseteq Z \subseteq F(X)$ and Z is countably compact then $Y \subseteq Z$. A space Z is a **pseudocompactification** of X if Z is pseudocompact and X is homeomorphic to a dense subset of Z. Bell [3] proved that the ***Pixley–Roy hyperspace*** of the Cantor set does not have a first-countable pseudocompactification and the Pixley–Roy hyperspace of a ***λ-set*** has a first-countable pseudocompactification which was used to produce a first-countable pseudocompact ccc non-separable space. Tree [21] proved that every zero-dimensional first-countable locally compact space X has a zero-dimensional first-countable locally compact pseudocompactification.

5. Supercompact spaces

A topological space X is said to be a **supercompact space** (de Groot (1967), see [24]) if there is a binary open ***subbase*** $\mathcal{B}$ for X, i.e., any open cover of X consisting of elements from $\mathcal{B}$ has a two-element subcover. By the ***Alexander Subbase Lemma***, supercompact implies compact. Strok and Szymanski (1975) proved that each compact metric space is supercompact (see [24]). Bell (1978) proved that if βX is supercompact then X is pseudocompact, hence, e.g., $\beta\mathbb{N}$ is not supercompact (see [24, 1.1.7]). Van Mill, see [24], proved that no compactification of $\mathbb{N} \cup \{p\}$ is supercompact when $p \in \mathbb{N}^*$ is a P-point.

A dual definition of supercompactness requires an existence of a subbase $\mathcal{D}$ for the closed sets such that every **linked system** $\mathcal{L}$ in $\mathcal{D}$ has a nonempty intersection (linked means that any two elements intersect). This gives a recipe for constructing supercompact extensions of topological spaces. For a ***normal*** topological space, put $\lambda X = \{\mathcal{L}\colon \mathcal{L}$ is a maximal linked system of closed subsets of $X\}$ and a base for λX is formed by all sets $U^+ = \{\mathcal{L}\colon (\exists L \in \mathcal{L})(L \subseteq U)\}$ where U is an open subset of X, for more information (see [25]). Notice that X does not have to be dense in λX; the closure of X in λX is called the **de Groot–Aarts compactification** of X. λX is an interesting geometric object. Thus van Mill proved that λX is homeomorphic to the ***Hilbert cube*** I^ω whenever X is a non-degenerate metrizable ***continuum*** [25]. For an infinite cardinal τ, Ivanov [12] proved that $\lambda(I^\tau) = I^\tau$. It is shown in the same paper that the superextension of a compact ***Absolute Retract*** is an Absolute Retract as well. Van de Vel proved [22] proved that λX has a ***fixed-point property*** for continuous maps if X is normal and connected.

6. A look outside completely regular spaces

Arkhangel'skiĭ [2] showed that each completely regular space X can be embedded in a Hausdorff countably compact space $\widetilde{X}$ such that $t(X) = t(\widetilde{X})$ where t denotes ***tightness***. Examples from [23] show very clearly that one cannot hope for a compact version of this fact even for countable tightness. Other, not first-countable, examples are mentioned in [2]: the countable Fréchet–Urysohn fan V_ω or $\mathbb{N} \cup \{p\}$, a subspace of $\beta\mathbb{N}$ with $p \in \mathbb{N}^*$, both these countable spaces have no compact extension of countable tightness. Nogura (1983) showed that under $\mathfrak{b} = \omega_1$, there is a completely regular first-countable space that cannot be embedded in a ***regular*** countably compact space with countable tightness. Matveev [14] showed that V_ω has even no completely regular pseudocompactification with countable tightness, in fact he proved that V_ω has no regular **feebly compact** extension with countable tightness; X is feebly compact if each locally finite open family is finite, for completely regular spaces it means precisely pseudocompactness, not otherwise (see [18]). On the other hand, Matveev showed that each Hausdorff space which is first-countable (has countable tightness, respectively) has a Hausdorff feebly compact extension which is first-countable (has countable tightness, respectively). Dikranjan and Pelant [8] obtained related results: σ denotes the ***sequential*** closure and σ^{∞} denotes the idempotent hull of σ. They gave an example of a completely regular zero-dimensional pseudocompact space X such that every countably compact space Y, containing X as σ^{∞}-dense subspace, is not Hausdorff. Furthermore, every Hausdorff space X admits a feebly compact Hausdorff extension Y such that X is σ-dense in Y and there exists a zero-dimensional (hence completely regular) space X having no feebly compact regular extension Y such that X is σ^{∞}-dense in Y.

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d-16 Remainders

In full generality a **remainder** of a space X is a space of the form $Y \setminus X$, where Y is an **extension** of X, i.e., a space containing (a homeomorphic copy of) X. Normally the extension comes from a class of nice spaces, e.g., compact Hausdorff or completely metrizable and we assume that X is dense in the extension.

1. Remainders in compactifications

In this section we consider remainders in ***compactifications***, so we assume X is ***completely regular*** and we consider compact Hausdorff extensions. In this case the remainders of X are the spaces $\gamma X \setminus X$ where γX runs through the family of compactifications of X. Every remainder is the image of $X^* = \beta X \setminus X$ (the **Čech–Stone remainder**) under a ***perfect map*** – to wit the restriction of the natural map from βX onto γX. Unfortunately the converse is not true in general. It is true if X is ***locally compact***, this is generally known as **Magill's theorem** (see [E, 3.5.13]) but a general characterization is still lacking.

Fundamental questions that have driven the research in this area through the years are: what are the properties of X^*, what are the remainders of specific X, what spaces have 'nice' remainders?

The first question is too ambitious as "every space is a remainder"; indeed, given Y put

$$X = \big((\omega_1 + 1) \times \beta Y\big) \setminus \big(\{\omega_1\} \times Y\big),$$

then $Y = X^*$. Therefore one should modify it with "as a function of X". Observe that the space X just constructed is ***pseudocompact***; this is why one looks for general results on X^* in classes that do not contain pseudocompact spaces. A first such result is that if X is not pseudocompact then X^* contains a copy of $\mathbb{N}^*$, which tells us something about the size of X^* – at least $2^{\mathfrak{c}}$ – and its structure – at least as complicated as $\mathbb{N}^*$.

This also implies that, for non-pseudocompact X at least, recognizing the remainders of X is at least as hard as recognizing those of $\mathbb{N}$. In the article on $\beta\mathbb{N}$ and $\beta\mathbb{R}$ in this volume Parovičenko's theorem is quoted, which says that under CH the continuous images of $\mathbb{N}^*$, and hence the remainders of $\mathbb{N}$, are precisely the compact spaces of weight $\mathfrak{c} = \aleph_1$. In ZFC alone one has to work harder. Thus far the following have been shown to be remainders of $\mathbb{N}$: all compact spaces of weight $\aleph_1$, all separable compact spaces, all perfectly normal compact spaces and certain products of spaces from these classes. If $f : X \to Y$ is a perfect surjection then $\beta f^{-1}[Y^*] = X^*$ (and conversely); in this case every remainder of Y is also a remainder of X. Therefore, if X is the sum of countably many compact spaces then every remainder of $\mathbb{N}$ is a remainder of X. This fact may also be used to see that, for example, $[0, \infty)$ and $\mathbb{R}^n$ (where $n > 2$) share the same remainders: the map $x \mapsto \|x\|$ is perfect from $\mathbb{R}^n$ to $[0, \infty)$ and there is also a perfect 'space-filling curve' from $[0, \infty)$ onto $\mathbb{R}^n$. The remainder $\mathbb{R}^*$ is not connected so the remainders of $\mathbb{R}$ form a larger family than those of $[0, \infty)$.

For locally compact spaces ***Wallman–Shanin compactifications*** offer a way to recognizing shared remainders: if $\mathcal{B}$ is a ***Wallman base*** for the locally compact space X and $w(X, \mathcal{B})$ is the associated compactification then the remainder $w(X, \mathcal{B}) \setminus X$ is the ***Wallman representation*** of the quotient lattice of $\mathcal{B}$ by the following equivalence relation: "every element of $\mathcal{B}$ contained in $A \triangle B$ is compact".

There are various other results on shared remainders but no general pattern has emerged. By way of example we mention that $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{S}$ share no remainders: all remainders of $\mathbb{Q}$ are ***topologically complete***, all remainders of $\mathbb{P}$ are of ***first category*** and all remainders of $\mathbb{S}$ are ***Baire space***s but not complete.

Discrete spaces offer an intriguing problem. It is known that if there are distinct cardinals κ and λ with κ^* and λ^* homeomorphic then also ω^* and ω_1^* are homeomorphic. It is still an open question whether "ω^* and ω_1^* are homeomorphic" is consistent with ZFC.

A very useful general result on remainders is that whenever X is ***σ-compact*** and ***locally compact*** (but not compact) then the remainder X^* is a compact ***F-space*** in which non-empty ***G_δ-sets*** have non-empty interior. This immediately implies that such X^* are not ***homogeneous***; in fact if X is not pseudocompact then X^* is never homogeneous.

2. Nice remainders

Another line of investigation is to start with a class of 'nice' spaces and to see which spaces have compactifications with a remainder from that class.

Locally compact spaces, and only those, are the spaces that have finite remainders. Demanding that *all* remainders of a space be finite brings us back to pseudocompactness again. The statement $|X^*| = n$ translates into: of every $n + 1$ mutually ***completely separated*** closed sets at least one must be compact. If $|X^*| = 1$ then X is called **almost compact**.

One of the first results on nice remainders is Zippin's theorem that a separable metric space has a compactification with a countable remainder (in short: it is a **CCR space**) iff it is both ***Čech-complete*** and ***rim-compact***. Freudenthal showed that a separable metric space has a compactification with a ***zero-dimensional*** remainder (in short: it is a

0-space) iff it is rim-compact. Both proofs provide implications in the class of completely regular spaces: all CCR spaces are Čech-complete and rim-compact, and all rim-compact spaces are 0-spaces (although rim-compact spaces needs not have ***strongly zero-dimensional*** remainders).

These implications are, in general, not reversible: there are 0-spaces that are not rim-compact and the product of the space of irrationals with an uncountable discrete space is completely metrizable and zero-dimensional but not a CCR space. There have been attempts to characterize CCR spaces and 0-spaces in terms of the set $R(X)$ of points without compact neighbourhoods. Adding separability of $R(X)$ to the necessary conditions of Čech-completeness and rim-compactness yields a characterization of CCR spaces among metrizable spaces. For general spaces adding the assumption that $R(X)$ is the continuous image of a separable metric space (a ***cosmic space***) ensures that X is a CCR space. However, intrinsic properties alone of $R(X)$ will not provide the definitive answer as there are two Čech-complete, even zero-dimensional, spaces X and Y, with both $R(X)$ and $R(Y)$ discrete and uncountable, yet X is a CCR space but Y is not.

For 0-spaces some conditions on $R(X)$ suffice to ensure rim-compactness: local compactness plus zero-dimensionality or ***scatteredness***. On the other hand there are spaces X and Y with $R(X)$ and $R(Y)$ homeomorphic to the space of irrationals of which X has a ***totally disconnected*** remainder but is not a 0-space and Y is a 0-space that is not rim-compact. Thus, again, a complete solution in terms of $R(X)$ seems unlikely.

Freudenthal's proof provides, for rim-compact spaces, a canonical construction of a compactification with zero-dimensional remainder, the ***Freudenthal compactification***. The Freudenthal compactification is the unique **perfect compactification** with zero-dimensional remainder and it is also the minimum perfect compactification. A compactification αX is perfect if $\operatorname{cl}\operatorname{Fr} O = \operatorname{Fr}\operatorname{Ex} O$ for all open subsets of X. This is equivalent to saying that the natural map from βX onto αX is ***monotone***. In general X has a minimum perfect compactification μX iff it has some compactification with a ***punctiform*** remainder and in that case μX is also the maximum compactification with punctiform remainder.

A particularly interesting family is formed by what are commonly referred to as **Ψ-spaces**. These are built by taking a set X and an ***almost disjoint family*** $\mathcal{A}$ consisting of countably infinite sets. The space $\Psi(X, \mathcal{A})$ has $X \cup \mathcal{A}$ as its underlying set, the points of X are isolated and for $A \in \mathcal{A}$ a typical basic neighbourhood is of the form $\{A\} \cup A \setminus F$ for some finite set F. Thus, the set A becomes a converging sequence with limit A. We concentrate on the case where $X = \omega$ and $\mathcal{A}$ is a maximal almost disjoint (MAD) family; we abbreviate $\Psi(\omega, \mathcal{A})$ by $\Psi(\mathcal{A})$. The space $\Psi(\mathcal{A})$ is pseudocompact and locally compact. Every compact metric space is the Čech–Stone remainder of some $\Psi(\mathcal{A})$. This implies that there are $\Psi(\mathcal{A})$ with infinite ***covering dimension*** as well as almost compact $\Psi(\mathcal{A})$. The Continuum Hypothesis implies that the families of all remainders of $\mathbb{N}$ and of all $\Psi(\mathcal{A})^*$ coincide. It is consistent, however, that there is an $\mathcal{A}$ for which some continuous image of $\Psi(\mathcal{A})^*$ is not of the form $\Psi(\mathcal{B})^*$ for any $\mathcal{B}$.

3. Dimension

An interesting line of research was opened by de Groot when he asked for an internal characterization of the minimum dimension of remainders in compactifications. This number is called the **compactness defect** or **compactness deficiency** of the space: $\operatorname{def} X = \min\{\dim Y \setminus X \colon Y$ is a compactification of $X\}$. The problem is usually studied in the class of separable metric spaces, so it does not matter what dimension function is used.

The **compactness degree** is defined just like the ***small inductive dimension***: $\operatorname{cmp} X \leqslant n + 1$ if for every point x and every open $O \ni x$ there is an open U with $x \in U \subseteq \operatorname{cl} U \subseteq O$ and $\operatorname{cmp}\operatorname{Fr} U \leqslant n$, the starting point however is not the empty set but the class of compact spaces: $\operatorname{cmp} X = -1$ means that X is compact. De Groot proved $\operatorname{cmp} X = \operatorname{def} X$ for values up to 0 and $\operatorname{cmp} X \leqslant \operatorname{def} X$ for all (separable metric) spaces. In 1982 R. Pol disproved de Groot's conjecture $\operatorname{cmp} = \operatorname{def}$ with an example of a space X with $\operatorname{cmp} X = 1$ and $\operatorname{def} X = 2$. Since then more examples appeared to show that the gap may be arbitarily large and that $\operatorname{Cmp} X$ (defined like the ***large inductive dimension***) does not characterize $\operatorname{def} X$ either. In 1988 T. Kimura showed that an invariant due to Skljarenko does characterize $\operatorname{def} X$. One has $\operatorname{def} X \leqslant n$ iff X has a base $\mathcal{B}$ such that whenever $\mathcal{B}'$ is an $n + 1$-element subfamily of $\mathcal{B}$ the intersection $\bigcap_{B \in \mathcal{B}'} \operatorname{Fr} B$ is compact.

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d-17 The Čech–Stone Compactification

1. Constructions

In [11] Tychonoff not only proved the ***Tychonoff Product Theorem*** (for powers of the unit interval), he also showed that every ***completely regular*** space of ***weight*** κ can be embedded into the space ${}^{\kappa}[0,1]$. This inspired Čech, in [1], to construct for every completely regular space S a ***compact Hausdorff*** space $\beta(S)$ that contains S as a ***dense subspace*** – a **compactification** of S – and such that every bounded real-valued continuous function on S can be extended to a continuous function on $\beta(S)$. The construction amounts to taking the family $\mathcal{C}$ of all continuous functions from S to $[0,1]$, the corresponding ***diagonal map*** $e = \triangle_{f\in\mathcal{C}} f : S \to [0,1]^{\mathcal{C}}$ and obtaining $\beta(S)$ as the closure of $e[S]$ in $[0,1]^{\mathcal{C}}$. Čech also proved that for any other compactification B of S there is a continuous map $h : \beta(S) \to B$ with $h(x) = x$ for $x \in S$ and $h[\beta(S) \setminus S] = B \setminus S$. From this he deduced that if B is a compactification of S in which ***functionally separated*** subsets in S have disjoint closures in B then the map h is a homeomorphism, whence $B = \beta(S)$.

Somewhat earlier, in [10], M.H. Stone applied his theory of representations of ***Boolean algebra***s to various topological problems. One of the major applications is the construction, using the ring $C^*(X)$ of bounded real-valued continuous functions, of a compactification $\mathcal{X}$ of X with the same extension property as the compactification that Čech would construct. The construction proceeds by taking the Boolean algebra $\mathbb{B}$ generated by all ***cozero sets*** and all ***nowhere dense*** subsets of X. As a first step Stone took the representing space $\mathfrak{S}(\mathbb{B})$ – the ***Stone space*** – of $\mathbb{B}$. Next, to every maximal ideal $\mathfrak{m}$ of $C^*(X)$ he associated the ideal $I_{\mathfrak{m}}$ of $\mathbb{B}$ consisting of those sets B in $\mathbb{B}$ for which there are $f \in C^*(X)$ and $a, b, c \in \mathbb{R}$ such that $A \subseteq f^{\leftarrow}[(a,b)]$; $f \equiv c$ (mod $\mathfrak{m}$) and $c < a$ or $b < c$. Finally $F_{\mathfrak{m}}$ is the closed subset of $\mathfrak{S}(\mathbb{B})$ determined by the filter that is dual to $I_{\mathfrak{m}}$. The space $\mathcal{X}$ is the quotient space of $\mathfrak{S}(\mathbb{B})$ by the decomposition consisting of the sets $F_{\mathfrak{m}}$. One obtains an embedding of X into $\mathcal{X}$ by associating x with the maximal ideal $\mathfrak{m}_x = \{f\colon f(x) = 0\}$. Stone also proved that every continuous map on X with a compact co-domain can be extended to $\mathcal{X}$.

The compactification constructed by Čech and Stone is nowadays called the **Čech–Stone compactification**;[1] Čech's β is still used, we write βX (without Čech's parentheses). The properties of βX that Čech and Stone established each characterize it among all compactifications of X: (1) it is the maximal compactification; (2) functionally separated subsets of X have disjoint closures in βX; and (3) every continuous map from X to a compact space extends to all of βX (the extension of $f : X \to K$ is denoted βf). Of the many other constructions of βX that have been devised we mention two. First, in [6], Gel'fand and Kolmogoroff showed that the ***hull-kernel topology*** on the set of maximal ideals of $C^*(X)$ immediately gives us βX and that one may also use the ring $C(X)$ of all real-valued continuous functions on X. Second, in [7], Gillman and Jerison gave what for many is the definitive construction of βX: the ***Wallman compactification*** of X with respect to the family $\mathcal{Z}(X)$ of all ***zero set***s of X. This means βX is the set of all ultrafilters on the family $\mathcal{Z}(X)$ – the z**-ultrafilter**s or **zero-set ultrafilters** – with the family $\{\overline{Z}\colon Z \in \mathcal{Z}(X)\}$ as a base for the closed sets, where $\overline{Z} = \{u \in \beta X\colon Z \in u\}$. One identifies a point x of X with the z-ultrafilter $u_x = \{Z\colon x \in Z\}$.

Property 2 above is usually reformulated as: (2′) disjoint zero sets of X have disjoint closures in βX. For ***normal spaces*** one can obtain βX as *the* Wallman compactification of X, i.e., using the family of all closed sets; property (2′) then becomes (2″) disjoint closed sets in X have disjoint closures in βX. The equality $\beta X = K$ should be taken to mean that K is compact and there is an embedding $f : X \to K$, for which $f[X]$ is dense in K and for which the extension βf is a homeomorphism – especially if X is dense in K. In this sense the notation βf is unambiguous: the graph of βf is the Čech–Stone compactification of the graph of f.

The assignment $X \mapsto \beta X$ is a ***covariant functor*** from the ***category*** of Tychonoff spaces to the category of compact Hausdorff spaces, both with continuous maps as **morphisms**. It is in fact the adjoint of the ***forgetful functor*** from compact Hausdorff spaces to Tychonoff spaces. This gives another way of proving that "βX exists": because the category of compact Hausdorff spaces is closed under products and closed subsets. In fact Čech's construction of βX may be construed as a forerunner of the Adjoint Functor Theorem.

2. Properties

We say that a subspace A is C**-embedded** (C^***-embedded**) in a space X if every (bounded) real-valued continuous function on A can be extended to a continuous real-valued function on X. Thus a completely regular space X is C^*-embedded in its Čech–Stone compactification βX and any compactification of X in which X is C^*-embedded must be βX. These remarks help us to recognize some Čech–Stone compactifications: if $A \subseteq X$ then $\mathrm{cl}_{\beta X} A = \beta A$ iff A is C^*-embedded in X and $\beta Y = \beta X$ whenever $X \subseteq Y \subseteq \beta X$. If X is normal then $\mathrm{cl}_{\beta X} A = \beta A$ whenever A is closed in X (by the ***Tietze–Urysohn theorem***).

[1] In Europe; elsewhere one speaks of the **Stone–Čech compactification**.

One can use C^*-embedding to calculate βX explicitly for a few X, by which we mean that βX is an already familiar space. The well-known fact that every continuous real-valued function on the ***ordinal space*** ω_1 is constant on a tail implies that $\beta\omega_1 = \omega_1 + 1$ – the same holds for every ordinal of uncountable cofinality. Other examples are provided by ***Σ-product***s: if κ is uncountable and $X = \{x: \{\alpha: x_\alpha \neq 0\}$ is countable$\}$ as a subspace of ${}^\kappa[0,1]$ (or ${}^\kappa 2$) then every continuous real-valued function on X depends on countably many coordinates and hence can be extended to the ambient product so that $\beta X = {}^\kappa[0,1]$ (or $\beta X = {}^\kappa 2$). Note that these spaces, with an easily identifiable Čech–Stone compactification, are ***pseudocompact***. In fact if X is not pseudocompact then it contains a C-embedded copy of $\mathbb{N}$, whence βX contains a copy of $\beta\mathbb{N}$. The space $\beta\mathbb{N}$ is, in essence, hard to describe: if u is a point in $\beta\mathbb{N} \setminus \mathbb{N}$ and so a free ***ultrafilter*** on $\mathbb{N}$ then the set $\{\sum_{n\in U} 2^{-n}: U \in u\}$ is a non-Lebesgue measurable set of reals. This illustrates that the construction of βX requires a certain amount of Choice, indeed, the existence of βX is equivalent to the Tychonoff Product Theorem for compact Hausdorff spaces, which, in turn, is equivalent to the Boolean Prime Ideal Theorem.

The map $f \mapsto \beta f$ from $C^*(X)$ to $C(\beta X)$ is an isomorphism of rings (or lattices, or Banach spaces ...); this explains why very often investigations into $C^*(X)$ assume that X is compact; this gives the advantage that ideals are fixed, i.e., if I is an ideal of $C^*(X)$ then there is a point x with $f(x) = 0$ for all $f \in I$. Furthermore the maximal ideals are precisely the ideals of the form $\{f: f(x) = 0\}$ for some x.

A **perfect map** is one which is continuous, closed and with compact fibers. A map $f: X \to Y$ between completely regular spaces is perfect iff its Čech-extension βf satisfies $\beta f[\beta X \setminus X] \subseteq \beta Y \setminus Y$ or equivalently $X = \beta f^{-1}[Y]$. One can use this, for instance, to show that complete metrizability is preserved by perfect maps (also inversely if the domain is metrizable). A metrizable space is ***completely metrizable*** iff it is a ***G_δ-set*** in its Čech–Stone compactification; the latter property is then easily seen to be preserved both ways by perfect maps. One calls a space a **Čech-complete space** (sometimes **topologically complete**) if it is a G_δ-set in its Čech–Stone compactification. This is an example of βX providing a natural setting for defining or characterizing topological properties of X – the best-known example being of course local compactness: a space is ***locally compact*** iff it is open in its Čech–Stone compactification. Other properties that can be characterized via βX are: the ***Lindelöf*** property (X is **normally placed** in βX, which means that for every open set $U \supseteq X$ there is an F_σ-set F with $X \subseteq F \subseteq U$) and ***paracompactness*** ($X \times \beta X$ is normal).

The product $\beta\mathbb{N} \times \beta\mathbb{N}$ is *not* $\beta(\mathbb{N} \times \mathbb{N})$: the characteristic function of the diagonal of $\mathbb{N}$ witnesses that $\mathbb{N} \times \mathbb{N}$ is not C^*-embedded in $\beta\mathbb{N} \times \beta\mathbb{N}$. The definitive answer to the question when $\beta\prod = \prod\beta$ was given in [8]: if both X and Y are infinite then $\beta X \times \beta Y = \beta(X \times Y)$ iff $X \times Y$ is pseudocompact and the same holds for arbitrary products, with a similar proviso: $\prod_i \beta X_i = \beta \prod_i X_i$ iff $\prod_i X_i$ is pseudocompact, provided $\prod_{i \neq i_0} X_i$ is never finite. If the product can be factored into two subproducts without isolated points then even the homeomorphy of $\prod_i \beta X_i$ and $\beta \prod_i X_i$ implies $\prod_i X_i$ is pseudocompact.

If X is not compact then no point of $X^* = \beta X \setminus X$ is a G_δ-set, in fact a G_δ-set of βX that is a subset of X^* contains a copy of $\mathbb{N}^*$. This implies that nice properties like metrizability, and second- or first-countability do not carry over to βX. Of course separability carries over from X to βX, but not conversely: a Σ-product in ${}^{\mathfrak{c}}2$ is not separable but ${}^{\mathfrak{c}}2$ is.

Properties that are carried over both ways are usually of a global nature. Examples are ***connectedness***, ***extremal disconnectedness***, ***basic disconnectedness*** and the values of the ***large inductive dimension*** (for normal spaces) and ***covering dimension***. These properties have in common that they can be formulated using the families of (co)zero sets and/or the ring $C^*(X)$, which makes it almost automatic for each that X satisfies it iff βX does. Interestingly βX is ***locally connected*** iff X is locally connected and pseudocompact; so, e.g., $\beta\mathbb{R}$ is connected but not locally connected.

A particularly interesting class of spaces in this context is that of the ***F*-space**s; it can be defined topologically (cozero sets are C^*-embedded) or algebraically (every finitely generated ideal in $C^*(X)$ is principal). The algebraic formulation shows that X is an F-space iff βX is, because $C(\beta X)$ and $C^*(X)$ are isomorphic. Also, X^* is an F-space whenever X is locally compact and σ-compact. Neither property by itself guarantees that X^* is F-space: $\mathbb{Q}^*$ is not an F-space, nor is $(\omega_1 \times [0,1))^*$ (which happens to be $[0,1]$).

This result shows that F-spaces are quite ubiquitous; for example, $\mathbb{N}^*$, $\mathbb{R}^*$, and $(\mathbb{R}^n)^*$ are F-spaces, as well as $(\bigoplus_n X_n)^*$ for any topological sum of countably many compact spaces. An F-space imposes some rigidity on maps having it for its range: if $f: X^\kappa \to Z$ is a continuous map from a power of the compact space X to an F-space Z then X^κ can be covered by finitely many clopen sets such that f depends on one coordinate on each of them. This implies, e.g., that a continuous map from a power of $[0,\infty)^*$ to $[0,\infty)^*$ itself depends on one coordinate only.

Some of the properties mentioned above have relationships beyond the implications between them. Every ***P-space*** is basically disconnected. But an extremally disconnected P-space that is not of a ***measurable cardinal*** number is discrete (the converse is clearly also true). As noted above X is extremally (or basically) disconnected iff βX is but there is more: if X is extremally disconnected or a P-space then βX can be embedded into βD for a large discrete space D. Every compact subset of βD is an F-space but a characterization of the compact subspaces of βD is not known. In the special case of $\beta\mathbb{N}$ there is a characterization under the assumption of the ***Continuum Hypothesis***, but it is also consistent that not all basically disconnected spaces embed into $\beta\mathbb{N}$ and that not every F-space embeds into a basically disconnected space.

As seen above, pseudocompactness is a property that helps give positive structural results about βX; this happens again in the context of topological groups. If X is a topological group then the operations can be extended to βX, making βX into a topological group, if and only if X is pseudocompact.

3. Special points

It is a general theorem that $X^* = \beta X \setminus X$ is not ***homogeneous*** whenever X is not pseudocompact [5]. This in itself very satisfactory result prompted further investigation into the structure of remainders and a search for more reasons for this nonhomogeneity. Many special points were defined that would exhibit different topological behaviour in X^* or βX. The best known are the remote points: a point p of X^* is a **remote point** of X if $p \notin \mathrm{cl}_{\beta X} N$ for all nowhere dense subsets N of X. If p is a remote point of X then βX is extremally disconnected at p.

Many spaces have remote points, e.g., spaces of countable π***-weight*** (or even with a σ-locally finite π***-base***) and $\omega \times {}^{\kappa}2$. If X is **nowhere locally compact**, i.e., when X^* is dense in βX, then X^* is ***extremally disconnected*** at every remote point of X. This gives another reason for the nonhomogeneity of, for example, $\mathbb{Q}^*$, as this space is not extremally disconnected.

Under CH all separable spaces have remote points but in the side-by-side Sacks model there is a separable space without remote points. Many spaces with the ***countable chain condition*** have remote points and it is unknown whether there is such a space without remote points. Proofs that certain spaces have remote points have generated interesting combinatorics; the proof for $\omega \times {}^{\kappa}2$ contains a crucial ingredient for one proof of the consistency of the ***Normal Moore Space Conjecture***.

Further types of points are obtained by varying on the theme of 'not in the closure of a small set'. Thus one obtains **far point**s: not in the closure of any closed discrete subset of X; requiring this only for countable discrete sets defines ω**-far points** – a **near point** is a point that is not far.

Of earlier vintage are P**-point**s, points for which the family of neighbourhoods is closed under countable intersections. They occured in the algebraic context: a point is a P-point iff every continuous function is constant on a neighbourhood of it. This means that for a P-point x the ideals $\{f\colon f(x) = 0\}$ and $\{f\colon x \in \mathrm{int}\, f^{\leftarrow}(f(x))\}$ coincide. Under CH or even MA one can prove that many spaces of the form X^* have P-points, thus obtaining witnesses to the nonhomogeneity of X^*. Many of these results turned out to be independent of ZFC. The search for a general theorem that would, once and for all, establish nonhomogeneity of X^* by means of a 'simple' topological property of some-but-not-all points lead to weak P-points. A **weak** P**-point** is one that is not an accumulation point of any countable set. Their advantage over P-points is that their existence is provable in ZFC, first for $\mathbb{N}^*$, later for more spaces. The final word has not been said, however. The weakest property that has not been ruled out by counterexamples is 'not an accumulation point of a countable discrete set'.

Further reading
Walker's book [12] gives a good survey of work on βX up to the mid 1970s. Van Douwen's [4, 2, 3] and van Mill's [9] laid the foundations for the work on βX in more recent years.

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d-18 The Čech–Stone Compactifications of ℕ and ℝ

It is safe to say that among the ***Čech–Stone compactifications*** of individual spaces, that of the space ℕ of natural numbers is the most widely studied. A good candidate for second place is $\beta\mathbb{R}$. This article highlights some of the most striking properties of both compactifications.

1. Description of $\beta\mathbb{N}$ and $\mathbb{N}^*$

The space $\beta\mathbb{N}$ (aka $\beta\omega$) appeared anonymously in [10] as an example of a ***compact Hausdorff*** space without non-trivial converging sequences. The construction went as follows: for every $x \in (0,1)$ let $0.a_1(x)a_2(x)\ldots a_n(x)\ldots$ be its dyadic expansion (favouring the one that ends in zeros). This gives us a countable set $A = \{a_n(x)\colon n \in \mathbb{N}\}$ of points in the ***Tychonoff cube*** $[0,1]^{(0,1)}$. The closure of A is the required space. To see that this closure is indeed $\beta\mathbb{N}$ one checks that the map $n \mapsto a_n$ induces a homeomorphism of $\beta\mathbb{N}$ onto $\operatorname{cl} A$. Indeed, it suffices to observe that whenever X is a coinfinite subset of ℕ one has $a_n(x) = 1$ iff $n \in X$, where $x = \sum_{n\in X} 2^{-n}$.

The present-day description of $\beta\mathbb{N}$ is as the ***Stone space*** of the ***Boolean algebra*** $\mathcal{P}(\mathbb{N})$. Thus the underlying set of $\beta\mathbb{N}$ is the set of all ***ultrafilters*** on ℕ with the family $\{\overline{X}\colon X \subseteq \mathbb{N}\}$ as a base for the open sets, where $\overline{X}$ denotes the set of all ultrafilters of which X is an element. The space $\beta\mathbb{N}$ is a ***separable*** and ***extremally disconnected*** compact Hausdorff space and its cardinality is the maximum possible, i.e., $2^{\mathfrak{c}}$.

Most of the research on $\beta\mathbb{N}$ concentrates on its ***remainder*** $\beta\mathbb{N} \setminus \mathbb{N}$, which, as usual, is denoted $\mathbb{N}^*$. By extension one writes $X^* = \overline{X} \setminus \mathbb{N}$ for subsets X of ℕ. The family $\mathcal{B} = \{X^*\colon X \subseteq \mathbb{N}\}$ is precisely the family of ***clopen*** sets of $\mathbb{N}^*$. Because $X^* \subseteq Y^*$ iff $X \setminus Y$ is finite the algebra $\mathcal{B}$ is isomorphic to the quotient algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ of $\mathcal{P}(\mathbb{N})$ by the ideal of finite sets – hence $\mathbb{N}^*$ is the Stone space of $\mathcal{P}(\mathbb{N})/\mathit{fin}$. Much topological information about $\mathbb{N}^*$ comes from knowledge of the combinatorial properties of this algebra. In practice one works in $\mathcal{P}(\mathbb{N})$ with all relations taken modulo finite. We use, e.g., $X \subseteq^* Y$ to denote that $X \setminus Y$ is finite, $X \subset^* Y$ to denote that $X \subseteq^* Y$ but not $Y \subseteq^* X$, and so on. In this context the word 'almost' is mostly used in place of 'modulo finite', thus 'A and B are almost disjoint' means $A \cap B =^* \emptyset$.

2. Basic properties of $\mathbb{N}^*$

Many results about $\mathbb{N}^*$ are found by constructing special families of subsets of ℕ, although the actual work is often done on a suitable countable set different from ℕ.

The proof that $\beta\mathbb{N}$ has cardinality $2^{\mathfrak{c}}$ employs an **independent family**, which we define on the countable set $\{\langle n,s\rangle\colon n \in \mathbb{N}, s \subseteq \mathcal{P}(n)\}$. For every subset x of ℕ put $I_x = \{\langle n,s\rangle\colon x \cap n \in s\}$. Now if F and G are finite disjoint subsets of $\mathcal{P}(\mathbb{N})$ then $\bigcap_{x\in F} I_x \setminus \bigcup_{y\in G} I_y$ is non-empty – this is what independent means. Sending $u \in \beta\mathbb{N}$ to the point $p_u \in {}^{\mathcal{P}(\mathbb{N})}2$, defined by $p_u(x) = 1$ iff $I_x \in u$, gives us a continuous map from $\beta\mathbb{N}$ onto ${}^{\mathfrak{c}}2$. Thus the existence of an independent family of size $\mathfrak{c}$ easily implies the well-known fact that the ***Cantor cube*** ${}^{\mathfrak{c}}2$ is separable; conversely, if D is dense in ${}^{\mathfrak{c}}2$ then setting $I_\alpha = \{d \in D\colon d_\alpha = 0\}$ defines an independent family on D. Any closed subset F of $\mathbb{N}^*$ such that the restriction to F of the map onto ${}^{\mathfrak{c}}2$ is irreducible is (homeomorphic with) the ***absolute*** of ${}^{\mathfrak{c}}2$. If a point u of $\mathbb{N}^*$ belongs to such an F then *every* compactification of the space $\mathbb{N} \cup \{u\}$ contains a copy of $\beta\mathbb{N}$.

Next we prove that $\mathbb{N}^*$ is very non-separable by working on the tree $2^{<\mathbb{N}}$ of finite sequences of zeros and ones. For every $x \in {}^{\mathbb{N}}2$ let $A_x = \{x\restriction n\colon n \in \mathbb{N}\}$. Then $\{A_x\colon x \in {}^{\mathbb{N}}2\}$ is an **almost disjoint family** of cardinality $\mathfrak{c}$ and so $\{A_x^*\colon x \in {}^{\mathbb{N}}2\}$ is a disjoint family of clopen sets in $\mathbb{N}^*$.

We can use this family also to show that $\mathbb{N}^*$ is not extremally disconnected. Let Q denote the points in ${}^{\mathbb{N}}2$ that are constant on a tail (these correspond to the endpoints of the ***Cantor set*** in $[0,1]$) and let $P = {}^{\mathbb{N}}2 \setminus Q$. Then $O_Q = \bigcup_{x\in Q} A_x^*$ and $O_P = \bigcup_{x\in P} A_x^*$ are disjoint open sets in $\mathbb{N}^*$, yet $\operatorname{cl} O_Q \cap \operatorname{cl} O_P \neq \emptyset$, for if A is such that $A_x \subseteq^* A$ for all $x \in P$ then the ***Baire Category Theorem*** may be applied to find $x \in Q$ with $A_x \subseteq^* A$. This example shows that $\mathbb{N}^*$ is not **basically disconnected**: O_Q is an open F_σ-set whose closure is not open.

The algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ has two countable (in)completeness properties. The first states that when $\langle b_n\rangle_n$ is a decreasing sequence of non-zero elements there is an x with $b_n > x > 0$ for all n; in topological terms: nonempty ***G_δ-sets*** on $\mathbb{N}^*$ have nonempty interiors and $\mathbb{N}^*$ has no isolated points. The second property says that when $\langle b_n\rangle_n$ is as above and, in addition, $\langle a_n\rangle_n$ is an increasing sequence with $a_n < b_n$ for all n there is an x with $a_n < x < b_n$ for all n; in topological terms: disjoint open F_σ-sets in $\mathbb{N}^*$ have disjoint closures, i.e., $\mathbb{N}^*$ is an **F-space**.

We now turn to sets A and B where no interpolating x can be found, that is, we look for A and B such that $\bigvee A' < \bigwedge B'$ whenever $A' \in [A]^{<\omega}$ and $B' \in [B]^{<\omega}$ but for which there is no x with $a \leqslant x \leqslant b$ for all $a \in A$ and $b \in B$. The minimum cardinalities of sets like these are called **cardinal characteristics of the continuum** and these cardinal numbers play an important role in the study of $\beta\mathbb{N}$ and $\mathbb{N}^*$.

We have already seen such a situation with a countable A and a B of cardinality $\mathfrak{c}$: the families $\{A_x^*\}_{x \in Q}$ and $\{\mathbb{N}^* \setminus A_x^*\}_{x \in P}$. However, this case, of a countably infinite A, is best visualized on the countable set $\mathbb{N} \times \mathbb{N}$. For $n \in \mathbb{N}$ put $a_n = (n \times \mathbb{N})^*$ and for $f \in {}^{\mathbb{N}}\mathbb{N}$ put $b_f = \{\langle m, n\rangle : n \geqslant f(m)\}^*$; then $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_f : f \in {}^{\mathbb{N}}\mathbb{N}\}$ are as required: if $a_n < x$ for all n then one readily finds an f such that $b_f < x$ (make sure $\{n\} \times [f(n), \infty) \subset x$). A pair like (A, B) above is called an (unfillable) **gap**; gap, because, as in a Dedekind gap, one has $a < b$ whenever $a \in A$ and $b \in B$, and unfillable because there is no x such that $a \leqslant x \leqslant b$ for all a and b. Because unfillable gaps are the most interesting one drops the adjective and speaks of gaps.

One does not need the full set ${}^{\mathbb{N}}\mathbb{N}$ to create a gap; it suffices to have a subset U such that for all $g \in {}^{\mathbb{N}}\mathbb{N}$ there is $f \in U$ such that $\{n : f(n) \geqslant g(n)\}$ is infinite. The minimum cardinality of such a set is denoted $\mathfrak{b}$.

The properties of $\mathfrak{b}$ and its bigger brother $\mathfrak{d}$ are best explained using the relation $<^*$ on ${}^{\mathbb{N}}\mathbb{N}$ where, in keeping with the notation established above, $f <^* g$ means that $\{n : f(n) \geqslant g(n)\}$ is finite. The definition of $\mathfrak{b}$ given above identifies it as the minimum cardinality of an unbounded – with respect to $<^*$ – subset of ${}^{\mathbb{N}}\mathbb{N}$; unbounded is not the same as dominating (cofinal), the minimum cardinality of a dominating subset is denoted $\mathfrak{d}$ and it is called the **dominating number**.

If one identifies $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, as above, then one recognizes $\mathfrak{d}$ as the character of the closed set $F = \mathrm{cl} \bigcup_n A_n^*$ and $\mathfrak{b}$ as the minimum number of clopen sets needed to create an open subset of $\mathbb{N}^* \setminus F$ whose closure meets F. In either characterization of $\mathfrak{b}$ the defining family can be taken to be well-ordered.

In case where A is finite one can simplify matters by letting $A = \{0\}$. If B is an ultrafilter then there is no x with $0 < x < b$ for all $b \in B$, hence there are filters whose only lower bound is 0; the minimum cardinality of a base for such a filter is denoted $\mathfrak{p}$. Alternatively one can ask for chains without positive lower bound: the minimum length of such a chain is denoted $\mathfrak{t}$ – it is called the **tower number**. A defining family for the cardinal $\mathfrak{t}$ can be used to create a separable, normal, and sequentially compact spaces that is not compact.

When we let A become uncountable we encounter two important types of objects of cardinality $\aleph_1$: Hausdorff gaps and Lusin families. A **Hausdorff gap** [5] is a pair of sequences $A = \langle a_\alpha : \alpha < \omega_1\rangle$ and $B = \langle b_\alpha : \alpha < \omega_1\rangle$ of elements such that $a_\alpha < a_\beta < b_\beta < b_\alpha$ whenever $\alpha < \beta$ but for which there is no x with $a_\alpha < x < b_\alpha$ for all α. A **Lusin family** [7] is an almost disjoint family $\mathcal{A}$ of cardinality $\aleph_1$ such that no two uncountable and disjoint subfamilies of $\mathcal{A}$ can be separated, i.e., if $\mathcal{B}$ and $\mathcal{C}$ are disjoint and uncountable then there is no set X such that $B \subseteq^* X$ and $X \cap C =^* \emptyset$ for all $B \in \mathcal{B}$ and $C \in \mathcal{C}$. One can parametrize the F-space property: an F_κ**-space** is one in which disjoint open sets that are the union of fewer than κ many closed sets have disjoint closures. The two families above show that $\mathbb{N}^*$ is not an $F_{\aleph_2}$-space.

By the Axiom of Choice every almost disjoint family can be extended to a **Maximal Almost Disjoint family** (a **MAD family**). If $\mathcal{A}$ is a MAD family then $\mathbb{N}^* \setminus \bigcup\{A^* : A \in \mathcal{A}\}$ is ***nowhere dense*** and every nowhere dense set is contained in such a 'canonical' nowhere dense set. The minimum number of nowhere dense sets whose union is dense in $\mathbb{N}^*$ is denoted $\mathfrak{h}$ and is called the **weak Novák number**. It is equal to the minimum number of MAD families without a common MAD refinement – in Boolean terms, it is the minimum κ for which $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is *not* (κ, ∞)**-distributive**.

Interestingly [1], one can find a sequence $\langle \mathcal{A}_\alpha : \alpha < \mathfrak{h}\rangle$ of MAD families, without common refinement and such that (1) $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ whenever $\alpha < \beta$; (2) if $A \in \mathcal{A}_\alpha$ then $\{B \in \mathcal{A}_{\alpha+1} : B \subseteq^* A\}$ has cardinality $\mathfrak{c}$; and (3) the family $\mathcal{T} = \bigcup_\alpha \mathcal{A}_\alpha$ is dense in $\mathcal{P}(\mathbb{N})/\mathit{fin}$ – topologically: $\{A^* : A \in \mathcal{T}\}$ is a π***-base***. This all implies that, with hindsight, there is an increasing sequence of closed nowhere dense sets of length $\mathfrak{h}$ whose union is dense and that $\mathfrak{h}$ is a regular cardinal; also note that $\mathcal{T}$ is a ***tree*** under the ordering $\supset^*$.

One can use such a tree, of minimal height, in inductive constructions, e.g., as in [2] to show that $\mathbb{N}^*$ is very non-extremally disconnected: every point is a $\mathfrak{c}$**-point**, which means that one can find a family of $\mathfrak{c}$ many disjoint open sets each of which has the point in its closure. An important open problem, at the time of writing, is whether this can be proved for *every* nowhere dense set, i.e., if for every nowhere dense subset of $\mathbb{N}^*$ one can find a family of $\mathfrak{c}$ disjoint open sets each of which has this set in its boundary – in short, whether every nowhere dense set is a $\mathfrak{c}$**-set**.

The latter problem is equivalent to a purely combinatorial one on MAD families: for every MAD family $\mathcal{A}$ the family $\mathcal{I}^+(\mathcal{A})$ should have an **almost disjoint refinement**. Here $\mathcal{I}(\mathcal{A})$ is the ideal generated by $\mathcal{A}$ and $\mathcal{I}^+(\mathcal{A}) = \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}(\mathcal{A})$; an **almost disjoint refinement** of a family $\mathcal{B}$ is an indexed almost disjoint family $\{A_B : B \in \mathcal{B}\}$ with $A_B \subseteq^* B$ for all B. Another characterization asks for enough (on even one) MAD families of true cardinality $\mathfrak{c}$, i.e., if $X \in \mathcal{I}^+(\mathcal{A})$ then $X \cap A \neq^* \emptyset$ for $\mathfrak{c}$ many members of $\mathcal{A}$. If the minimum size of a MAD family, denoted $\mathfrak{a}$, is equal to $\mathfrak{c}$ then this is clearly the case, however there are various models with $\mathfrak{a} < \mathfrak{c}$.

3. Homogeneity

Given that the algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is homogeneous it is somewhat of a surprise to learn that the space $\mathbb{N}^*$ is not a **homogeneous space**, i.e., there are two points x and y for which there is no autohomeomorphism h of $\mathbb{N}^*$ with $h(x) = y$.

The first example of this phenomenon is from [9]: the ***Continuum Hypothesis*** (CH) implies that $\mathbb{N}^*$ has P**-point**s; a P-point is one for which the family of neighbourhoods is closed under countable intersections. Clearly $\mathbb{N}^*$ has non-P-points (as does every infinite compact space), so CH implies $\mathbb{N}^*$ is not homogeneous. To construct P-points one does not need the full force of CH, the equality $\mathfrak{d} = \mathfrak{c}$ suffices.

In [4] one finds a proof of the non-homogeneity of $\mathbb{N}^*$ that avoids CH; it uses the **Rudin–Frolík order** on $\mathbb{N}^*$, which is defined by: $u \sqsubset v$ iff there is an embedding $f : \beta\mathbb{N} \to \mathbb{N}^*$ such that $f(u) = v$. If $u \sqsubset v$ then there is no autohomeomorphism of $\mathbb{N}^*$ that maps u to v. This proof works to show that no compact F-space is homogeneous: if X is such a space then one can embed $\beta\mathbb{N}$ into it, no autohomeomorphism of X can map (the copy of) u to (the copy of) v.

That this proof avoiding CH was really necessary became clear when the consistency of "$\mathbb{N}^*$ has no P-points" was proved. In [6] the P-point proof was salvaged partially: $\mathbb{N}^*$ has **weak P-point**s, i.e., points that are not accumulation points of any countable subset.

Though not all points of $\mathbb{N}^*$ are P-points one may still try to cover $\mathbb{N}^*$ by nowhere dense P-sets. Under CH this cannot be done but it is consistent that it can be done. The principle NCF implies that $\mathbb{N}^*$ is the union of a chain, of length $\mathfrak{d}$, of nowhere dense P-sets. The principle NCF (**Near Coherence of Filters**) says that any two ultrafilters on $\mathbb{N}$ are **nearly coherent**, i.e., if $u, v \in \mathbb{N}^*$ then there is a finite-to-one $f : \mathbb{N} \to \mathbb{N}$ such that $\beta f(u) = \beta f(v)$.

This in turn implies that the **Rudin–Keisler order** is downward directed; we say $u \prec v$ if there is some $f : \mathbb{N} \to \mathbb{N}$ such that $u = f(v)$. This is a preorder on $\beta\mathbb{N}$ and a partial order on the **type**s of $\beta\mathbb{N}$: if $u \prec v$ and $v \prec u$ then there is a permutation of $\mathbb{N}$ that send u to v. The Rudin–Frolík and Rudin–Keisler orders are related: $u \sqsubset v$ implies $u \prec v$, so $\sqsubset$ is a partial order on the types as well.

Both orders have been studied extensively; we mention some of the more salient results. There are $\sqsubset$-minimal points in $\mathbb{N}^*$: weak P-points are such. Points that are $\prec$-minimal in $\mathbb{N}^*$ are called **selective ultrafilters**; they exist if CH holds but they do not exist in the ***random real*** model. There are $\prec$-incomparable points (even a family of $2^{\mathfrak{c}}$ many $\prec$-incomparable points) but it is not known whether for every point there is another point $\prec$-incomparable to it. The order $\sqsubset$ is tree-like: types below a fixed type are linearly ordered.

4. The continuum hypothesis and $\beta\mathbb{N}$

The behaviour of $\beta\mathbb{N}$ and, especially, $\mathbb{N}^*$ under the assumption of the Continuum Hypothesis (CH) is very well understood. The principal reason is that the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is characterized by the properties of being atomless, countably saturated and of cardinality $\mathfrak{c} = \aleph_1$. Topologically, $\mathbb{N}^*$ is, up to homeomorphism, the unique compact zero-dimensional without isolated points, which is an F-space in which nonempty G_δ-sets have non-empty interior and which is of weight $\mathfrak{c}$. This is known as Parovičenko's characterization of $\mathbb{N}^*$ and it implies that whenever X is compact zero-dimensional and of weight $\mathfrak{c}$ (or less) the remainder $(\mathbb{N} \times X)^*$ is homeomorphic to $\mathbb{N}^*$. This provides us with many incarnations of $\mathbb{N}^*$, e.g., as $(\mathbb{N} \times {}^{\mathfrak{c}}2)^*$, which immediately provides us with $2^{\mathfrak{c}}$ many autohomeomorphisms of $\mathbb{N}^*$, or as $(\mathbb{N} \times (\omega+1))^*$, which gives us a P-set in $\mathbb{N}^*$, viz. $(\mathbb{N} \times \{\omega\})^*$, that is homeomorphic to $\mathbb{N}^*$ itself.

Parovičenko's 'other theorem' states that every compact space of weight $\aleph_1$ (or less) is a continuous image of $\mathbb{N}^*$, whence under CH the space $\mathbb{N}^*$ is a universal compactum of weight $\mathfrak{c}$, in the mapping onto sense.

Virtually everything known about $\mathbb{N}^*$ under CH follows from Parovičenko's theorems. To give the flavour we show that a compact zero-dimensional space can be embedded into $\mathbb{N}^*$ if (and clearly only if) it is an F-space of weight $\mathfrak{c}$ (or less). Indeed, let X be such a space and take a P-set A in $\mathbb{N}^*$ that is homeomorphic to $\mathbb{N}^*$ and a continuous onto map $f : A \to X$. This induces an upper-semi-continuous decomposition of $\mathbb{N}^*$ whose quotient space, the ***adjunction space*** $\mathbb{N}^* \cup_f X$, can be shown to have all the properties that characterize $\mathbb{N}^*$, hence it is $\mathbb{N}^*$ and we have embedded X into $\mathbb{N}^*$, even as a P-set.

In the absence of CH very few of the consequences of Parovičenko's theorems remain true. The characterization theorem is in fact equivalent to CH and for many concrete spaces, like $\mathbb{N}^* \times \mathbb{N}^*$, $\mathbb{R}^*$, $(\mathbb{N} \times (\omega+1))^*$ and the Stone space of the ***measure algebra*** it is *not* a theorem of ZFC that they are continuous images of $\mathbb{N}^*$. It is also consistent with ZFC that all autohomeomorphisms of $\mathbb{N}^*$ are **trivial**, i.e., induced by bijections between cofinite sets. This is a far cry from the $2^{\mathfrak{c}}$ autohomeomorphisms that we got from CH. It is also consistent that $\mathbb{N}^*$ cannot be homeomorphic to a nowhere dense P-subset of itself; this leaves open a major question: is there a non-trivial copy of $\mathbb{N}^*$ in itself, i.e., one not of the form $\operatorname{cl} D \setminus D$ for some countable discrete subset of $\mathbb{N}^*$.

A good place to start exploring $\beta\mathbb{N}$ is van Mill's survey [KV, Chapter 11].

5. Cardinal numbers

The cardinal numbers mentioned above are all uncountable and not larger than $\mathfrak{c}$, so the Continuum Hypothesis (CH) implies that all are equal to $\aleph_1$. More generally, ***Martin's Axiom*** implies all are equal to $\mathfrak{c}$. One can prove certain inequalities between the characteristics, e.g., $\aleph_1 \leqslant \mathfrak{p} \leqslant \mathfrak{t} \leqslant \mathfrak{h} \leqslant \mathfrak{b} \leqslant \mathfrak{d}$. Intriguingly it is as yet unknown whether $\mathfrak{p} = \mathfrak{t}$ is provable, what is known is that $\mathfrak{p} = \aleph_1$ implies $\mathfrak{t} = \aleph_1$.

One proves $\mathfrak{b} \leqslant \mathfrak{a}$ but neither $\mathfrak{a} \leqslant \mathfrak{d}$ nor $\mathfrak{d} \leqslant \mathfrak{a}$ is provable in ZFC.

Two more characteristics have received a fair amount of attention, the **splitting number** $\mathfrak{s}$ is the minimum cardinality of a splitting family, that is, a family $\mathcal{S}$ such that for every infinite X there is $S \in \mathcal{S}$ with $X \cap S$ and $X \setminus S$ infinite; and its dual, the **reaping number** $\mathfrak{r}$, which is the minimum cardinality of a family that cannot be reaped (or split), that is, a family $\mathcal{R}$ such that for every infinite X there is $R \in \mathcal{R}$ such that one of $X \cap R$ and $X \setminus R$ is not infinite.

Topologically $\mathfrak{s}$ is the minimum κ for which ${}^{\kappa}2$ is not ***sequentially compact*** and $\mathfrak{r}$ is equal to the minimum ***π-character*** of points in $\mathbb{N}^*$.

Further inequalities between these characteristics are, e.g., $\mathfrak{h} \leqslant \mathfrak{s} \leqslant \mathfrak{d}$ and $\mathfrak{b} \leqslant \mathfrak{r}$.

These 'small' cardinals are the subject of ongoing research; good introductions are [KV, Chapter 3] and [vMR, Chapter 11].

6. Properties of $\beta\mathbb{R}$

Instead of $\beta\mathbb{R}$ one usually considers $\beta\mathbb{H}$, where $\mathbb{H}$ is the positive half line $[0, \infty)$. This is because $x \mapsto -x$ induces an autohomeomorphism of $\beta\mathbb{R}$ that shows that $\beta[0, \infty)$ and $\beta(-\infty, 0]$ are the same thing.

In a sense $\beta\mathbb{H}$ looks like $\beta\mathbb{N}$ in that it is a thin locally compact space with a large compact lump at the end; this remainder $\mathbb{H}^*$ has some properties in common with $\mathbb{N}^*$: both are F-spaces in which nonempty G_δ-sets have nonempty interior, both have tree π-bases and neither is an $F_{\aleph_2}$-space. Under CH the spaces $\mathbb{H}^*$ and $\mathbb{N}^*$ have homeomorphic dense sets of P-points. That is where the superficial similarities end because $\beta\mathbb{H}$ and $\mathbb{H}^*$ are connected and $\beta\mathbb{N}$ and $\mathbb{N}^*$ most certainly are not.

A deeper similarity is a version of Parovičenko's universality theorem: every continuum of weight $\aleph_1$ (or less) is a continuous image of $\mathbb{H}^*$, so that, under CH, the space $\mathbb{H}^*$ is a universal continuum of weight $\mathfrak{c}$. A version of Parovičenko's characterization theorem for $\mathbb{H}^*$ is yet to be found.

The structure of $\mathbb{H}^*$ as-a-continuum is very interesting. Most references for what follows can be found in [HvM, Chapter 9].

The following construction is crucial for our understanding of the structure of the continuum $\mathbb{H}^*$: take a discrete sequence $\langle [a_n, b_n] \rangle_n$ of non-trivial closed intervals and an ultrafilter u on $\mathbb{N}$. The intersection

$$I = \bigcap_{U \in u} \mathrm{cl}\left(\bigcup_{n \in U} [a_n, b_n] \right)$$

is a continuum, often called a **standard subcontinuum** of $\mathbb{H}^*$. What is striking about this construction is not its simplicity but that these (deceptively) simple continua govern the continuum-theoretic properties of $\mathbb{H}^*$. Every proper subcontinuum (in particular every point) is contained in a standard subcontinuum – it is in fact the intersection of some family of standard subcontinua. From this one shows that $\mathbb{H}^*$ is hereditarily ***unicoherent*** – every finite intersection of standard subcontinua is a standard subcontinuum or a point – and ***indecomposable*** – standard subcontinua are nowhere dense.

From the other direction every subcontinuum is also the union of a suitable family of standard subcontinua. Thus, no subcontinuum of $\mathbb{H}^*$ is hereditarily indecomposable, as standard subcontinua have ***cut points***. Indeed, I_u contains the ultraproduct $\prod_n [a_n, b_n]/u$, as a dense set: the equivalence class of a sequence $\langle x_n \rangle_n$ corresponds to its own u-limit, denoted x_u. The subspace topology of this ultraproduct coincides with its ***order topology*** and every point of the ultraproduct (except a_u and b_u) is a cut point of I_u and so a **weak cut point** (i.e., cut point of some subcontinuum) of $\mathbb{H}^*$. Although the continuum I_u is an **irreducible continuum**, i.e., no smaller continuum contains its ***end point***s a_u and b_u, it is definitely not an ordered continuum. It is an F-space so the closure of an increasing sequence of points in the ultraproduct is homeomorphic to $\beta\mathbb{N}$ (in an ordered continuum it would have to be $\omega + 1$). The 'supremum' of such a sequence of points in I_u is an irreducibility layer of I_u; this layer is non-trivial (it contains $\mathbb{N}^*$) and indecomposable. Adding all these facts together we can deduce that a maximal chain of indecomposable subcontinua of $\mathbb{H}^*$ has a one-point intersection; such a point is not a weak cut point of $\mathbb{H}^*$. Thus $\mathbb{H}^*$ is shown to be not homogeneous by purely continuum-theoretic means. There is a natural quasi-order on an irreducible continuum: in the present case $x \leqslant y$ means "every continuum that contains a_u and y also contains x". An **irreducibility layer** is an equivalence class for the equivalence relation "$x \leqslant y$ and $y \leqslant x$".

The weak cut points constructed above are ***near point***s and, in fact, every near point is a weak cut point of $\mathbb{H}^*$. Under CH one can construct different kinds of weak cut points: ***far*** but not ***remote*** and even remote. Under CH it is also possible to map a remote weak cut point to a near point by an autohomeomorphism of $\mathbb{H}^*$. On the other hand it is consistent that all weak cut points are near points and hence that the far points are topologically invariant in $\mathbb{H}^*$. A similar consistency result for remote points is still wanting.

The ***composant*** of a point x of $\mathbb{H}^*$ meets $\mathbb{N}^*$ in the ultrafilter $\{U : x \in \mathrm{cl} \bigcup_{n \in U} [n, n+1]\}$ and two points from $\mathbb{N}^*$ are in the same composant of $\mathbb{H}^*$ iff they are nearly coherent. Therefore, the structure of the set of composants of $\mathbb{H}^*$ is determined by the family of finite-to-one maps of $\mathbb{N}$ to itself. This implies that the number of composants may be $2^{\mathfrak{c}}$ (e.g., if CH holds), 1 (this is equivalent to the NCF principle) or 2 (in other models of set theory); whether other numbers are possible is unknown.

The number of (homeomorphism types of) subcontinua of $\mathbb{H}^*$ is as yet a function of Set Theory: in ZFC one can establish the lower bound of 14. Under CH there are $\aleph_1$ types even though in one respect CH act as an equalizer: CH is equivalent to the statement that all standard subcontinua are mutually homeomorphic. Most of the ZFC continua are found as intervals in an I_u with points and non-trivial layers at their ends and with varying cofinalities.

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d-19 Wallman–Shanin Compactification

1. Compactifications

A **compactification** of a ***topological space*** X is a ***compact*** space Y that contains X as a ***dense subspace***; more formally, a compactification of X is a pair (Y, e) of a compact space Y and a ***topological embedding*** $e: X \to Y$ such that $e(X)$ is dense in Y. It should be noted that in this article the space Y is not required to be ***Hausdorff*** (cf. [E, p. 177]). The Wallman–Shanin compactification is a method for constructing compactifications which may be used for obtaining compactifications of X with specified properties. In the 1960s it was shown that all of the known compactifications could also be obtained as Wallman–Shanin compactifications. It was not until 1977 that Ul'janov [13] exhibited an example of a space X and a Hausdorff compactification Y of X which is not a Wallman–Shanin compactification of X.

For the description of the Wallman–Shanin compactification we employ subbases for the ***closed set***s of a topological space. A family $\mathcal{B}$ of closed subsets of a topological space X is called a **base for the closed sets** of X if each closed subset of X is the intersection of some subfamily of $\mathcal{B}$ or, equivalently, for each closed subset G of X and each point x of $X \setminus G$ there exists a B in $\mathcal{B}$ such that $G \subseteq B$ and $x \notin B$. A family $\mathcal{S}$ of closed subsets of a topological space X is said to be a **subbase for the closed subsets** of X if the family of all unions of finite subfamilies of $\mathcal{S}$ is a base for the closed subsets of X. Suppose that $\mathcal{S}$ is a subbase for the closed subsets of the topological space X. A subfamily $\mathcal{T}$ of $\mathcal{S}$ is called a **centered system** of $\mathcal{S}$ if $S_1 \cap \cdots \cap S_n \neq \emptyset$ for each finite subfamily $\{S_1, \ldots, S_n\}$ of $\mathcal{T}$. Then, from the ***Alexander Subbase Lemma*** [E, 3.12.2] it follows that the space X is compact if and only if each centered system of $\mathcal{S}$ has non-empty intersection.

Let X be a topological space and $\mathcal{S}$ a subbase for its closed subsets. The basic idea of the Wallman–Shanin compactification (with respect to $\mathcal{S}$) is forcing the intersection of each centered system of $\mathcal{S}$ to be non-empty. This is achieved by adding a new point to the space X whenever a centered system of $\mathcal{S}$ happens to have an empty intersection. A more accurate description follows (cf. [12, 2, 7]). By the Teichmuller–Tukey lemma [E, p. 22] each centered system of $\mathcal{S}$ is contained in a maximal one. The set of all maximal centered systems of $\mathcal{S}$ is denoted by $\omega(X, \mathcal{S})$. The set $\omega(X, \mathcal{S})$ is the underlying set of the Wallman–Shanin compactification. Its points are denoted by small Greek letters. For each S in $\mathcal{S}$ the subset S^* of $\omega(X, \mathcal{S})$ is defined by $S^* = \{\xi \in \omega(X, \mathcal{S})\colon S \in \xi\}$. A ***topology*** on $\omega(X, \mathcal{S})$ is defined by taking $\mathcal{S}^* = \{S^*\colon S \in \mathcal{S}\}$ as a subbase for the closed sets. Thus a subset G of $\omega(X, \mathcal{S})$ is closed if and only if for each ξ in $\omega(X, \mathcal{S})$ such that $\xi \notin G$ there exist $S_1, \ldots, S_n$ in $\mathcal{S}$ such that $G \subseteq S_1^* \cup \cdots \cup S_n^*$ and $\xi \notin S_i^*$ for $i = 1, \ldots, n$. The key property of $\omega(X, \mathcal{S})$ is: for all $S_1, \ldots, S_n$ in $\mathcal{S}$ one has $S_1 \cap \cdots \cap S_n = \emptyset$ if and only if $S_1^* \cap \cdots \cap S_n^* = \emptyset$. The proof of the compactness of $\omega(X, \mathcal{S})$ by the Alexander subbase lemma follows. Suppose that $\mathcal{T}^*$ is a centered system of $\mathcal{S}^*$. We may assume that $\mathcal{T}^*$ is maximal. From the key property it follows that the family $\mathcal{T} = \{S\colon S^* \in \mathcal{T}^*\}$ is a maximal centered system of $\mathcal{S}$. Consequently $\mathcal{T}$ is a point of $\omega(X, \mathcal{S})$ that belongs to every set in $\mathcal{T}^*$. It follows that $\bigcap \mathcal{T}^* \neq \emptyset$, whence $\omega(X, \mathcal{S})$ is compact. To obtain a topological embedding $e: X \to \omega(X, \mathcal{S})$ we define $e(x) = \{S \in \mathcal{S}\colon x \in S\}$. Without any further restriction on $\mathcal{S}$ the family $e(x)$ need not be a maximal centered system for any x, and even if e is a map, it need not be an embedding. We shall say that the subbase $\mathcal{S}$ is **disjunctive** if for each S of $\mathcal{S}$ and each point x of X such that $x \notin S$ there exists a T in $\mathcal{S}$ such that $x \in T \subseteq X \setminus S$. If the subbase $\mathcal{S}$ of X is disjunctive, the space X as well as $\omega(X, \mathcal{S})$ are ***T_1-spaces*** and e is an embedding. Thus for a disjunctive subbase $\mathcal{S}$ the space $\omega(X, \mathcal{S})$ is a compactification of X. Further it can be shown that S^* coincides with the ***closure*** of S in $\omega(X, \mathcal{S})$. In general, the compactification $\omega(X, \mathcal{S})$ need not be Hausdorff. We shall formulate sufficient conditions for $\omega(X, \mathcal{S})$ to be Hausdorff.

A family $\mathcal{V}$ of subsets of a set X is called a **ring** if for every finite subfamily $\{V_1, \ldots, V_n\}$ of $\mathcal{V}$ both the intersection $V_1 \cap \cdots \cap V_n$ and the union $V_1 \cup \cdots \cup V_n$ belong to $\mathcal{V}$. Suppose that a subbase $\mathcal{S}$ of the closed subsets of a topological space X is a ring. Note that because of the ring property $\mathcal{S}$ is a base for the closed sets. The family $\mathcal{S}$ is said to be **normal** if for any two disjoint elements A and B of $\mathcal{S}$ there are elements C and D of $\mathcal{S}$ such that $A \cap D = \emptyset$, $B \cap C = \emptyset$ and $C \cup D = X$; the pair (C, D) is called a **screening** the pair (A, B). A base $\mathcal{B}$ for the closed subsets of a topological space X is called a **Wallman–Shanin base**, or **WS-base**, if (1) $\mathcal{B}$ is a ring, (2) $\mathcal{B}$ is disjunctive, (3) $\mathcal{B}$ is normal. If $\mathcal{B}$ is a Wallman–Shanin base the compactification $\omega(X, \mathcal{B})$ is called a **Wallman–Shanin compactification**, or **WS-compactification**, of X. Note that in view of property (1) distinct points of $\omega(X, \mathcal{B})$ are contained in disjoint elements of $\mathcal{B}^*$ and that by property (3) these disjoint elements of $\mathcal{B}^*$ have disjoint neighbourhoods in $\omega(X, \mathcal{B})$. In other words, the space $\omega(X, \mathcal{B})$ is a compact Hausdorff space. Also observe that in view of the ring property of $\mathcal{B}$ the maximal centered systems of $\mathcal{B}$ are ***ultrafilter***s. The WS-compactification first appeared in the paper [14] of Wallman. We shall discuss this somewhat more in detail in the next section. In [9, 8, 10] Shanin indicated that the compactification method of Wallman had much wider applications and introduced what is now called the WS-compactification. The work of Shanin has remained unnoticed for a long time.

That explains why the WS-compactification is often referred to as **Wallman compactification**. In 1961 Frink [5] rediscovered some of Shanin's results and posed the question whether each Hausdorff compactification Y of a space X is a WS-compactification, i.e., whether Y can be represented as $\omega(X, \mathcal{B})$ for some base $\mathcal{B}$ for the closed subsets of X. The question of Frink has triggered many partial answers. We shall discuss this further in the final section of the article.

2. Wallman representation

In [14] Wallman developed a theory of representing lattices by bases for the family of all closed subsets of topological spaces. This work is related to Stone's theory of representing ***Boolean rings*** by the family of ***closed-and-open*** subsets of compact ***zero-dimensional*** spaces. In this section the spaces under discussion are T_1-spaces. Wallman observed that much information about a space X, more specifically its ***covering dimension*** and its ***Čech homology***, is related to the lattice of some base for the closed subsets rather than its topology. A **lattice** is a nonempty set L with a ***reflexive partial order*** $\leqslant$ such that for each pair (x, y) of elements of L there is a unique smallest element $x \vee y$, called the **join** of x and y, such that $x \leqslant x \vee y$ and $y \leqslant x \vee y$ and there is a unique largest element $x \wedge y$, called the **meet** of x and y, such that $x \wedge y \leqslant x$ and $x \wedge y \leqslant y$ [2, 4, 14]. The family of all closed subsets of a topological T_1-space X with the partial order $\subseteq$ is a lattice which is denoted by $\mathcal{L}(X)$. The join and meet of two closed subsets are the union and intersection respectively. The lattice $\mathcal{L}(X)$ is ***distributive***, has a largest element 1 (the set X) and a smallest element 0 (the empty set). Also the lattice $\mathcal{L}(X)$ has the **disjunction property**: for all x and y of $\mathcal{L}(X)$ that are distinct there exists a z in $\mathcal{L}(X)$ such that one of $x \wedge z$ and $y \wedge z$ in 0 and the other is not. (For example, z is a one point subset of the symmetric difference of the subsets x and y of X.)

In the first part of Wallman's paper [14] it is shown that each distributive lattice L with 0 and 1 that has the disjunction property can be represented as the lattice of a base for the closed subsets of some compact T_1-space ωL. The construction of the space ωL is similar to the construction of the compactifications in Section 1. A nonempty subset ξ of L is called a **dual ideal** (or **filter**) if for all x and y in L we have: (1) if x and y in ξ then $x \wedge y \in L$, (2) if $x \in \xi$ and $x \leqslant y$ then $y \in \xi$. Every dual ideal is contained in a maximal one. Note that if $L = \mathcal{L}(X)$ for some T_1-space X then a dual ideal of L is a centered system of the family of closed sets of X and a maximal dual ideal of L is just an ultrafilter. The points of ωL are the maximal dual ideals of L. For each u in L a subset B_u of $\omega(L)$ is defined by $B_u = \{\xi\colon u \in \xi\}$. The topology on ωL is defined by taking the family $\mathcal{F} = \{B_u\colon u \in L\}$ as a base for the closed subsets. Then ωL is a compact T_1 space and is referred to as the **Wallman representation** of the lattice L. The proof of the compactness is essentially the same as that in Section 1. It can be shown that the closed base $\mathcal{F}$ has the following properties: (1) if u and v are distinct elements of L, then $B_u \neq B_v$, (2) $B_u \cup B_v = B_{u \vee v}$ and $B_u \cap B_v = B_{u \wedge v}$ for all u and v of L. In other words, the map of L to $\mathcal{F}$ defined by sending u to B_u is an ***isomorphism*** of lattices. The space ωL is Hausdorff if and only if for all s and t in L with $s \wedge t = 0$ there exist u and v in L such that $s \wedge v = 0$, $u \wedge t = 0$ and $u \vee v = 1$.

The topological representation of lattices can be used to construct a compactification of a topological space via the lattice of its closed sets. The **Wallman compactification** of a T_1-space X is the topological representation of the lattice $\mathcal{L}(X)$ of all closed subsets of X. The space is denoted by $\omega\mathcal{L}(X)$ or, as in Section 1, by $\omega(X, \mathcal{L}(X))$. The embedding of X in $\omega(X, \mathcal{L}(X))$ is defined by sending the point x to the ultrafilter of all closed sets of X that contain x. The Wallman compactification of a T_1-space X is Hausdorff if and only if X is normal. The Wallman compactification is a **maximal compactification** in the following sense. If $f: X \to K$ is a ***continuous map*** of a T_1-space X to a compact Hausdorff space K, then f can be extended to a continuous map $\tilde{f}: \omega(X, \mathcal{L}(X)) \to K$.

3. Wallman–Shanin compactifications

Suppose that $\mathcal{B}$ is a WS-base of a topological space X. Then the topological representation of the lattice $\mathcal{B}$ coincides with the space $\omega(X, \mathcal{B})$ as defined in Section 1. The problem of deciding whether a Hausdorff compactification Y of a space X is a WS-compactification $\omega(X, \mathcal{B})$ for some WS-base $\mathcal{B}$ was already considered by Shanin. He showed that $Y = \omega(X, \mathcal{B})$ for some base $\mathcal{B}$ for the closed subsets if and only if (1) $\{\overline{B}\colon B \in \mathcal{B}\}$ is a base for the closed subsets of Y and (2) $\overline{B_1} \cap \overline{B_2} = \overline{B_1 \cap B_2}$ for all B_1, B_2 in $\mathcal{B}$, where the an upper bar denotes the closure in the space Y. The following more general statement can be found in [1]. Suppose that Y is a Hausdorff compactification of a space X. Then $Y = \omega(X, \mathcal{S})$ for some subbase $\mathcal{S}$ for the closed sets if and only if the following two conditions hold: (1) $\{\overline{S}\colon S \in \mathcal{S}\}$ is a subbase for the closed subsets of Y and (2) $\overline{S_1} \cap \dots \cap \overline{S_n} = \emptyset$ if and only if $S_1 \cap \dots \cap S_n = \emptyset$ for all $S_1, \dots, S_n$ in $\mathcal{S}$. And if the conditions (1) and (2) are satisfied, then $Y = \omega(X, \mathcal{B})$, where $\mathcal{B}$ is the family of all finite unions and intersections of $\mathcal{S}$. This result is related to Steiner's result [12] that $\omega(X, \mathcal{B}) = \omega(X, \mathcal{S})$ and the following observation. If $\mathcal{S}$ is a subbase for the closed sets of a compact Hausdorff space, then the family of all finite unions and intersections of elements of $\mathcal{S}$ is a WS-base.

In Section 2 we have mentioned the maximality of the Wallman compactification. As the ***Čech–Stone compactification*** is characterized by its maximality [E, 3.6.6], it is a natural question whether the Čech–Stone compactification is a WS-compactification. Note that the Wallman compactification of a normal T_1-space X is Hausdorff. Then, by maximality it must be the Čech–Stone compactification of X. We shall exhibit a WS-base of X. A subset Z of a topological T_1-space X is called a **zero set** if $Z = f^{-1}[\{0\}]$ for some

continuous real-valued function $f : X \to R$. The set Z is closed by continuity of f. A T_1-space X is **completely regular** if the family $\mathcal{Z}$ of all zero sets is a base for the closed subsets of X. If X is a completely regular T_1-space then the family $\mathcal{Z}$ of all zero sets is a WS-base and the compactification $\omega(X, \mathcal{Z})$ is the same as the Čech–Stone compactification [6, Section 6.5]. In connection with the characterization of completely regular T_1-spaces as subspaces of compact Hausdorff spaces [E, 3.5.1], we make the observation that a T_1-space X is completely regular if and only if X has a WS-base [5].

A Hausdorff space X is ***locally compact*** if each of its points has an ***open neighbourhood*** the closure of which is compact. A non-compact Hausdorff space X is locally compact if and only if X has a Hausdorff **one-point compactification** $\alpha(X)$ (i.e., $\alpha(X)$ is a Hausdorff compactification such that $\alpha(X) \setminus X$ consists of one point). It may be seen that the one-point compactification $\alpha(X)$ of a non-compact, locally compact Hausdorff space X is a WS-compactification. An appropriate WS-base consists of all compact subsets of X and all sets of the form $\overline{X \setminus K}$ for some compact subset K.

WS-compactifications may be used to show the existence of compactifications with specified properties. It can be shown, for example, that for a completely regular T_1-space X with $\dim X \leqslant n$ and a family $\mathcal{A}$ of closed subsets such that the ***cardinality*** of the family does not exceed the ***weight*** $w(X)$ of the space there exists a compactification Y such that $w(Y) = w(X)$, $\dim Y \leqslant \dim X$ and $\dim \bar{A} \leqslant \dim A$ for all A in $\mathcal{A}$. The idea of the proof is that the dimension of a space or of one of its closed subsets can be formulated in terms of finite families from a given base for the closed subsets of the space. Then an appropriate WS-base is built in an inductive way. We refer to [2, Chapter 6] for various examples of applications of WS-compactifications along these lines

In investigating Frink's problem whether every Hausdorff compactification of a space is a WS-compactification we may restrict our attention to compactifications of discrete spaces. This is a consequence of a theorem of Bandt [3]: if all Hausdorff compactifications of the ***discrete space*** of cardinality $\leqslant \gamma$ are WS-compactifications, then so are all Hausdorff compactifications of spaces of density $\leqslant \gamma$.

In addressing Frink's problem it is natural to study compactifications with increasing complexity. A measure for the complexity is the weight of the compactification. It was shown by Steiner and Steiner [11] and Aarts [1] that all compactifications with weight $\leqslant \aleph_0$ are WS-compactifications. This statement can be rephrased as: all metrizable compactifications are Wallman. This result was substantially strengthened by Bandt [3] in showing that all compactifications with weight $\leqslant \aleph_1$ are WS-compactifications. Under the ***Continuum Hypothesis*** it follows that every compactification of a separable space is a WS-compactification. In a sense this result is the best possible: Ul'janov [13] showed that for any cardinal τ such that $2^\tau \geqslant \aleph_2$, there exists a completely regular space S of cardinality τ and a connected Hausdorff compactification νS of weight 2^τ which is not a WS-compactification.

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d-20 *H*-Closed Spaces

A result taught in a first course in topology is that a ***compact*** subspace of a ***Hausdorff space*** is closed. A Hausdorff space with the property of being closed in every Hausdorff space containing it as a subspace is called ***H*-closed** (short for **Hausdorff-closed**). *H*-closed spaces were introduced in 1924 by Alexandroff and Urysohn. They produced an example of an *H*-closed space that is not compact, showed that a regular *H*-closed space is compact, characterized a Hausdorff space as *H*-closed precisely when every open cover has a finite subfamily whose union is dense, and posed the question of which Hausdorff spaces can be densely embedded in an *H*-closed space. Let $\mathbb{I}$ denote the unit interval $[0, 1]$ and $\mathbb{Q}$ the set of rationals. Enlarge the usual topology of $\mathbb{I}$ by adding $\mathbb{Q} \cap \mathbb{I}$ as an open set. The resulting space, denoted as **AU**, is *H*-closed but not compact. More details about *H*-closed spaces, including historical notes and references, are available to the interested reader in the text by Porter and Woods [7].

H-closed spaces enjoy many of the same properties of compact Hausdorff spaces. In 1941, Chevally and Frink established that the product of nonempty spaces is *H*-closed iff each coordinate space is *H*-closed. In 1940, Katětov proved that the continuous Hausdorff image of an *H*-closed space is *H*-closed. Unlike compact spaces, *H*-closed spaces are not closed hereditary, e.g., the closed set $\mathbf{AU} \setminus \mathbb{Q}$, a copy of the irrationals with the usual topology, is not *H*-closed. In 1947, Katětov provided a new characterization of compact spaces by showing that a Hausdorff space is compact iff every closed subset is *H*-closed. However, certain types of closed subsets of an *H*-closed space are again *H*-closed. Recall that a ***regular closed*** subset of a space is the closure of an open set. Katětov proved that regular closed subsets of an *H*-closed space are *H*-closed. The size of an infinite, first-countable compact Hausdorff space is ω or 2^ω. Dow and Porter have shown that if X is *H*-closed, then $|X| \leqslant 2^{\chi(X)}$. In particular, a first-countable *H*-closed space has cardinality at most 2^ω. Dow and Porter (1982) and Gruenhage (1993) have shown that the existence of an uncountable, first-countable *H*-closed space of cardinality less than 2^ω is consistent.

In the same 1930 article that Tychonoff characterized those spaces that can be densely embedded in a compact Hausdorff space, he shows that an arbitrary Hausdorff space can be embedded in an *H*-closed space. Tychonoff knew that unlike the compactness situation, the closure of this embedding may not be *H*-closed and that the Alexandroff–Urysohn question was only partial answered. Tychonoff's result did open up the possibility that every Hausdorff space could be densely embedded in an *H*-closed space. Within a few years, five separate topologists – Stone, Katětov, Fomin, A.D. Alexandroff, and Šanin – proved that every Hausdorff space has an *H*-closed extension. The ironical twist of this story is that Tychonoff would have seceded had he continued. It was proved some 60 years later by Porter that the closure of Tychonoff's embedding is *H*-closed.

By now, the reader has noticed the theory of *H*-closed spaces is quite rich. What is missing is the existence of an *H*-closed $\mathbf{H}$ (for compact spaces it is $\mathbb{I}$) such that every Hausdorff space can be embedded in a product of $\mathbf{H}$'s, i.e., for each Hausdorff space X the set $C(X, \mathbf{H})$ of continuous maps separates points and closed sets. Herrlich showed there is no such space Y such that $C(X, Y)$ separates points for each Hausdorff space X by showing that given any Hausdorff space Y, there is a Hausdorff space X (which uses Y in its construction) such that $C(X, Y)$ consists of only the constant functions. In the theory of compact spaces, the space $\mathbb{I}$ is used to construct a "largest" compact Hausdorff extension of a Tychonoff space. The 1940 construction by Katětov of an *H*-closed extension κX of a Hausdorff space is also the largest of the *H*-closed extensions of X in this sense: if Y is an *H*-closed extension of X, then there is a continuous function $f : \kappa X \to Y$ such that $f \upharpoonright X = \mathrm{Id}_X$ (we denote this situation by $\kappa X \geqslant Y$ or more accurately by $\kappa X \geqslant_X Y$). This latter fact was established Porter and Thomas in 1969 in response to a question asked by P. Alexandroff in 1960. For a Hausdorff space X, the order $\leqslant_X$ is a partial order on the set $\mathcal{H}(X)$ of *H*-closed extensions of X; moreover, $(\mathcal{H}(X), \leqslant_X)$ is a complete upper semi-lattice. A Hausdorff space X is **locally *H*-closed** (every point has an *H*-closed neighbourhood) iff X has a one-point *H*-closed extension iff $(\mathcal{H}(X), \leqslant_X)$ is a complete lattice.

For a Hausdorff space X, let $X^* = X \cup \{\mathcal{U}: \mathcal{U}$ is a free open ultrafilter on $X\}$. Let κX be the set X^* with the topology generated by the base $\tau(X) \cup \{U \cup \{\mathcal{U}\}: U \in \mathcal{U} \in X^* \setminus X\}$, and σX be the set X^* with the topology generated by $\{o(U): U \in \tau(X)\}$ where $o(U) = U \cup \{\mathcal{U} \in X^* \setminus X: U \in \mathcal{U}\}$. Both spaces κX and σX are *H*-closed extensions of X. κX is called the **Katětov *H*-closed extension** of X, and σX is called the **Fomin *H*-closed extension** of X. The identity function $\mathrm{Id} : \kappa X \to \sigma X$ is continuous. The remainder of κX ($\kappa X \setminus X$) is discrete and closed in κX, and the remainder of σX ($\sigma X \setminus X$) is a zero-dimensional subspace of σX. If X is a ***Tychonoff space***, then $\kappa X \geqslant \sigma X \geqslant \beta X$. When X is Tychonoff, $\kappa X = \beta X$ iff X is compact and $\sigma X = \beta X$ iff every closed nowhere dense subset of X is compact. If hX is an *H*-closed extension of X and $f_h : \kappa X \to hX$ is a continuous function such that $f_h \upharpoonright X = \mathrm{Id}_X$, then $P_h = \{f_h^{\leftarrow}(y): y \in hX \setminus X\}$ is a partition of $\kappa X \setminus X = \sigma X \setminus X$ (recall that $\kappa X \setminus X$ and $\sigma X \setminus X$ are the same set). In terms of $\sigma X \setminus X$, P_h is a partition of compact sets. Conversely, if P is a partition of $\sigma X \setminus X$ into compact sets, there is at least one H-closed extension hX of X such that $P = P_h$. Recall that an extension Y of a space X has **relatively zero-dimensional remainder** if there is a base $\mathcal{B}$ for Y such that $\mathrm{cl}_Y B \setminus B \subseteq X$ for each $B \in \mathcal{B}$. Flachsmeyer has generated, using Boolean algebra bases, all the H-closed extensions with relatively zero-dimensional remainders. A consequence of his theory is the existence of an H-closed extension hX of X such that $w(hX) = w(X)$. Dow and Porter have shown that if hX is an H-closed extension of X and Z is a space such that $\tau(Z) \supseteq \tau(hX \setminus X)$, there is an H-closed extension Y of X such that $Y \setminus X = Z$.

A compact Hausdorff space has no strictly coarser Hausdorff topology, i.e., it is **minimal Hausdorff**. Recall that a ***regular open*** set is the complement of a regular closed set and that the collection of regular open sets (denoted as $\mathcal{RO}(X)$) of a space X form a complete Boolean algebra. A space X with the topology generated by $\mathcal{RO}(X)$ is denoted as X_s and called the **semiregularization** of X. A space is **semiregular** iff $\tau(X) = \tau(X_s)$; the space X_s is semiregular. If X is H-closed, then X_s is minimal Hausdorff, and a space is minimal Hausdorff iff it is H-closed and semiregular. Banaschewski has shown that a Hausdorff space can be densely embedded in a minimal Hausdorff space iff it is semiregular. As every Hausdorff space can be embedded as a closed nowhere dense subset in a semiregular Hausdorff space, it follows, by Banaschewski's result, that every Hausdorff space can be embedded as a closed nowhere dense subset of a minimal Hausdorff space. When X is semiregular, $(\kappa X)_s = (\sigma X)_s$ is a minimal Hausdorff extension of X and is denoted as μX. If X is Tychonoff, then $\mu X = \beta X$ iff $\mathrm{cl}_X U \setminus U$ is compact for every regular open subset U of X. Let $\mathbb{Z}$ denote the set of all integers with the discrete topology and $\mathbb{N}$ denote the subspace of positive integers. For the set $\mathbf{U} = \mathbb{N} \times \mathbb{Z} \cup \{\pm\infty\}$, a subset $U \subseteq \mathbf{U}$ is defined to be open if $+\infty \in U$ (or$-\infty \in U$) implies for that some $k \in \mathbb{N}$, $\{(n,m)\colon n \geqslant k, m \in \mathbb{N}\} \subseteq U$ (or $\{(n,-m)\colon n \geqslant k, m \in \mathbb{N}\} \subseteq U$, respectively) and if $(n,0) \in U$ implies for some $k \in \mathbb{N}$, $\{(n,\pm m)\colon m \geqslant k\} \subseteq U$. The space $\mathbf{U}$, defined by Urysohn in 1925, is minimal Hausdorff but is not compact as $\{(n,0)\colon n \in \mathbb{N}\}$ is an infinite, closed discrete subset. Now, $\mathbf{U}$ is a minimal Hausdorff extension of $\mathbb{N}$ but, as $\beta\mathbb{N} = \mu\mathbb{N}$ we have $\mu\mathbb{N} \not\geqslant_{\mathbb{N}} \mathbf{U}$.

Finding an internal characterization of the closed subsets of an H-closed space remains an unsolved problem. A subset A of an Hausdorff space X is called an ***H*-set** if whenever $A \subseteq \bigcup \mathcal{C}$ for some $\mathcal{C} \subseteq \tau(X)$, there is a finite subcollection $\mathcal{F} \subseteq \mathcal{C}$ such that $A \subseteq \bigcup\{\mathrm{cl}_X U\colon U \in \mathcal{F}\}$. An H-set is always closed, and an H-closed set is an H-set. The discrete subset $\{(n,0)\colon n \in \mathbb{N}\} \cup \{+\infty\}$ of $\mathbf{U}$ is an H-set but is not H-closed. The remainder of an H-closed extension of a locally H-closed space is an H-set. Joseph has shown that for a subset A of an H-closed space X, $\{p \in X\colon p \in U \in \tau(X), \mathrm{cl}_X U \cap A \neq \emptyset\}$, denoted as $\mathrm{cl}_\theta A$, is an H-set. Note that for an open set U of a space X, $\mathrm{cl}_X U = \mathrm{cl}_\theta U$; this provides another proof that a regular open subset of an H-closed space is an H-set. A Hausdorff space is a **Katětov space** if it has a coarser minimal Hausdorff topology. A space is Katětov iff it is the remainder of an H-closed extension of a discrete space. This shows that every Katětov space is an H-set in some H-closed space. The converse of whether an H-set of an H-closed space is necessarily Katětov still remains unsolved, some 15 years after it was asked by Vermeer. Complete metric spaces, as well as regular, ***Lindelöf***, ***Čech-complete*** spaces are Katětov. Bourbaki showed that a countable H-closed space has a dense set of isolated points. An immediate consequence is that $\mathbb{Q}$ is not Katětov. A Hausdorff space is Katětov iff it is the perfect (not necessarily continuous) image of an H-closed space.

There is a natural link between H-closed spaces and ***extremally disconnected***, compact Hausdorff spaces. Let X be a Hausdorff space and θX denote the set of all open ultrafilters on X. For each open set U in X, let $OU = \{\mathcal{U}\colon U \in \mathcal{U}\}$. The topology on θX generated by $\{OU\colon U \in \tau(X)\}$ is extremally disconnected and compact Hausdorff (θX is in fact the ***Stone space*** generated by $\mathcal{RO}(X)$). The subspace $EX = \{\mathcal{U} \in \theta X\colon a(\mathcal{U}) \neq \emptyset\}$ is dense and extremally disconnected, $\theta X = \beta EX$ (the ***Čech–Stone compactification*** of EX), $\beta EX \setminus EX = \sigma X \setminus X$, and $\{Y \setminus EX\colon Y \in \mathcal{H}(EX)\} = \{Y \setminus X\colon Y \in \mathcal{H}(X)\}$. A Hausdorff space $Z \in \{Y \setminus X\colon Y \in \mathcal{H}(X)\}$ iff Z is the perfect (not necessarily continuous) image of $\sigma X \setminus X$. If X is locally H-closed and $|\sigma X \setminus X| \geqslant \omega$ and Z is a separable H-closed space, then $Z \in \{Y \setminus X\colon Y \in \mathcal{H}(X)\}$. A Hausdorff space X is H-closed iff EX is compact. If EX is Katětov, then so is X; the converse is false.

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d-21 Connectedness

The concept of connectedness appeared in Greek philosophy in the run of discussions on the inner structure of physical substances and geometrical figures. Although for physical substances the atomistic ideas prevailed, the geometrical figure was always treated by Greeks as a *continuum*, i.e., an object which is infinitely divisible and connected in a sense near to our intuitive meaning. It was Aristotle who penetrated the problem. The final conclusion of his reasoning was the claim that continuum cannot be composed of points. According to this conviction Euclid's *Elements* were written. For Euclid the points were nothing more than signs of places on the figure. The scholastic philosophers confirmed the Aristotle conviction, which was accepted by mathematicians till the nineteenth century rebuilding of mathematics.

The set point of view for the continuum appeared in Cauchy's *Analyse* and it was strongly emphasized in Bolzano's work. However, only Dedekind (1858, 1872) definitively rejected the Aristotle's objections concerning the point structure of continuum by constructing his *arithmetical continuum* composed of numbers which are called now *real*. The set of real numbers is linearly ordered and has neither jumps nor gaps. Intuitively these latter properties together imply connectedness of the linearly ordered set, but a precise description of the general notion of connectednes had to wait till the first years of the twentieth century, when point-set topology was developed.

The precise formulation of connectedness appeared in the Hausdorff's *Mengenlehre* (1914), but the notion was formulated earlier by N.J. Lennes (1905, 1906) and Frederic Riesz (1906, 1967). For these and other historical notes the reader is referred to the article of R.L. Wilder (1978) [15]. The book *Topology* by J.G. Hocking and G.S. Young [5] is the source of notions, references and basic theorems which are not explained here.

According to Riesz, and this is our contemporary definition, a *set* – now we prefer to say a ***space*** – is **connected** if it cannot be decomposed into two disjoint nonempty closed subsets. Since the complement of a closed subset is open, an equivalent formulation claims that the connected space cannot be decomposed into disjoint nonempty open subsets, or that it cannot contain proper nonempty subsets which are simultaneously closed and open, in short ***closed-and-open*** or ***clopen***. A space which is not connected is called **disconnected**. The term ***continuum*** is reserved for ***compact*** connected spaces.

We regard as non-trivial the proof that the real line, i.e., the set of real numbers with the usual topology – its ***order topology*** – is connected [5, Theorems 1–12]. For the same reasons other **lines**, i.e., sets ordered linearly without jumps and gaps and – in order to agree with commonly used terminology – without ends, endowed with the order topology are connected. A **jump** in an ordered set is a pair (a, b) of points with $a < b$ and with no x such that $a < x < b$ and a **gap** is a pair (A, B) of sets that cover the set and are such that $a < b$ whenever $a \in A$ and $b \in B$ *and* A has no maximum and B has no minimum. The real line is topologically distinguished among other lines by the property of being ***separable***. The ***long line***, i.e., the set of ordinal numbers less than ω_1, whose gaps are completed by segments of reals, is an example of a non-separable line. There are lines of arbitrarily high ***density***. The real line is ***homogeneous***. If a line has the power greater than continuum then it is not homogeneous; if a line is homogeneous then it is ***first-countable***; see M.A. Maurice (1970) [9] for a survey.

Continuous images of connected spaces are connected. The union of two connected sets is connected if the sets have points in common. A space is connected if it contains a connected dense subset. Products of connected spaces are connected, even if the number of spaces is infinite. So, since lines are connected, products of lines, in particular Euclidean spaces, are connected. The intersection of a descending chain of connected sets need not be connected.

Maximal connected subsets are called **component**s of the space. Components are closed and pairwise disjoint. Being connected, components are contained in, or are disjoint with, closed-open sets. The intersection of all closed-open sets containing a component is called a **quasi-component** of the space. Quasi-components need not be connected and then (and only in this case) can be bigger than the corresponding components. For compact Hausdorff spaces the difference between quasi-components and components vanishes.

A space is said to be **locally connected at a point** (Hahn 1915, Mazurkiewicz 1916), if each neighbourhood of the point contains connected neighbourhoods of this point. The well-known **$\sin(\frac{1}{x})$-curve** is not locally connected at points of the segment of condensation, and there are easy examples of plane sets, even continua, which are not locally connected at any point. The $\sin(\frac{1}{x})$-curve (or **topologists' sine curve**) is the closure of the graph $\{(x, \sin(\frac{1}{x})\colon 0 < x \leqslant 1\}$.

A space is said to be a **locally connected space** if it is locally connected at each point; the connectedness of the space is not assumed, so discrete spaces are locally connected. The real line is locally connected, as well Euclidean spaces. The product of infinitely many locally connected spaces need not be locally connected, unless the spaces are connected. For locally connected spaces quasi-components coincide with components. The components of open subsets are open and this property characterizes locally connected spaces.

Spaces whose all components are trivial are called **hereditarily disconnected**. If this is true for quasi-components, then the space is called **totally disconnected**. The notions are distinct; see [K II, 46 VI] for Sierpiński's (1921) and other examples.

An even stronger level of disconnectedness, in the realm of Hausdorff spaces, is given by the ***zero-dimensional*** spaces, i.e., those with a base consisting of closed-open subsets, with the Cantor set and the set of irrational numbers as non-trivial examples. There exist separable metric separable totally disconnected that are not zero-dimensional, see [E, 6.2.23–24].

Hereditarily disconnected spaces contain, by definition, no non-trivial connected subsets. Much wider is the class of spaces called **punctiform** which, by definition, contain no non-trivial continua. There exist non-trivial connected punctiform spaces, for instance the graph of the derivative of the famous Pompéiu function; see [K II, 47 IX] for references and comments (other examples will be mentioned later).

Using the non-effective method of Bernstein (1908) [5, p. 110], the plane can be divided into two sets without perfect subsets, thus into two punctiform sets. These sets are connected; Sierpiński (1920).

The existence of connected punctiform spaces shows that connectedness does not prevent singularities, even pathologies, in the realm of sets which do not lie on the real line (or on other line). In the run of this article, we will see much more such examples.

Continuity of a real function evidently implies the connectedness of the graph. The ***Darboux property*** is not sufficient to that purpose. But if a function is of the ***first Baire class*** and has the Darboux property, then its graph is connected. So, graphs of derivatives are connected, for instance, the just mentioned graph of the derivative of the Pompéiu function (see [K II, 47 IX]).

The graph of a solution of the Cauchy equation $f(x+y) = f(x) + f(y)$ must be totally disconnected if it is disconnected. Both possibilities can be realized, also in the case of discontinuous solutions; see the papers by F.B. Jones (1942) [6].

A deep insight into subtlety of connectedness was given by Knaster and Kuratowski (1922) [8], who studied connectedness in a general situation of an arbitrary topological space (called, as a rule, a set). One of the theorems proved there says that if a connected subset A is removed from a connected set X and $X \setminus A = U \cup V$, where U and V are open in $X \setminus A$, then $A \cup U$ and $A \cup V$ are connected. This theorem plays important role in many reasonings concerning connected sets and continua; see for instance Moore's theorem on the existence of non-separating points in continua, [5, Theorems 2–18]. According to another theorem of this paper, if a connected set consists of more than two points then it contains a proper non-trivial connected subset. A mysterious circumstance is that this theorem does not assure the existence of connected proper subsets which differ from the whole by more than finitely many points. This situation was not clear for a long time until Paul Erdős (1944) showed – attributing this result to A.H. Stone – that the plane connected sets (they must be infinite, in fact of the power of continuum) must contain connected subsets which differ from the whole by an infinite number of points. This is the best result in this direction, as M.E. Rudin (1958) [11] constructed – using CH – a connected set on the plane, each connected subset of which has a countable complement.

As an example illustrating the difficulties in the appearance of non-trivial small connected subsets, Knaster and Kuratowski constructed in the same 1921 paper their famous connected set, the **Knaster–Kuratowski fan**, with a **dispersion point**, see [E, 6.3.23] or [KII, § 46, II]. This set is a dense subset of the **Cantor fan** (the ***cone*** over the Cantor set). The vertex of the fan belongs to the set, on the lines ending in 'end points' of the Cantor set one takes the points with rational second coordinate and on the other lines those with irrational second coordinate. After removing the vertex, the set disconnects into single components, i.e., the complement of the vertex is hereditarily disconnected; a space cannot have more than one dispersion point, J.R. Kline (1922). R.L. Wilder constructed later (1927) yet more singular example in which the complement of the dispersion point is totally disconnected. The constructions given by Knaster and Kuratowski and that of Wilder were effective.

The non-effective method of Bernstein was later used by Swingle for developing (1931) the theory of **widely connected** spaces, i.e., connected spaces whose all non-trivial connected subsets are dense. They cannot have dispersion points. The widely connected sets were constructed by Swingle on the plane as subsets of ***indecomposable continua*** of the Knaster type (see [K II, 48 V] for examples).

The Knaster–Kuratowski fan – as it was noticed by the authors – is **biconnected**, which means that it is not the union of two disjoint non-trivial connected proper subsets (each such subset goes through the vertex). The existence of biconnected sets without a dispersion point was shown by E.W. Miller (1937) [10] by use of CH. Miller's set is widely connected in addition. It is not hardly to observe that widely connected sets are not necessarily biconnected. However, only M.E. Rudin (1995) [12] showed by using CH (in fact, the weaker MA suffices) that there exist biconnected sets without dispersion points which are not widely connected. Hence, the realms of widely connected sets and biconnected sets without dispersion points overlap. The sets of Miller and Rudin are constructed as subsets of indecomposable continua of the Knaster type. The same concerns the connected set of Rudin (1958) [11], which is easily seen to be widely connected.

The plane can be expressed as the sum of m disjoint biconnected sets for every cardinal m, $2 < m \leqslant$ continuum. This was shown by Erdös (1944) in continuation the research of Swingle (1932), who showed that the Euclidean space E^n, $n > 1$, can decomposed for every natural number r, $r \geqslant n + 1$, into r disjoint biconnected sets, and that for $r \leqslant n$ this is impossible.

For a survey and quotations concerning biconnected sets see [10] (till 1937) and [1].

A metric non-trivial connected space must be of the power of continuum and this remains true for ***completely regular*** spaces. Non-trivial ***regular*** connected spaces must be uncountable (Urysohn 1924). Among these spaces are examples of regular spaces on which each continuous real function is constant (see [E, 1.5.17, 2.7.17]).

However there exist countable connected Hausdorff spaces. The first examples came from Urysohn (1924). Among many other examples the best-known is that of Bing (1950) (see [5, p. 88]), but the most spectacular is that of Golomb (1961) [3]. The points in Golomb's example are the natural numbers and the topology is generated the arithmetic progressions $\{a + bn\colon n \in \mathbb{N}\}$ with $(a, b) = 1$. Dirichlet's famous theorem implies that the prime numbers are dense in this topology. The aim of several other constructions is to get additional properties, in particular local connectedness and the separation properties stronger than T_2 (but, what is clear, weaker than regularity). There are countable connected Hausdorff spaces with a dispersion point; P. Roy (1966), J.M. Martin (1966). For a survey of connected countable Hausdorff spaces see F.B. Jones and A.H. Stone (1971) [7].

If the topology is expanded, connectedness can be lost. However, the usual topology on the real line can be expanded non-trivially to the ***density topology*** which is connected. Not always a connected topology can be expanded to a topology which is **maximally connected**, i.e., which is connected and which can not be expanded non-trivially with preserving the connectedness; Baggs (1974). In maximal connected spaces – i.e., spaces with maximal connected topologies – dense subsets must be open, and the topologies with this property are called **submaximal**. Submaximal connected topology need not be maximally connected. Metric (non-trivial) spaces cannot be submaximal, as they can be always divided (non-effectively) into disjoint dense subsets. Known examples of submaximal Hausdorff topologies were obtained by the use of the axiom of choice. The density topology is not submaximal, thus not maximally connected. Surprisingly, the usual topology of the real line has an expansion to a maximally connected topology, as was shown by P. Simon (1978) [13] and J.A. Guthrie, H.E. Stone and M. Wage (1978) [4]. Connected spaces with a dispersion point are maximally connected. So, in view of the examples mentioned above, there exist maximal connected topologies on countable sets. Widely connected space are not maximally connected. A survey of maximally connected Hausdorff spaces can be found in [4]. There is also significant interest in finite T_0-spaces with these properties, see J.P. Thomas (1968) [14].

The real line can be treated as a ***connectification*** of its dense subsets. However – as was shown by A. Emeryk and W. Kulpa (1977) [2] – the Sorgenfrey line has no regular connectification, although it has Hausdorff connectifications. For more recent results see W.S. Watson and R.G. Wilson (1993); a survey is given in *Connectifications* in this volume.

Sierpiński (1918) (see [K II, 47 III) constructed a plane set which is not **σ-connected**, which means that it is the sum of countable collection of closed disjoint subsets. This is a counterpart to his theorem (from the same paper) according to which metric continua are σ-connected (there are no obstacles to extend the theorem to Hausdorff continua).

A space is said to be ***extremally disconnected*** if the closures of open subsets are open. Metric spaces cannot be extremally disconnected, unless they are discrete. Easy examples (for instance, Cantor discontinua) show that the existence of a base of closed–open sets does not imply extremal disconnectedness. The ***Čech–Stone compactification*** of the set of natural numbers with the discrete topology is extremally disconnected, but its remainder is not (see [E, 6.2.31]). Regular, even ***semiregular***, extremally disconnected spaces have bases of closed–open sets. Although the Hausdorff condition is not sufficient (R. Stephenson, personal communication) for this purpose, extremal disconnectedness presents, in a rather wide realm of spaces, a higher level of discontinuity of spaces than those mentioned before.

In the realm of compact (even locally compact) Hausdorff spaces most of pathologies of connectedness vanish. Biconnected and widely connected sets, thus the sets with dispersion points, do not appear. This is due to the Janiszewski Lemma (1912) (see [K II, 47 III]), which states that the closures of the components of open subsets of Hausdorff continua intersect the boundaries of these open sets (see also [5, Theorem 2.16]). This ensures the existence of non-trivial connected subsets which non-trivially differ from the whole space.

At point one enters a new chapter of the theory of connected spaces, namely the theory of ***continua*** with the emphasis on ***metric continua***.

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d-22 Connectifications

A topological space is said to be **connected** if the only open and closed subsets are the whole space and the empty set. This property was defined early in the development of Set-theoretic Topology and was extensively studied by the Polish school of topology and others in the 1920s and 1930s, but later studies of the topic were mostly confined to ***compact*** connected spaces otherwise known as continua. Another much studied concept of topology is that of a ***dense embedding*** and it is rather surprising that so little attention has been paid to dense embeddings in connected spaces. Prior to 1993, virtually the only reference to dense embeddings in connected spaces was a question asked by E. van Douwen at a conference in 1977 as to whether the ***Sorgenfrey line*** has a connected (Hausdorff) ***compactification***, a problem that was solved by Emeryk and Kulpa [4] in the negative shortly thereafter. Before we continue, in order to simplify the terminology, let us say that a connected space Y is a **connectification** of a space X, if X can be embedded as a dense subspace of Y (X is then said to be **connectifiable** and if $Y \setminus X$ is countable, then X is said to be **ω-connectifiable**). The questions then arise as to which spaces are connectifiable and ω-connectifiable. Without imposing further conditions, this question has a trivial answer: Every topological space X can be densely embedded in a connected topological space simply by adding one point $p \notin X$ and requiring that $X \cup \{p\}$ be the only neighbourhood of p. This connectification of X is even T_0 if X is. To make the problem interesting and more elegant, we thus require that if the space X satisfies some separation axiom T, then so should its connectification. For example, which T_1 (respectively, ***Hausdorff***, ***Tychonoff***, ***metric***) spaces have T_1 (respectively, Hausdorff, Tychonoff, metric) connectifications?

The first question is easily answered, indeed, it is a simple exercise to show that a T_1 space has a T_1 connectification if and only if it has no isolated points [11]. However, none of the other questions has a trivial answer and there is still no global characterization of the existence of connectifications in the classes of Hausdorff, Tychonoff and metric spaces.

Before we proceed further, we need the following definitions: A family of open sets $\mathcal{P}$ is a **π-base** for a topology τ if each non-empty element of τ contains an element of $\mathcal{P}$ and a Hausdorff space is said to be **H-closed** if it is closed in every Hausdorff space in which it is embedded; regular H-closed spaces are compact. Finally, a space is **feebly compact** if every ***locally finite*** family of open sets is finite. An H-closed space is feebly compact and a Hausdorff space with a ***σ-locally finite*** base is feebly compact if and only if it is H-closed.

The existence of Hausdorff connectifications was first studied in [11], where it was shown that a Hausdorff space X which has a ***σ-discrete*** π-base and such that every non-empty open subset has closure which is not H-closed has an ω-connectification. As immediate corollaries, it can be seen that the Sorgenfrey line has a Hausdorff ω-connectification (a result which was known earlier from [4]) and further, that every metric space which is not ***locally compact*** at any point is ω-connectifiable. The latter result was extended to metric spaces with no compact open subspaces in [10]. The requirement that a Hausdorff space have no H-closed open subspaces is easily seen to be necessary for the existence of a Hausdorff connectification, but it is not sufficient; in [11] and [10], many examples are given to illustrate this fact. Indeed, in [10], there is an example under CH of a first-countable ***T_4-space*** of cardinality $\mathfrak{c}$ with no compact open sets which has no Hausdorff connectification.

Connectifications of countable spaces were also considered in [11], where it was shown that a countable Hausdorff space has a Hausdorff connectification if and only if it has no isolated points. This is the only class of Hausdorff spaces for which a characterization of connectifiability is known.

It was shown by Bowers in [3] that each nowhere locally compact separable metric space can be densely embedded in the Hilbert space, and hence has a metric connectification, cf. [10]. This result was extended in [1], where it was proved that a separable (or even locally separable) metric space with no compact open subspaces has a metric connectification. The result of [10] mentioned above was further extended by Gruenhage et al. in [8], where it was shown that each nowhere locally compact metric space has a metric connectification. However, the above mentioned theorem of [1] does not extend to all metric spaces with no compact open subspaces: Using a novel construction, the authors of [8] have produced a metrizable space of weight $\mathfrak{c}^+$ with no compact open subspaces but which has no metric connectification. Somewhat surprisingly, this space has a Tychonoff connectification Y such that $Y \setminus X$ is a single point.

The problem of preserving ***dimension*** was considered in [2], where it was shown that if X is a separable metric space of dimension $n \geqslant 0$ with no compact open subspaces, then there exists a connected metric compactification of X of the same dimension if $n > 0$ or of dimension 1 if $n = 0$. This last result should be contrasted with that of a much earlier paper [9] (an article not directly concerned with connectifications), where it was shown that a ***strongly zero-dimensional*** metric space (with more than one point) has a compact ***linearly ordered*** (hence of dimension 1) connectification if and only if it has no compact open sets.

Given that the problem of connectifications arose from a question concerning the Sorgenfrey line S, it is not surprising that connectifications of this space have been studied by a number of authors. As mentioned previously, Emeryk and Kulpa [4], showed that this space has a Hausdorff but no regular connectification. It was further shown in [2] that S has a Urysohn connectification and in [7] that is has a ***locally pathwise connected*** Hausdorff connectification. The existence of ***locally connected*** Hausdorff connectifications was studied more generally in [6] and it was proved there that each Hausdorff space X with $\pi w(X) \leqslant 2^{\mathfrak{c}}$ and such that every open set has closure which is not feebly compact, has a locally connected Hausdorff connectification. As a consequence, a countable Hausdorff space has a locally connected Hausdorff connectification if and only if it has no isolated points. On the other hand, an example was given in [2] of a separable metric space with no compact open sets but which has no locally connected Hausdorff connectification.

Pathwise connected connectifications have also received some attention. In [5], it was shown that a countable, first-countable Hausdorff space has a connectification which is pathwise connected if and only if it has no isolated points. In this same article, many examples of spaces which are not pathwise connectifiable are given, but no other positive results are known.

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d-23 Special Constructions

General Topology is rife with examples; the article on *Special spaces* contains a selection of spaces that 'every young topologist should know'. Many of the articles in this volume describe or refer to further examples, some specific to a problem and others of general use. The present article describes some techniques for constructing spaces that have become part of the topologist's standard toolkit.

1. Spaces of maximal systems

A versatile and oft-used method for constructing topological spaces is by means of maximal systems. Various instances of this method can be found in other articles in the volume, we mention the ***Stone space*** of a ***Boolean algebra*** and the ***Wallman representation*** of a distributive lattice. Both have as points the maximal ***filter***s on the structure. For an element a of the structure let a^+ denote the set of all maximal filters that contain a, i.e., $a^+ = \{x\colon a \in x\}$; the family of all a^+ is used as a ***subbase*** for the closed sets of the space, in the cases of lattices (and hence Boolean algebras) one even obtains a ***base***.

The Stone space
The ***Stone duality*** between Boolean algebras and ***compact zero-dimensional*** spaces provides a rich source of objects on either side, with neither side the clear winner. The ***measure algebra*** gives us a compact ccc non-separable space courtesy of this duality.

Various compactifications of $\mathbb{N}$ have been constructed by constructing suitable subalgebras of $\mathcal{P}(\mathbb{N})$, see [KV, Chapter 11] for Bell's compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ with $\gamma\mathbb{N} \setminus \mathbb{N}$ ccc non-separable and [14] for more examples.

One way of defining the **absolute** (or **Gleason space**) of a topological space X is by way of the Stone space of the algebra of ***regular open*** sets. The absolute EX is the set of converging ultrafilters and the natural map $\pi_X : EX \to X$ simply assigns the limit to each ultrafilter. See [12] for a thorough treatment of this subject.

Cliques in graphs and hypergraphs
The Stone space of a Boolean algebra B can in a natural way be identified with a subspace of the ***Cantor cube*** $\{0,1\}^B$; indeed, the ultrafilter x corresponds to the associated homomorphism $\varphi_x : B \to \{0,1\}$, defined by $\varphi_x(a) = 1$ iff $a \in x$. The subspace topology corresponds to the Stone space topology.

This idea can be used for other structures as well; we consider one very successful instance, that of cliques in graphs and hypergraphs.

A **graph** consists of a set V of points and a subset E of $[V]^2$, called the edges. A **clique** is a subset C of V such that $[C]^2 \subseteq E$. One immediately obtains the space of maximal cliques, as the points x of 2^V for which $x^{\leftarrow}(1)$ is a maximal clique. This space is zero-dimensional but not necessarily compact; its properties are, however, intimately connected with the partial order of finite cliques.

In [10] one finds a construction, using CH, of a two complementary graphs E_1 and E_2 on ω_1 such that the spaces of maximal cliques satisfy the ccc, but their product does not. The ccc translates into the following statement: whenever $\mathcal{F}$ is an uncountable family of finite cliques there are $F, G \in \mathcal{F}$ such that $F \cup G$ is a clique as well. Because the graphs are complementary the basic clopen sets $\{(x, y)\colon x(\alpha) = y(\alpha) = 1\}$ in the product are disjoint.

The space of *all* cliques (plus the empty set) is compact [1] and it has been used to produce a first-countable compact space that is not a continuous image of $\mathbb{N}^*$ (the graph is a Cohen-generic subset of ω_2).

Whether a set is a clique depends on its finite subsets. A family of sets with this property has been called an **adequate family**: a family $\mathcal{A}$ of subsets of a set V is adequate if every singleton belongs to it, it is closed under taking subsets and $A \in \mathcal{A}$ iff every finite subset of A belongs to $\mathcal{A}$. Such families have been used to construct various interesting compact spaces, see [13] and [KV, Chapter 23] for applications to the theory of Banach spaces.

Maximal threads in complexes
It is possible to put a compact topology on the family of maximal elements of an adequate family; this was done in [16] and is more in the spirit of the Stone representation: given a set V, an **abstract complex** is a family K of finite subsets of V such that $t \in K$ and $s \subseteq t$ always implies $s \in K$. A **thread** in the complex is a subset W of V with $[V]^{<\aleph_0} \subseteq K$ (so the set of threads is an adequate family). The set of all maximal threads can be topologized by using the family $\mathcal{S} = \{v^+\colon v \in V\}$ as a subbase for the closed sets, where v^+ denotes the set of maximal threads that contain v. The resulting space is compact and T_1 (not necessarily Hausdorff) and every compact T_1***-space*** can be obtained in this way (see the article on *Wallman–Shanin compactification* in this volume).

In case V is a graph and K is the family of all finite cliques this yields a compact T_1 topology on the set of all maximal cliques; in fact this topology is ***supercompact***: the natural subbase $\mathcal{S}$ is a ***binary subbase***, for a family $\{v^+\colon v \in W\}$ is linked iff W is a clique. See *Topological Characterizations of Spaces* for an application.

The Hull–Kernel topology
Given a ring R let M_R denote the set of its maximal ideals. The **hull-kernel topology** is defined by specifying a ***closure operator***, thus: if I is a set of maximal ideals then its closure is defined to be $\{m \in M: \bigcap I \subseteq m\}$, the 'hull' of the 'kernel'.

When applied to the ring $C(X)$ of real-valued continuous functions on the ***completely regular*** space X one obtains the ***Čech–Stone compactification*** of X.

2. Recursive constructions

Lots of examples have been constructed by recursion. The basic idea is to construct a topology on a well-ordered set X by specifying neighbourhoods at each point, using the neighbourhoods of the points that come before in the well-order. We mention some well-known examples.

Ostaszewski's space
This is a ***perfectly normal countably compact*** noncompact space. The underlying set is ω_1 with its natural order; the main ingredients in the construction are an enumeration of $[\omega_1]^{\aleph_0} = \{A_\alpha: \alpha \text{ a limit}\}$ (with $A_\alpha \subseteq \alpha$) and a ***♣-sequence*** $\{B_\alpha: \alpha \text{ a limit}\}$ – this construction is under the assumption of the $\diamondsuit$-principle. We define a topology τ_α on α, such that (β, τ_β) is an open subspace of (α, τ_α) whenever $\beta < \alpha$.

When α is a limit we first let τ_α be the ***direct limit*** topology of the topologies τ_β with $\beta < \alpha$, i.e., $O \in \tau_\alpha$ iff $O \cap \beta \in \tau_\beta$ for all $\beta < \alpha$. Then we consider A_α and B_α; the latter set is closed and discrete in α. If A_α has an accumulation point in α we ignore it; otherwise we enumerate $A_\alpha \cup B_\alpha$ (in a one-to-one fashion) as $\{x_n: n \in \mathbb{N}\}$. Find a disjoint clopen cover $\{C_n: n \in \mathbb{N}\}$ of α such that $x_n \in C_n$ for all n. Finally then write $\mathbb{N}$ as a disjoint union $\bigcup_i D_i$ of infinite sets such that each D_i has infinitely many n with $x_n \in A_\alpha$ and with $x_n \in B_\alpha$. Finally then a local base at $\alpha + i$ consists of the sets $\{\alpha + i\} \cup \bigcup\{C_n: n \in D_i,\ n \geqslant m\}$ ($m \in \mathbb{N}$).

One checks that $\operatorname{cl} B_\alpha = B_\alpha \cup [\alpha, \omega_1)$ for all α, that every countably infinite subset has an accumulation point and that every set α is open. This gives us the desired properties of the space. In the course of the construction one should verify that (α, τ_α) is always metrizable and, if one starts in the right way, locally compact; this will enable one to keep going. Observe that the ♣-principle also implies that the resulting space is hereditarily separable and hence an S-space.

Kunen's line
This is an S-space topology on the real line. Its construction is like the previous one, but using a bit of the structure of $\mathbb{R}$. The main ingredient is an enumeration $\{A_\alpha: \alpha < \omega_1\}$ of the family of countably infinite subsets of $\mathbb{R}$ and an enumeration $\{x_\alpha: \alpha < \omega_1\}$ of $\mathbb{R}$ itself – this construction works under CH. With the aid of the metric structure one can now recursively define local bases at x_α so that the following holds in the end: if $\beta < \alpha$, $A_\beta \subseteq \{x_\gamma: \gamma < \alpha\}$ and $x_\alpha \in \operatorname{cl}_{\mathbb{R}} A_\beta$ then $x_\alpha \in \operatorname{cl} A_\beta$.

This makes the resulting space hereditarily separable: if $A \subseteq \mathbb{R}$ then $A \subseteq \operatorname{cl}_{\mathbb{R}} A_\alpha$ for some α and then $A_\alpha \cup (A \cap \{x_\beta: \beta < \alpha\})$ is dense in A.

Van Douwen's line
This is a topology on the real line that is constructed much like in the previous example but in ZFC only. The main ingredient now is a listing $\{\langle K_{\alpha,n}: n \in \mathbb{N}\rangle: \alpha < \mathfrak{c}\}$ of all sequences of countable subsets of $\mathbb{R}$ with $\bigcap_n \operatorname{cl}_{\mathbb{R}} K_{\alpha,n}$ uncountable. As above, the construction is set up to retain this property and to make the topology locally compact and locally countable. The resulting space is ***normal***, ***countably paracompact***, ***separable*** but not ***paracompact*** and not hereditarily normal. This idea has proved very versatile; van Douwen's [5] contains many more applications and offers a good introduction to this method.

3. Resolutions

A totally different type of construction is the resolution, which is a way of replacing each point in a space by a copy of some other space, possibly a different one for each point. This is done as follows. We are given a space X, and for each $x \in X$ a space Y_x and a continuous map $f_x: X \setminus \{x\} \to Y_x$. The **resolution** of X (at each x, into Y_x, by f_x) is the set $R(X, Y_x, f_x) = \bigcup_{x \in X}\{x\} \times Y_x$.

To define the topology we define for each pair (U, V), where U is open in X and V is open in Y_x for some $x \in U$, the set

$$U \otimes V = \{x\} \times V \cup \bigcup\bigl\{\{x'\} \times Y_{x'}: x' \in U \cap f_x^{-1}[V]\bigr\}.$$

The family of all such sets is a base for a topology on $R(X, Y_x, f_x)$, the **resolution topology**; we usually suppress mention of the spaces Y_x and the maps f_x and write $R(X)$ for the resolution. Generally the space X is assumed to be ***completely regular*** and the spaces Y_x are assumed to be compact.

The natural map $\pi: R(X) \to X$ (we shall call this a resolution too) is ***continuous*** and ***closed*** (if each Y_x is compact), and for each x the map $y \mapsto (x, y)$ is an embedding of Y_x into $R(X)$, so $R(X)$ is indeed obtained by replacing each x with Y_x. When all spaces involved are compact Hausdorff then so is the resolution (and conversely). The familiar $\sin\frac{1}{x}$-curve is a resolution: one takes $X = [0, 1]$, $Y_0 = [-1, 1]$ and $f_0(y) = \sin\frac{1}{y}$, and for $x > 0$ simply $Y_x = \{x\}$ and $f_x(y) = x$. The resolution process enables one to do this at all points of $[0, 1]$ at once: just take $Y_x = [-1, 1]$ and $f_x(y) = \sin\frac{1}{y-x}$ for all x. The resulting space is a first-countable ***chainable continuum***, whose ***small inductive dimension*** is two.

If each Y_x is compact, then the resolution map $\pi: R(X) \to X$ is a **fully closed map** (also called a **strongly closed map**); a map $f: X \to Y$ is fully closed if for every $y \in Y$ and every finite open cover $\mathcal{U}$ of $f^{-1}(y)$ the set $\{y\} \cup \bigcup\{f^{\#}[U]: U \in$

$\mathcal{U}\}$ is open, where $f^{\#}[U]$ denotes the **small image**, i.e., the set $\{z\colon f^{-1}(z) \subseteq U\} = Y \setminus f[X \setminus U]$.

A fully closed surjection $f : X \to Y$ between compact Hausdorff spaces is almost a resolution. The added requirement comes from considering, for every $y \in Y$ the decomposition $\{\{x\}\colon f(x) = y\} \cup \{f^{-1}(z)\colon z \neq y\}$ of X and the quotient space Y^y (this is Y with just y replaced by $f^{-1}(y)$). If f is fully closed *and* if for every y the fibre $f^{-1}(y)$ is a retract of Y^y then f is a resolution map. This operation may also be used to characterize fully closed maps themselves: a perfect map $f : X \to Y$ with regular domain is fully closed iff every Y^y is regular.

Fully closed maps almost preserve ***covering dimension*** between domain and range. If $f : X \to Y$ is a fully closed map between normal spaces then

$$\dim Y \leqslant \dim X + 1$$

and

$$\dim X \leqslant \max\{\dim Y, \dim f\},$$

where

$$\dim f = \sup\{\dim f^{-1}(y)\colon y \in Y\}.$$

The inductive dimensions of resolutions can be made high by using so-called ring maps: a surjective map $f : X \to Y$ is a **ring map** at $y \in Y$ if for every $x \in f^{-1}(y)$ and neighbourhoods O_x of x and O_y of y the set $O_y \cap f^{\#}[O_x]$ contains a partition between y and $Y \setminus O_y$; we say f is a ring map if it is a ring map at every point of Y. The resolution map $\pi : R(X) \to X$ is a ring map at $x \in X$ iff for every $y \in Y_x$ and neighbourhoods O_y of y and O_x of x the intersection $O_x \cap f_x^{-1}[O_y]$ contains a partition between x and $X \setminus O_x$.

The $\sin \frac{1}{x}$ resolutions described above are ring resolutions. In raising inductive dimensions the following result is often used. If $f : X \to Y$ is a monotone ring map of the compact space onto an n-dimensional ***Cantor manifold*** Y, where $n \geqslant 2$, then every partition of X contains some fibre of f. Thus resolving each point of $[0, 1]^n$ into $[0, 1]^n$ by means of ring maps yields an n-dimensional first-countable compact space without $(n - 1)$-dimensional partitions. To do so fix a dense subset $\{d_n\colon n \in \mathbb{N}\}$ of $[0, 1]^n$ and for each x a local base $\{U_n^x\colon n \in \mathbb{N}\}$ with $\operatorname{cl} U_{n+1}^x \subseteq U_n^x$; by ***Urysohn's Lemma*** we can find $f_x : [0, 1]^n \setminus \{x\} \to [0, 1]^n$ such that $f_x[U_n^x] = \{d_n\}$. This can be iterated to produce an inverse sequence of spaces whose limit is first-countable and n-dimensional but whose closed subsets are either n- or zero-dimensional.

One can extend such constructions to inverse sequences of length ω_1; using the ***Diamond Principle*** one can then construct compact S-spaces of cardinality $2^{\mathfrak{c}}$, higher-dimensional versions of Ostaszewski's space and perfectly normal n-manifolds with $n < \dim < \operatorname{Ind}$.

It is clear from the definition that for each x the map f_x need only be defined on a neighbourhood of x. The resolution process can be generalized to replaced subsets by products. We are given a space X, a family $\{O_\alpha\}_\alpha$ of open subsets of X and for each α a subset G_α closed in O_α and a map $f_\alpha : O_\alpha \setminus G_\alpha \to Y_\alpha$. For each α let $X_\alpha = X \setminus G_\alpha \cup G_\alpha \times Y_\alpha$, topologized by using all sets of the form $U \setminus G_\alpha \cup (U \cap G_\alpha) \times Y_\alpha$ (U open in X) and $U \cap f_\alpha^{-1}[V] \cup (U \cap G_\alpha) \times V$ (U open in X and V open in Y_α) as a base for the open sets. The obvious map $\pi_\alpha : X_\alpha \to X$ is continuous; the resolution of X by all the maps f_α is the subspace of the product $\prod_\alpha X_\alpha$ consisting of those points x for which $\pi_\alpha(x_\alpha) = \pi_\beta(x_\beta)$ for all α and β. This type of resolution was used to construct a compactification that is not a Wallman–Shanin compactification.

Resolutions were defined by Fedorchuk in [7]; a comprehensive introduction with many examples is given in [HvM, Chapter 20]. Some more applications are described in the references. The generalized resolution was introduced by Ul'janov in [15].

4. Elementary substructures

Strictly speaking this is not a method of constructing examples but rather a general stratagem that helps one to avoid laborious inductive proofs and recursive constructions. An **elementary substructure** (or **elementary submodel**) of the universe V is a set M with the following property: if $n \in \mathbb{N}$, $(a_1, \ldots, a_n) \in M^n$ and φ is a set-theoretic formula such that there is some x in V for which $\varphi(x, a_1, \ldots, a_n)$ holds then there is a c in M for which $\varphi^M(c, a_1, \ldots, a_n)$ holds. Here φ^M denotes the formula φ but with every existential quantifier $\exists z$ replaced by $\exists z \in M$.

This notion takes some time to get used to, mainly because of its seemingly abstract nature. A helpful analogy is that of algebraic closure: every algebraic equation with algebraic numbers for parameters has its solutions, if any, in the set of algebraic numbers. If one interprets $\varphi(x, a_1, \ldots, a_n)$ as an equation with parameters from M, then elementarity says that if the equation has a solution then at least one of these solutions is in M and the fact that it is a solution can be checked within M. The Löwenheim–Skolem Theorem gives us a large supply of elementary substructures of V: for every set X there is an M with $X \subseteq M$ and $|M| \leqslant \aleph_0 \cdot |X|$. The proof of this theorem simply subsumes all inductions and recursions that we may wish to perform. Thus once the phrase "let M be an elementary substructure of the universe" is uttered we have already performed the construction we intended to perform. It takes a slightly different mindset to work this way but the two examples below may be contrasted with the standard proofs by recursion. Experience shows that once one gets into this 'elementarity mindset' one gains a powerful tool for discovering results and proofs that would otherwise stay out of reach. The main new aspect here is the interplay between the external and internal views of the substructure; this is something that is very hard to obtain in 'normal' inductive and recursive situations.

Dow's articles [6] and [HvM, Chapter 4] are a good places to start acquiring the mindset needed to work with elementary substructures of the universe.

Continuous functions on ω_1

The well-known theorem that any continuous function $f:\omega_1 \to \mathbb{R}$ is constant on a tail can easily be proved by elementarity. Simply take a *countable* elementary substructure of the universe with $f \in M$. The sets ω, ω_1 and $\mathbb{R}$ automatically belong to M because they are unique solutions to equations without any parameters at all. In fact even $\omega \subset M$ because each individual integer is also unique solution to such an equation. Therefore if $A \in M$ and A is countable then $A \subset M$: there must be a solution to "x is a surjection from ω onto A" in M, but then, again by uniqueness, $x(n) \in M$ for all n. This all shows that $\delta_M = M \cap \omega_1$ is a countable ordinal. Given $n \in \omega$ there is $\beta < \delta_M$ such that $|f(\gamma) - f(\delta_M)| < 2^{-n-1}$ for $\gamma \in [\beta, \delta_M]$. Let $\varphi(x, \beta, f)$ denote $x \in [\beta, \omega_1) \wedge |f(x) - f(\beta)| \geqslant 2^{-n}$. There is no solution to $\varphi^M(x, \beta, f)$ in M, hence there is none to $\varphi(x, \beta, f)$ in V. Therefore $|f(x) - f(\beta)| < 2^{-n}$ for all $x \geqslant \beta$. Combining this we find that f is constant on $[\delta_M, \omega_1)$ (and, in restrospect and by elementarity, even on $[\gamma, \omega_1)$ for some $\gamma \in M$).

Arkhangel'skiĭ's Theorem

Consider a first-countable compact Hausdorff space X; we wish to show that $|X| \leqslant \mathfrak{c}$. The proof sketched in the article *Cardinal functions I* is tailor-made for the elementary-submodel approach. One takes an elementary substructure M of the universe, of cardinality $\mathfrak{c}$ with $X \in M$ and such that all countable subsets of M are elements of M. One proves first that $X \cap M$ is closed in X: if $x \in \mathrm{cl}(X \cap M)$ then x is the limit of a sequence from $X \cap M$, this sequence is an element of M and hence so is its limit, i.e., $x \in M$. Second one proves that if $y \in X \setminus M$ then there is a finite family $\mathcal{O}$ of open sets *in* M that covers $X \cap M$ but with $y \notin \bigcup \mathcal{O}$ – this is possible because M must contain a countable local base at each point of $X \cap M$. But now $x \in X \setminus \bigcup \mathcal{O}$ has no solution in M, whereas it does have a solution in V.

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E: Spaces with richer structures

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e-1 Metric Spaces

We can intuitively understand a distance from one point to another point. M. Fréchet [5] gave a mathematical formulation, called a metric, of a distance by abstracting the real line, plane and the three-dimensional space. Metric spaces appear in all areas of mathematics as a fundamental concept.

Let X be a set. A non-negative real valued function ρ defined on $X \times X$ is called a **metric** on X if ρ satisfies the following conditions: (1) $\rho(x, y) = 0$ if and only if $x = y$, (2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$, (3) $\rho(x, y) + \rho(y, z) \geqslant \rho(x, z)$ for all $x, y, z \in X$. The third condition is called the **triangle inequality**. A **metric space** is a pair (X, ρ) of a set X and a metric ρ on X. A metric ρ is called an **ultrametric** (or a **non-Archimedean metric**) if ρ satisfies the following **strong triangle inequality**: (3)′ $\max\{\rho(x, y), \rho(y, z)\} \geqslant \rho(x, z)$ for all $x, y, z \in X$. Some results on ultrametric spaces will be discussed later. A metric ρ is called a **bounded metric** if $\sup\{\rho(x, y)\colon x, y \in X\} < \infty$.

For a metric space (X, ρ), elements of X are called **point**s and the value $\rho(x, y)$ is called the **distance** between x and y. We give some examples of metric spaces. (a) Let $\mathbb{R}^n = \{(x_1, x_2, \ldots x_n)\colon x_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n\}$ and $\rho(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$, where $\mathbb{R}$ denotes the set of real numbers. The metric space $(\mathbb{R}^n, \rho)$ is the n-dimensional **Euclidean space** and its topology is called the **Euclidean topology**. (b) Let H be the set of sequences $(x_1, x_2, \ldots)$ of real numbers with $\sum_{i=1}^\infty x_i^2 < \infty$. For $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots) \in H$, we define $\rho(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^\infty (x_i - y_i)^2}$. Then ρ is a metric on H and the metric space (H, ρ) is called the **Hilbert space**. (c) Let τ be an infinite cardinal number and A a set of cardinality τ. Let I_α, $\alpha \in A$, be the copies of the unit interval $[0, 1]$. We identify all 0's in $\bigcup\{I_\alpha\colon \alpha \in A\}$ and denote the quotient set by $J(\tau)$. We define a metric ρ on $J(\tau)$ as $\rho(x, y) = |x - y|$ if $x, y \in I_\alpha$ for some $\alpha \in A$, and $\rho(x, y) = |x + y|$ if $x \in I_\alpha$, $y \in I_\beta$ with $\alpha \neq \beta$. The metric space $(J(\tau), \rho)$ is called the **hedgehog** (or **star space**) of ***weight*** τ. (d) Let τ be an infinite cardinal number and A a set of the cardinality τ. Let $\mathbb{R}_\alpha$, $\alpha \in A$, be copies of the real line $\mathbb{R}$. Let $H(A) = \{(x_\alpha) \in \prod_{\alpha \in A} \mathbb{R}_\alpha\colon \sum_\alpha x_\alpha^2 < \infty\}$ and $\rho((x_\alpha), (y_\alpha)) = \sqrt{\sum_{\alpha \in A}(x_\alpha - y_\alpha)^2}$, for $(x_\alpha), (y_\alpha) \in H(A)$. Then $(H(A), \rho)$ is a metric space called the **generalized Hilbert space** of weight τ. The generalized Hilbert space of weight τ is also written as H^τ. (e) Let κ be an infinite cardinal number and A a set of the cardinality κ. Let $B(\kappa)$ be the Cartesian product of countably many copies of A, i.e., $B(\kappa) = \prod_{i=1}^\infty A_i$, where each A_i is a copy of A. For each $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in B(\kappa)$ we define $\rho(x, y) = \frac{1}{\min\{i\colon x_i \neq y_i\}}$ if $x \neq y$, and $\rho(x, y) = 0$ if $x = y$. Then ρ is an ultrametric on $B(\kappa)$. The metric space $(B(\kappa), \rho)$ is called the **Baire space of weight** κ, $B(\kappa)$. It is well known that $B(\aleph_0)$ (**Baire's zero-dimensional space**) is ***homeomorphic*** to the space of irrationals. In fact, the map f of $B(\aleph_0)$ to the space of irrationals defined by use of continued fraction, $f(x_1, x_2, \ldots) = x_1 + \frac{1}{x_2+} \frac{1}{x_3+} \frac{1}{x_4+} \cdots$, is a ***homeomorphism***. (f) Let X be a set and (Y, ρ) a bounded metric space. Let $F(X, Y)$ be a set of maps from X to Y. For each $f, g \in F(X, Y)$ we define $d(f, g) = \sup\{\rho(f(x), g(x))\colon x \in X\}$. Then d is a metric on $F(X, Y)$.

Let (X, ρ) be a metric space and A a subset of X. It is trivial that the restriction of ρ on $A \times A$ is a metric on A. The metric space $(A, \rho|_{A \times A})$ is called a **metric subspace** of (X, ρ). We usually write (A, ρ) as a metric subspace of (X, ρ) without any confusion. Let A and B be subsets of a metric space (X, ρ). We define the distance $\rho(A, B)$ between A and B by $\rho(A, B) = \inf\{\rho(a, b)\colon a \in A$ and $b \in B\}$ and the **diameter** $\operatorname{diam} A$ of A by $\operatorname{diam} A = \sup\{\rho(a, a')\colon a, a' \in A\}$.

A most important property of metrics that every metric induces a ***topology***. For a metric space (X, ρ), each point $x \in X$, and $\varepsilon > 0$, the set $S_\varepsilon(x) = \{y \in X\colon \rho(x, y) < \varepsilon\}$ is called an **ε-ball** (or **ε-neighbourhood**) about (of) x. We can induce a topology $\mathcal{T}_\rho$ as follows: $\mathcal{T}_\rho = \{U \subset X$: for each point $x \in U$, there exists an ε-ball about x such that $S_\varepsilon(x) \subset U\}$. The topology $\mathcal{T}_\rho$ is called the **topology induced by** ρ. Usually, we consider a metric space (X, ρ) as a topological space endowed with the topology induced by ρ, and a metric space (X, ρ) is simply written by X in cases where there is no confusion. Two metrics on the same set are called **equivalent** if the induced topologies coincide.

Conversely, a ***topological space*** X is said to be a **metrizable space** if there is a metric ρ on X such that ρ induces the original topology of X. Then the metric ρ is called a **compatible metric** on a metrizable space X, and X is said to be **admit** the metric ρ. Every ***discrete space*** X is metrizable by the metric $\rho(x, y) = 1$ if $x \neq y$, $\rho(x, x) = 0$. The ***product space*** $\prod_{i=1}^\infty (X_i, \rho_i)$ of countable many metrizable spaces (X_i, ρ_i), $i = 1, 2, \ldots$ is also metrizable. In fact, it is easy to see that the topology induced by the metric $\rho((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sqrt{\sum_{i=1}^\infty \frac{1}{2^i}(\rho_i(x_i, y_i))^2}$ coincides with the ***product topology***. On the other hand, the product of uncountably many metrizable spaces all of whose factor spaces have at least two points can not be metrizable.

One of the most interesting problems in general topology is the *metrization problem*: When is a topological space metrizable? Many metrization theorems were obtained by P.S. Alexandroff, P.S. Urysohn, Yu.M. Smirnov, J. Nagata, R.H. Bing, R.L. Moore, etc. The class of metrizable spaces is the one of the most important classes of spaces in general topology, because this class is the origin of many branches of general topology, for example, generalized metric spaces, covering properties, and uniform spaces. The topic of metrization, and the other areas mentioned above, are separately discussed in other parts of the Encyclopedia. The theory of metrizable spaces is also closely related to dimension theory. We refer the reader to [13, Chapter V] for the relationship of the metrization theorems and dimension theory.

Metrization theorems are closely related to the concept of universal spaces. Let $\mathcal{P}$ be a class of spaces. We call a space X is a **universal space** for the class $\mathcal{P}$ provided that $X \in \mathcal{P}$ and every $Y \in \mathcal{P}$ is homeomorphic to a ***subspace*** of X. The Hilbert space is a universal space for ***separable*** metrizable spaces and both of the generalized Hilbert space of weight τ and the product of countably many copies of hedgehog spaces $J(\tau)$ of weight τ are the universal spaces for metrizable spaces of weight τ. Furthermore, the Baire's zero-dimensional space $B(\tau)$ is a universal space for ultrametrizable spaces of weight τ.

Metrizable spaces have nice topological properties. It is easy to see that every metrizable space is ***Hausdorff*** and satisfies the ***first axiom of countability***. Moreover, A.H. Stone [15] proved that every metrizable space is ***paracompact***. In the class of metrizable spaces, ***second-countability***, ***separability*** and ***Lindelöfness*** are equivalent. Further, a metrizable space X is separable if and only if there is a compatible metric ρ which is totally bounded, where a metric ρ is **totally bounded** if for each $\varepsilon > 0$ there are finite points $x_1, x_2, \ldots, x_n$ of X such that $X = S_\varepsilon(x_1) \cup S_\varepsilon(x_2) \cup \cdots \cup S_\varepsilon(x_n)$.

Let (X, ρ) and (Y, σ) be metric spaces, $f : X \to Y$ a map and $x \in X$. Then f is **continuous at** x if for each $\varepsilon > 0$ there is $\delta > 0$ such that $\sigma(f(x), f(y)) < \varepsilon$ whenever $\rho(x, y) < \delta$. If f is continuous at every point of X, then f is said to be a **continuous map**. If we can choose $\delta > 0$ independently of the choice of $x \in X$, then we say f is a **uniformly continuous map**, i.e., f is said to be uniformly continuous if for each $\varepsilon > 0$ there is $\delta > 0$ such that $\sigma(f(x), f(y)) < \varepsilon$ whenever $\rho(x, y) < \delta$. It is well-known that a map $f : X \to Y$ of a metric space X to a metric space Y is continuous if and only if $f^{-1}(U)$ is an ***open set*** of X for each open set U of Y. The continuity of maps between metric spaces is also characterized in terms of the convergence of sequences. A sequence $\{x_n : n = 1, 2, \ldots\}$ in a metric space (X, ρ) **converges** to $x \in X$ if for every $\varepsilon > 0$ there is an n_0 such that $\rho(x_n, x) < \varepsilon$ whenever $n \geqslant n_0$. Then the point x is called a **limit point** of the sequence $\{x_n : n = 1, 2, \ldots\}$ and we write as $x = \lim_{n\to\infty} x_n$. A map $f : (X, \rho) \to (Y, \sigma)$ is continuous at x if and only if for every sequence $\{x_n : n = 1, 2, \ldots\}$ with $\lim_{n\to\infty} x_n = x$, $f(x) = \lim_{n\to\infty} f(x_n)$. It is clear that every uniformly continuous map is continuous and every continuous map from a ***compact*** metric space is uniformly continuous. A map $f : (X, \rho) \to (Y, \sigma)$ is called a **Lipschitz continuous** with constant $\alpha \geqslant 0$ if $\sigma(f(x), f(y)) \leqslant \alpha\rho(x, y)$ for every $x, y \in X$. Furthermore, a map f of a metric space (X, ρ) into itself is called a **contraction** map if f is a Lipschitz continuous map with constant α for some $\alpha \in [0, 1)$.

Perfect maps preserve metrizability, but closed maps do not. More precisely, for a closed continuous map $f : X \to Y$ from a metrizable space X onto a topological space Y the following are equivalent ([12] and [16]): (a) Y is metrizable, (b) Y is first-countable, (c) the ***boundary*** of $f^{-1}(y)$ is compact for each $y \in Y$. K. Morita also proved that every metric space of weight τ is a perfect image of a subspace of the Baire's zero-dimensional space of weight τ.

The ***Nagata–Smirnov Metrization Theorem*** leads to the notion of a strong metrizability. A ***regular*** space X is called **strongly metrizable** if X has a base which is the union of countably many ***star-finite*** open coverings. It follows from Nagata–Smirnov metrization theorem that every strongly metrizable space is metrizable. Every separable metrizable (more generally, ***strongly paracompact*** metrizable) space is strongly metrizable. The product space $B(\tau) \times (0, 1)$ of the Baire's zero-dimensional space $B(\tau)$ and the open interval $(0, 1)$ is an example of a strongly metrizable space which is not strongly paracompact. The product $B(\tau) \times I^\omega$ of the Baire's zero-dimensional space $B(\tau)$ and the Hilbert cube I^ω is a universal space for strongly metrizable (and strongly paracompact metrizable) spaces of weight τ.

We shall discuss ultrametric spaces. Ultrametric spaces are also called non-Archimedean metric spaces or **isosceles spaces**. Ultrametric spaces were topologically characterized by J. de Groot [HM, Chapter 10, Theorem 2.1]: A metrizable space X admits an ultrametric if and only if $\dim X = 0$, where $\dim X$ denotes the ***covering dimension*** of X. The theorem was extended to higher dimension by J. Nagata and P. Ostrand (see Chapter "Special metrics" in the Encyclopedia for the details). Ultrametric spaces are also studied in dimension theory from more geometrical points of view. Let $\mathbb{R}_+$ denote the set of non-negative real numbers and $C, s \in \mathbb{R}_+$. A metric space (X, ρ) is (C, s)**-homogeneous** if the inequality $|X_0| \leqslant C(b/a)^s$ holds for $a > 0$, $b > 0$ and $X_0 \subset X$ provided that $b \geqslant a$ and that $a \leqslant \rho(x, y) \leqslant b$ holds for every pair of distinct points x and y of X_0. The space (X, ρ) is said to be s**-homogeneous** if it is (C, s)-homogeneous for some $C \in \mathbb{R}_+$. We define the **Assouad dimension** $\dim_A(X, \rho)$ of a metric space (X, ρ) as follows: $\dim_A(X, \rho) = \inf\{s \in \mathbb{R}_+ : (X, \rho) \text{ is } s\text{-homogeneous}\}$, if the infimum exists. Otherwise, we define $\dim_A(X, \rho) = \infty$. J. Luukkainen and H. Movahedi-Lankarani [11] and K. Luosto [10] proved that an ultrametric space (X, ρ) can be bi-Lipschitz embedded in the n-dimensional Euclidean space $\mathbb{R}^n$ if and only if $\dim_A(X, \rho) < n$, where a map $f : X \to \mathbb{R}^n$ is said to be a **bi-Lipschitz embedding** if there exists a real number $\alpha \geqslant 1$ such that for every $x, y \in X$, $\frac{1}{\alpha}\rho(x, y) \leqslant \|f(x) - f(y)\| \leqslant \alpha\rho(x, y)$ holds. Several metrically universal properties of ultrametric spaces are known. A.J. Lemin [8] proved that every ultrametric space of weight τ is isometrically embedded in the generalized Hilbert space H^τ. A.J. Lemin and V.A. Lemin [9] also proved that for every cardinal τ there is an ultrametric space (LW_τ, ρ) such that every ultrametric space of weight $\leqslant \tau$ is isometrically embedded in (LW_τ, ρ). The weight of the space (LW_τ, ρ) is $\tau^{\aleph_0}$ and this is the best possible. In fact, if an ultrametric space (X, ρ) isometrically contains all two-point ultrametric spaces, then the weight of $X \geqslant \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the continuum. In the case of spaces consisting of finite points, we have the following [9]: Every ultrametric space consisting of $n+1$ points is isometrically embedded in the n-dimensional Euclidean space $\mathbb{R}^n$ and there is no ultrametric space X consisting of $n+1$ points such that X is isometrically embedded in the k-dimensional Euclidean space $\mathbb{R}^k$ for $k < n$. A.J. Lemin further studied some relations between ultrametric spaces and Boolean algebras, lattices, Lebesgue measure and Lebesgue integral theory. Recently, the theory of ultrametric spaces has been developing applications in several branches of mathematics as well as physics, biology and information sciences. In particular, the theory of ultrametric spaces can be applied to domain theory and logic programming. A **computational model** for a topological space X is an ω-continuous ***directed complete*** partially ordered set P with an embedding $i : X \to \mathrm{Max}(P)$ which satisfy the following conditions: (1) The restrictions of the ***Scott topology*** and the ***Lawson topology*** to $\mathrm{Max}(P)$ coincide. (2) The embedding $i : X \to \mathrm{Max}(P)$ is a homeomorphism. (A directed complete partially ordered set is often abbreviated as a **DCPO**.) B. Flagg and R. Kopperman [3] proved that a topological space X has an ω-algebraic computational model if and only if X is a complete separable ultrametric space. We refer the reader to [vMR, Chapter 22] for terminology and recent developments of domain theory related to topology. We also refer the reader to [9] for a brief historical introduction to ultrametric spaces.

We shall discuss some topological spaces determined by functions similar to a metric. A non-negative real valued function ρ defined on $X \times X$ is called a **pseudometric** on X (and (X, ρ) is a **pseudometric space**) if ρ satisfies the conditions (2) and (3) mentioned in the definition of metrics as well as $(1)'$ $\rho(x, x) = 0$ for each $x \in X$. Pseudometrics induce topologies in a manner similar to metrics: $\mathcal{T}_\rho = \{U \subset X$: for each point $x \in U$, there exists an ε-ball of x such that $S_\varepsilon(x) \subset U\}$. Analogous to the metric case, a topological space X is called a **pseudometrizable space** if there is a pseudometric ρ on X such that $\mathcal{T}_\rho$ coincides with the original topology of X. Pseudometrizable spaces need not be ***T_0-space***s, for example, every ***indiscrete space*** with more than one point is pseudometrizable space but it is not a T_0-space. The ***quotient space*** of a pseudometric space identifying the points whose distance measured by the pseudometric are zero is a metric space. The theory of pseudometrizable spaces is similar to the case of metrizable spaces without separation axioms (see [14, §2.3]).

A non-negative real valued function ρ defined on $X \times X$ is called a **symmetric** on X if ρ satisfies the conditions (1) and (2) in the definition of a metric. As similar to the case of metrics, we define an **ε-ball** of $x \in X$ as $S_\varepsilon(x) = \{y \in X$: $\rho(x, y) < \varepsilon\}$ for a symmetric ρ, a point $x \in X$ and $\varepsilon > 0$. Since symmetrics do not satisfy the triangle inequality, $\{S_\varepsilon(x)$: $\varepsilon > 0\}$ does not form a ***neighbourhood base*** at x. Hence we can not topologize symmetric spaces similar to metric spaces. However, we have a topology closely related to a symmetric. A topological space X is said to be a **symmetrizable** space if X admits a symmetric ρ such that $U \subset X$ is open if and only if for each $x \in U$ there is $\varepsilon > 0$ with $S_\varepsilon(x) \subset U$. Equivalently, a topological space X is symmetrizable if X admits a symmetric ρ such that $F \subset X$ is closed if and only if $\rho(x, F) > 0$ for each $x \in X \setminus F$. A topological space X is called a **semi-metrizable** space if X admits a symmetric ρ such that $x \in \bar{A}$ if and only $\rho(x, A) = 0$, where $\bar{A}$ denotes the ***closure*** of $A \subset X$. Equivalently, a topological space X is semi-metrizable if X admits a symmetric ρ such that $\{S_\varepsilon(x)$: $\varepsilon > 0\}$ is a neighbourhood base (but not necessarily open) at x. It is clear that every semi-metrizable space is symmetrizable and first-countable. There is an example of a symmetrizable space which is not semi-metrizable. In fact, let X be the quotient space of the real line obtained by identifying n and $1/n$ for each natural number n. Then X is a symmetrizable space but it is not first-countable, hence it is not semi-metrizable [1]. If a space is first countable, then the converse holds, more precisely, the following are equivalent for a Hausdorff space X: (a) X is semi-metrizable, (b) X is symmetrizable and first-countable, (c) X is symmetrizable and ***Fréchet***. There are several metrization theorems for symmetrizable and semi-metrizable spaces (cf. [1] and [KV, Chapter 10]). Both of the following spaces are metrizable; (i) countably compact symmetrizable Hausdorff spaces; and (ii) ***collectionwise normal*** symmetrizable ***p-spaces***. We have another characterization of semi-metrizable spaces [2]: A regular T_1-space is semi-metrizable if and only if it is first countable and ***semi-stratifiable***. Semi-metrizable spaces have many nice properties related to covering properties and maps; for example, every semi-metrizable space is ***subparacompact***, and a ***collectionwise normal*** semi-metrizable space is paracompact. A continuous map $f : X \to Y$ of a space X onto a space Y is called **hereditarily quotient** if for every subset S of Y $f|_{f^{-1}(S)}$ is a ***quotient map***. Let $f : X \to Y$ be a hereditarily quotient map of a metric space X onto a space Y such that $f^{-1}(y)$ is compact for each $y \in Y$. Then Y is semi-metrizable. We note that the example of symmetrizable space which is not semi-metrizable mentioned above shows that we cannot replace a hereditarily quotient map by a quotient map in the theorem above.

A non-negative real valued function ρ defined on $X \times X$ is called a **quasi-metric** on X (and (X, ρ) a **quasi-metric space**) if ρ satisfies the conditions (1) and (3) in the definition of a metric. A topological space X is said to be **quasi-metrizable** if X admits a quasi-metric ρ such that $\{S_\varepsilon(x)$: $\varepsilon > 0\}$ forms a neighbourhood base at each point $x \in X$. The ***Sorgenfrey line*** $\mathbb{S}$ is a quasi-metrizable space by the quasi-metric defined by $\rho(x, y) = y - x$, if $y \geqslant x$, and $\rho(x, y) = 1$, otherwise. We have a useful characterization of quasi-metrizable spaces in terms of **g-functions**: A topological space $(X, \mathcal{T})$ is quasi-metrizable if and only if there is a function $g : \omega \times X \to \mathcal{T}$ satisfying (i) $\{g(n, x)$: $n \in \omega\}$ is a neighbourhood base at x, and (ii) if $y \in g(n+1, x)$, then

$g(n+1, y) \subset g(n, x)$, where ω denotes the set of natural numbers. We notice that there are many characterizations of generalized metric spaces in terms of g-functions (see [KV, Chapter 10] and [MN, Chapter 9]). A quasi-metric ρ on X is called non-Archimedean if ρ satisfies the condition $(3)'$ mentioned after the definition of a metric. A space X is non-Archimedean quasi-metrizable if and only if X has a ***σ-interior-preserving*** base [KV, Theorem 10.3].

A non-negative real valued function ρ defined on $X \times X$ is called a **γ-metric** on X if $\rho(x, z_n) \to 0$ whenever $\rho(x, y_n) \to 0$ and $\rho(y_n, z_n) \to 0$. A topological space X is said to be a **γ-space** if X admits a γ-metric ρ such that $\{S_\varepsilon(x): \varepsilon > 0\}$ forms a neighbourhood base at each point $x \in X$. Every quasi-metrizable space is a γ-space, but there is a paracompact Hausdorff γ-space which is not quasi-metrizable [4].

We refer the reader to Gruenhage's articles [KV, Chapter 10], [HvM, Chapter 7] and Kofner's survey [6] for the theory of symmetrizable, semi-metrizable, quasi-metrizable and γ-spaces. We also refer the reader to [E] and [N] for more details on metric (metrizable) spaces, and to [7] for a comprehensive development of the theory of separable metric spaces including continuum theory, descriptive set-theory, and the theory of hyperspaces on compact and separable metric spaces.

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e-2 Classical Metrization Theorems

A topological space $\langle X, \mathcal{T}\rangle$ is said to be ***metrizable*** if there is a ***metric*** d on X such that the topology induced by d is $\mathcal{T}$. A **metrization theorem** gives (necessary and) sufficient conditions for a space $\langle X, \mathcal{T}\rangle$ to be metrizable. The heart of metrization theory is captured by the following four theorems. (Throughout we assume that all spaces are T_1.)

THEOREM 1 (Alexandroff and Urysohn). *A topological space is metrizable if and only if it has a regular development.*

THEOREM 2 (Urysohn). *A **regular** space with a countable **base** is metrizable.*

THEOREM 3 (Nagata–Smirnov). *A topological space is metrizable if and only if it is regular and has a σ**-locally finite** base.*

THEOREM 4 (Bing). *A topological space is metrizable if and only if it is regular and has a σ**-discrete** base.*

The honor of proving the first metrization theorem goes to Alexandroff and Urysohn; however, their proof did use an earlier result of Chittenden on distance functions. The Alexandroff–Urysohn Theorem clearly established the concept of a development as fundamental in metrization theory. A **development** for X is a sequence $\{\mathcal{G}_n\colon n \in \mathbb{N}\}$ of open covers of X such that for all $x \in X$, $\{\mathrm{St}(x, \mathcal{G}_n)\colon n \in \mathbb{N}\}$ is a local base for x; $\mathrm{St}(x, \mathcal{G}) = \bigcup\{G\colon G \in \mathcal{G} \text{ and } x \in G\}$. X is a **developable space** if X has a development.

Urysohn's Theorem, which gives a simple sufficient condition for metrizability, is perhaps the best known metrization theorem. The proof uses an important result known as **Urysohn's Lemma**: Let X be a ***normal*** space and let H and K be disjoint closed sets in X. Then there is a continuous function $f : X \to [0, 1]$ such that $f(H) = 0$ and $f(K) = 1$. (Strictly speaking, Urysohn assumed normality in his metrization theorem; one year later, Tychonoff proved that every regular ***Lindelöf*** space is normal.)

Urysohn's Theorem is not easily derived from the Alexandroff–Urysohn Theorem, and this gave rise to the famous *metrization problem*: Find a necessary and sufficient condition for metrizability that gives Urysohn's result as a easy corollary. This problem was finally solved in the early 1950s independently by Nagata, Smirnov, and Bing. It should be noted that A.H. Stone's important result that every metric space is ***paracompact*** played a key role in the Nagata–Smirnov–Bing solution of the metrization problem.

These four theorems illustrate the two basic methods of constructing a metric on X that is compatible with the topology of X. We now outline these two constructions.

Construction I
Let $\{\mathcal{G}_n\colon n \in \mathbb{N}\}$ be a regular development for $\langle X, \mathcal{T}\rangle$. (**Regular development** means: $\{\mathcal{G}_n\colon n \in \mathbb{N}\}$ is a development such that, for all $n \in \mathbb{N}$, if $G_1, G_2 \in \mathcal{G}_{n+1}$ and $G_1 \cap G_2 \neq \emptyset$, then $(G_1 \cup G_2) \subset G$ for some $G \in \mathcal{G}_n$.) Define a distance function $d : X \times X \to [0, \infty)$ as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } y \in \mathrm{St}(x, \mathcal{G}_n) \text{ for all } n \in \mathbb{N}; \\ 2^{-n} & \text{where } n = \min\{m\colon y \notin \mathrm{St}(x, \mathcal{G}_m)\}. \end{cases}$$

This distance function satisfies the following properties for all $x, y, z \in X$:

(1) $d(x, y) = 0 \iff x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leqslant 2\max\{d(x, y), d(y, z)\}$;
(4) $B_d(x, \frac{1}{2^n}) = \mathrm{St}(x, \mathcal{G}_n)$.

By (4), the distance function d induces the topology $\mathcal{T}$. However, d need not satisfy the triangle inequality. So define $\rho : X \times X \to [0, \infty)$ by

$$\rho(x, y) = \inf\bigl\{d(x, x_1) + d(x_1, x_2) + \cdots + d(x_n, y)\bigr\},$$

where the infimum is taken over all finite sequences of points $x, x_1, \ldots, x_n, y$. The distance function ρ is a metric on X such that for all $x, y \in X$: $d(x, y)/4 \leqslant \rho(x, y) \leqslant d(x, y)$. From this it follows that ρ induces the topology $\mathcal{T}$ of X. The construction of d is due to Alexandroff and Urysohn and the construction of ρ is due to A. Frink.

Construction II
Let $\{\mathcal{B}_n\colon n \in \mathbb{N}\}$ be a σ-locally finite base for the normal space X. Fix $\langle m, n\rangle \in \mathbb{N} \times \mathbb{N}$, and for each $B \in \mathcal{B}_n$ let B_0 be the union of all elements of $\mathcal{B}_m$ whose closure is contained in B. By Urysohn's Lemma, there is a continuous function $f_B : X \to [0, 1]$ such that $f_B(\overline{B_0}) = 1$ and $f_B(X - B) = 0$. Let

$$d_{m,n}(x, y) = \min\Biggl\{1, \sum_{B \in \mathcal{B}_n} \bigl| f_B(x) - f_B(y)\bigr|\Biggr\}.$$

It is easy to check that $d_{m,n}$ is a pseudometric on X. Now relabel the set $\{d_{m,n}\colon \langle m, n\rangle \in \mathbb{N} \times \mathbb{N}\}$ as $\{d_k\colon k \in \mathbb{N}\}$, and define $\rho : X \times X \to [0, \infty)$ by

$$\rho(x, y) = \sum_{k=0}^{\infty} \frac{d_k(x, y)}{2^k};$$

ρ is a metric on X that induces the topology of X.

With these two constructions firmly established, metrization theory reduces to the problem of proving the existence of a regular development or a σ-locally finite base. For example, A. Frink obtained the following neighbourhood characterization of metrizability by constructing a regular development.

THEOREM 5 (Frink). *A space X is metrizable if and only if for each $x \in X$, there is a countable collection $\{V_n(x): n \in \mathbb{N}\}$ of open neighbourhoods of x such that the following hold*:

(1) $\{V_n(x): n \in \mathbb{N}\}$ *is a local base for* x;
(2) *for all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that if $V_k(x) \cap V_k(y) \neq \emptyset$, then $V_k(y) \subset V_n(x)$.*

As noted earlier, the concept of a development is fundamental in the early history of metrization theory. This is illustrated by the following list of classical results.

THEOREM 6. *For any space X the following are equivalent*:

(1) *X is metrizable*;
(2) *X is regular and there is a sequence $\{\mathcal{G}_n: n \in \mathbb{N}\}$ of* ***closure-preserving*** *open collections in X such that for all $x \in X$, $\{\mathrm{St}(x, \mathcal{G}_n): n \in \mathbb{N}$ and $\mathrm{St}(x, \mathcal{G}_n) \neq \emptyset\}$ is a local base for x (Bing)*;
(3) *there is a sequence $\{\mathcal{G}_n: n \in \mathbb{N}\}$ of open covers of X such that for all $x \in X$, $\{\mathrm{St}^2(x, \mathcal{G}_n): n \in \mathbb{N}\}$ is a local base for x (Moore, Morita)*;
(4) *there is a sequence $\{\mathcal{G}_n: n \in \mathbb{N}\}$ of open covers of X such that for all $x \in X$ and every neighbourhood R of x, there is a neighbourhood V of x and some $n \in \mathbb{N}$ such that $\mathrm{St}(V, \mathcal{G}_n) \subset R$ (Arhangel'skiĭ, Stone)*;
(5) *there is a sequence $\{\mathcal{G}_n: n \in \mathbb{N}\}$ of open covers of X such that given disjoint closed sets H and K in X with K compact, there exists $n \in \mathbb{N}$ such that $\mathrm{St}(K, \mathcal{G}_n) \cap H = \emptyset$ (Jones)*;
(6) *there is a sequence $\{\mathcal{F}_n: n \in \mathbb{N}\}$ of locally finite closed covers of X such that for all $x \in X$ and every neighbourhood R of x, there exists $n \in \mathbb{N}$ such that $\mathrm{St}(x, \mathcal{F}_n) \subset R$ (Morita)*;
(7) *there is a sequence $\{\mathcal{F}_n: n \in \mathbb{N}\}$ of closure-preserving closed covers of X such that for all $x \in X$ and every neighbourhood R of x, there exists $n \in \mathbb{N}$ such that $\mathrm{St}(x, \mathcal{F}_n) \subset R$ (Nagami).*

There are two metrization theorems, due to Bing and Nagata, that deserve special emphasis. These two theorems clearly mark the transition from classical metrization theory to modern metrization theory and will be starting point of the article *Modern Metrization Theorems*.

THEOREM 7 (Bing). *A space X is metrizable if and only if it is* ***collectionwise normal*** *and developable.*

THEOREM 8 (Nagata). *A space X is metrizable if and only if for each $x \in X$, there are two countable collections $\{U_n(x): n \in \mathbb{N}\}$ and $\{V_n(x): n \in \mathbb{N}\}$ of open neighbourhoods of x such that*

(1) $\{U_n(x): n \in \mathbb{N}\}$ *is a local base for* x;
(2) *if $y \notin U_n(x)$, then $V_n(x) \cap V_n(y) = \emptyset$*;
(3) *if $y \in V_n(x)$, then $V_n(y) \subset U_n(x)$.*

Bing's Theorem is influential for a number of reasons. First of all, it introduces into topology a new and important separation axiom, namely collectionwise normality. In addition, it is the first metrization theorem of the factorization type. Prior to 1950, metrization theorems were stated in terms of strong base conditions. But Bing's Theorem factors metrizability into an elementary base condition (developability) and a separation axiom.

Nagata's Theorem, stated in terms of neighbourhood assignments, also gives a factorization of metrizability into the two conditions (2) and (3) (later called **Nagata space**s and **γ-space**s). In Nagata's paper we see for the first time the idea of formulating a metrization theorem of a very general nature and then deriving from it many others, thereby giving a unified approach to metrization theory. In his paper, Nagata proves the existence of a regular development from his list of three conditions. But he then shows that it is possible to derive a number of metrization theorems from his list of conditions, including the Alexandroff–Urysohn Theorem, Frink's Theorem, the Nagata–Smirnov Theorem, Morita's Theorem on locally finite closed covers, and Bing's Theorem in terms of closure-preserving open collections.

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e-3 Modern Metrization Theorems

We assume that all spaces are T_1 and ***regular***. It is not unreasonable to claim that the modern era of metrization theory began with the publication of two fundamental theorems.

THEOREM B (Bing, 1951). *A space X is **metrizable** if and only if it is **collectionwise normal** and **developable**.*

THEOREM N (Nagata, 1957). *A space X is metrizable if and only if for each $x \in X$ there are two countable collections $\{U_n(x): n \in \mathbb{N}\}$ and $\{V_n(x): n \in \mathbb{N}\}$ of open neighbourhoods of x such that*

(1) *$\{U_n(x): n \in \mathbb{N}\}$ is a local base for x;*
(2) *if $y \notin U_n(x)$, then $V_n(x) \cap V_n(y) = \emptyset$;*
(3) *if $y \in V_n(x)$, then $V_n(y) \subset U_n(x)$.*

The importance of these two results is discussed in the article *Classical Metrization Theorems*. We can summarize the discussion there by noting that the theorems of Bing and Nagata stimulated research in generalized metric spaces, which in turn led to significant contributions to metrization theory.

It is impossible to do justice to the topic of modern metrization theory in the space allowed, and so we will be content with a selection of representative results. The article is divided into four parts: base conditions, neighbourhood assignments, collectionwise normal spaces, ***G_δ-diagonals*** and ***point-countable bases***.

1. Base conditions

See the article *Classical Metrization Theorems* for a long list of characterizations of metrizability in terms of a ***development*** and for a discussion of the Nagata–Smirnov–Bing solution of the metrization problem in terms of a σ-locally finite or a σ-discrete base. In this section we further emphasize the important role of base conditions in metrization theory. We begin with variations of a ***σ-locally finite*** or a ***σ-discrete*** base.

THEOREM 1. *The following are equivalent for a space X:*

(1) *X is metrizable;*
(2) *X has a σ-locally finite base (Nagata–Smirnov);*
(3) *X has a σ-discrete base (Bing);*
(4) *X has a σ-locally countable base and is **paracompact** (Fedorčuk);*
(5) *X has a σ-point-finite base and is collectionwise normal and **perfectly normal** (Arhangel'skiĭ);*
(6) *X has a σ-disjoint base and is perfectly normal (Aull);*
(7) *X has a σ-**hereditarily closure-preserving** base (Burke–Engelking–Lutzer);*
(8) *X has a σ-locally finite k-network and is a q-space (O'Meara);*
(9) *X has a σ-closure-preserving k-network $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, where each $\mathcal{B}_n$ is pseudo-interior preserving (Nagata);*
(10) *X has a base $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ such that for all $x \in X$ and all $n \in \mathbb{N}$, $V_n(x)$ is a neighbourhood of x, where $V_n(x) = \bigcap\{\operatorname{int}\overline{B}: x \in B \in \mathcal{B}_n\} \cap [\bigcap(X - \overline{B}): B \in \mathcal{B}_n$ and $x \notin \overline{B}]$ (Hung).*

A collection $\mathcal{B}$ of subsets of X is a ***k*-network** if given any compact subset K of X and any open set R with $K \subset R$, there is a finite subcollection $\{B_1, \dots, B_n\}$ of $\mathcal{B}$ such that $K \subset \bigcup_{k=1}^{n} B_k \subset R$. Note that elements of $\mathcal{B}$ need not be open and that every base is a k-network. Michael introduced this new base-like condition and then proved the following analogue of Urysohn's Theorem: a q-space with a countable k-network has a countable base and hence is metrizable (q-spaces are defined in the next section). A collection $\mathcal{C}$ of sets is **pseudo-interior preserving** if $\bigcap \mathcal{C}_0 \subset \operatorname{int}(\bigcap\{\overline{C}: C \in \mathcal{C}_0\})$ for every subcollection $\mathcal{C}_0$ of $\mathcal{C}$.

Another characterization of metrizability in terms of a base condition is the following somewhat surprising result of Arhangel'skiĭ (see [E]).

THEOREM 2. *The following are equivalent for a space X:*

(1) *X is metrizable;*
(2) *X has a base $\mathcal{B}$ such that for every $x \in X$ and every neighbourhood R of x, there is an open neighbourhood V of x such that the number of elements of $\mathcal{B}$ that intersect both V and $X - R$ is finite;*
(3) *X has a base $\mathcal{B}$ such that for every compact subset K of X and every neighbourhood R of K, the number of elements of $\mathcal{B}$ that intersect both K and $X - R$ is finite.*

2. Neighborhood assignments

This has been a very active area of research in metrization theory, mainly because so many generalizations of metric spaces can be characterized in terms of neighbourhood assignments. Theorem N of Nagata plays a key role here; also recall A.H. Frink's Metrization Theorem. Another influence is the suggestion of F.B. Jones that theorems about developable spaces often extend to semi-metric spaces (but with harder proofs); Heath's contributions here are especially important. We begin with a long list of definitions.

DEFINITION 1. Let X be a topological space, and for each $x \in X$ let $\{V_n(x): n \in \mathbb{N}\}$ be a countable collection of open neighbourhoods of x. Consider the following conditions on $\{V_n(x): x \in X,\ n \in \mathbb{N}\}$:

(N) if $V_n(x) \cap V_n(x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x_n \to x$;
(wN) if $V_n(x) \cap V_n(x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point;
(γ) if $y_n \in V_n(x)$ and $x_n \in V_n(y_n)$ for all $n \in \mathbb{N}$, then $x_n \to x$;
(dev) if $x, x_n \in V_n(y_n)$ for all $n \in \mathbb{N}$, then $x_n \to x$;
(wΔ) if $x, x_n \in V_n(y_n)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point;
(σ) if $x \in V_n(y_n)$ and $y_n \in V_n(x_n)$ for all $n \in \mathbb{N}$, then $x_n \to x$;
(θ) if $x, x_n \in V_n(y_n)$ and $y_n \in V_n(x)$ for all $n \in \mathbb{N}$, then $x_n \to x$;
(SS) if $x \in V_n(x_n)$ for all $n \in \mathbb{N}$, then $x_n \to x$;
(β) if $x \in V_n(x_n)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point;
(q) if $x_n \in V_n(x)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

A metric space $\langle X, d \rangle$ satisfies all of these conditions; take $V_n(x) = B_d(x, \frac{1}{2^n})$. A space that satisfies (N) is called a **Nagata space** (see condition (2) in Nagata's Theorem); a space that satisfies (γ) is called a **γ-space** (see condition (3) in Nagata's Theorem). Every ***quasi-metric space*** is a γ-space (but not conversely). Nagata's Theorem can be restated as follows: Nagata space + γ-space $\Longleftrightarrow$ metrizable.

Developable spaces can be characterized in terms of the condition (dev); σ-***space***s (spaces with a σ-discrete network) can be characterized in terms of the condition (σ). A space that satisfies (θ) is called a **θ-space**; this class of spaces unifies the developable and the γ-spaces. A space that satisfies (SS) (respectively, (β), (q)) is said to be **semi-stratifiable** (respectively, a **β-space**, **q-space**). Countably compact spaces satisfy all of the conditions (wN), (wΔ), (β), and (q). For θ-***refinable*** spaces, wΔ-spaces and ***p-spaces*** are equivalent (but first replace Arhangel'skiĭ's original definition with a characterization due to D. Burke; for details, see [KV, Chapter 10]).

We begin with two variations of Nagata's 1957 result.

THEOREM 3. *The following are equivalent for a space X*:

(1) *X is metrizable*;
(2) *X is a Nagata space and a γ-space*;
(3) *X is a Nagata space and a developable space*;
(4) *X is a Nagata space and a θ-space.*

Two classical results from metrization theory are:

(a) every countably compact quasi-metrizable space is metrizable;
(b) every countably compact developable space is metrizable.

The next result of Hodel extends Theorem 3 and in addition has (a) and (b) as corollaries.

THEOREM 4. *The following are equivalent for a space X*:

(1) *X is metrizable*;
(2) *X is a γ-space and satisfies property* (wN);
(3) *X is a developable space and satisfies property* (wN).

In 1984 Collins and Roscoe obtained the following very interesting neighbourhood characterization of metrizability.

THEOREM 5. *A space X is metrizable if and only if every $x \in X$ has a decreasing neighbourhood assignment $\{V_n(x)\colon n \in \mathbb{N}\}$ such that given $x \in X$ and a neighbourhood R of x, there exists $n \in \mathbb{N}$ and an open neighbourhood $W[x, R]$ of x such that for all $y \in X$: if $y \in W[x, R]$, then $x \in V_n(y) \subset R$.*

We emphasize that the sets $V_n(x)$ need not be open; on the other hand, $\{V_n(x)\colon n \in \mathbb{N}\}$ must be a decreasing collection. Also note that for each x and each neighbourhood R of x, there is a 'uniform' n. The Collins–Roscoe Theorem can be derived from Nagata's Theorem (and vice versa) without too much difficulty. A generalization of Theorem 5 that is much harder to prove is the following result of Collins, Reed, Roscoe, and Rudin. (This time the sets $V_n(x)$ must be open. On the other hand, there is no uniform n; rather, $n(y)$ depends on y.)

THEOREM 6. *A space X is metrizable if and only if every $x \in X$ has a decreasing open neighbourhood assignment $\{V_n(x)\colon n \in \mathbb{N}\}$ such that given $x \in X$ and a neighbourhood R of x, there is an open neighbourhood $W[x, R]$ of x such that for all $y \in X$: if $y \in W[x, R]$, then there exists $n(y) \in \mathbb{N}$ such that $x \in V_{n(y)}(y) \subset R$.*

The next theorem is a very general result that is the culmination of research by Gao, Hodel, Hung, Nagata, and Ziqiu (see [12] for references). Note that every σ-space and every θ-space satisfies condition (1) of the theorem.

THEOREM 7. *A space X is metrizable if and only if every $x \in X$ has a neighbourhood assignment $\{V_n(x)\colon n \in \mathbb{N}\}$ such that*

(1) *if $x \in V_n(y_n)$, $y_n \in V_n(x_n)$, $x_n \in V_n(y_n)$, and $y_n \in V_n(x)$ for all $n \in \mathbb{N}$, then $x_n \to x$*;
(2) *for all $n \in \mathbb{N}$ and all $Y \subset X$, $\overline{Y} \subset \bigcup\{V_n(x)\colon x \in Y\}$.*

There are several metrization theorems for symmetrizable spaces that are in the spirit of neighbourhood assignments. A space X is **symmetrizable** if there is a symmetric distance function d on X that induces the topology of X; symmetrizable spaces can be characterized in terms of a weak base that satisfies (SS). Note the similarity between condition (2) below and (γ).

THEOREM 8. *Let X be symmetrizable with symmetric distance function d. The following are equivalent*:

(1) *X is metrizable*;
(2) *if $d(x, y_n) \to 0$ and $d(x_n, y_n) \to 0$, then $d(x, x_n) \to 0$* (*Arhangel'skiĭ*);

(3) *if* $H \cap K = \emptyset$, *where H is closed and K is compact, then* $d(H, K) > 0$;
(4) *if H is closed and* $x \notin H$, *then there exists* $\varepsilon > 0$ *such that* $B_d(x, \varepsilon) \cap B_d(H, \varepsilon) = \emptyset$ (*Harley–Faulkner*).

3. Collectionwise normal spaces

The results in this section are related to Theorem B (Bing's fundamental factorization of metrizability).

THEOREM 9. *Let X be a collectionwise normal space. The following are equivalent*:

(1) *X is metrizable*;
(2) *X is developable* (*Bing*);
(3) *X is a semistratifiable space with a point-countable base* (*Heath*);
(4) *X is a* Σ*-space with a point-countable base* (*Shiraki*);
(5) *X is a symmetrizable p-space* (*Arhangel'skiĭ*);
(6) *X is a* ***perfect space*** *with a* θ*-base* (*Wicke–Worrell*).

Each of the conditions (3), (4), (5) and (6) implies developability. Characterization (3) is based on a deep theorem of Heath that gives conditions under which a semi-metric space is developable. A space X is a Σ**-space** if there is a sequence $\{\mathcal{F}_n\colon n \in \mathbb{N}\}$ of locally finite closed covers of X such that if

$$x_n \in c(x, \mathcal{F}_n) = \bigcap\{F\colon F \in \mathcal{F}_n \text{ and } x \in F\}$$

for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point; every σ-space and every countably compact space is a Σ-space. Condition (4) is proved by first showing that every Σ-space with a point-countable base is a σ-space and then using Heath's Theorem.

Characterization (5) is remarkable for the following reason: it gives a list of conditions equivalent to metrizability, yet it is not obvious that these conditions are strong enough to prove that every point of X is a G_δ!

A collection $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ of open subsets of a space X is a θ**-base** if given $x \in R$ with R open, there exists $n \in \mathbb{N}$ and $B_x \in \mathcal{B}_n$ such that

(1) $x \in B_x \subset R$;
(2) $\{B\colon B \in \mathcal{B}_n$ and $x \in B\}$ is finite.

A σ-point finite base is a θ-base and so Theorem 9(6) generalizes Theorem 1(5). If we weaken condition (2) in the definition of a θ-base by replacing 'finite' with 'countable', we obtain a $\delta\theta$**-base**. This generalizes the concept of a point-countable base and will be discussed in the next section.

Bing's Theorem factors metrizability into two parts: collectionwise normality and developability. There is a similar factorization due to Arhangel'skiĭ [1] in which collectionwise normality is strengthened to paracompactness but developability is replaced by a weaker property. A space X has a **base of countable order** (**BCO**) if there is a sequence $\{\mathcal{B}_n\colon n \in \mathbb{N}\}$ of bases for X such that for all $x \in X$: if $x \in B_n \in \mathcal{B}_n$ for all $n \in \mathbb{N}$, and $\{B_n\}$ is decreasing, then $\{B_n\}$ is a local base for x. A space is developable if and only if it is θ-refinable and has a BCO; a space is paracompact if and only if it is collectionwise normal and θ-refinable. Using these two results, the Bing factorization and the Arhangel'skiĭ factorization can each be derived from the other. In summary:

THEOREM 10. *The following are equivalent for a space X*:

(1) *X is metrizable*;
(2) *X is paracompact and has a BCO* (*Arhangel'skiĭ*);
(3) *X is collectionwise normal,* θ*-refinable, and has a BCO* (*Wicke–Worrell*).

4. G_δ-diagonals and point-countable bases

The results in this section are motivated by two 'classical' results due to Šneider and Miščenko (see [KV, Chapter 10]):

(Š) compact + G_δ-diagonal $\Longrightarrow$ metrizable;
(M) compact + point-countable base $\Longrightarrow$ metrizable.

The problem is to find characterizations of metrizability that yield these two results as easy corollaries. To solve the problem, we consider two classes of spaces: paracompact p-spaces and M-spaces. Both generalize metric spaces; moreover, every compact space is a paracompact p-space and every countably compact space is an M-space. The two classes are related to metric spaces as follows.

THEOREM 11 (Arhangel'skiĭ, Morita). *A space X is a paracompact p-space* (***M-space***) *if and only if there is a metric space Y and a* ***perfect map*** (***quasi-perfect map***) *from X onto Y*.

First let us consider spaces with a G_δ-diagonal.

THEOREM 12. *The following are equivalent for a space X*:

(1) *X is metrizable*;
(2) *X is a paracompact p-space with a* G_δ*-diagonal*;
(3) *X is an M-space with a* G_δ*-diagonal.*

Borges and Okuyama independently proved Theorem 12(2); this is the first characterization of metrizability that gives (Š) as an immediate corollary. Theorem 12(3) on M-spaces can be derived from the Borges–Okuyama result and the following deep extension of (Š) due to Chaber: every countably compact space with a G_δ-diagonal is compact (see [KV, Chapter 10]).

There is an interesting metrization theorem due to Heath, Lutzer, and Zenor that is closely related to the Borges–Okuyama Theorem (but it does not give (Š)).

THEOREM 13. *A space X is metrizable if and only if it is a* ***monotonically normal*** *p-space with a* G_δ*-diagonal.*

In 1984 Gruenhage obtained a generalization of Šneider's result by proving that if X is a compact space such that $X^2 - \Delta$ is paracompact, then X is metrizable. He also noted that "compact" can be replaced by "countably compact". Somewhat later, he and Pelant generalized this result by proving that if X is a Σ-space such that $X^2 - \Delta$ is paracompact, then X has a G_δ-diagonal. This theorem, when combined with the Borges–Okuyama result, gives:

THEOREM 14. *A space X is metrizable if and only if it is a p-space such that $X^2 - \Delta$ is paracompact.*

Now let us turn to spaces with a point-countable base.

THEOREM 15. *The following are equivalent for a space X*:

(1) *X is metrizable*;
(2) *X is a paracompact p-space with a point-countable base* (*Filippov*);
(3) *X is a paracompact p-space with a point-countable separating open cover* (*Nagata*);
(4) *X is an M-space with a point-countable separating open cover.*

Filippov gave the first characterization of metrizability that also has Miščenko's Theorem as an immediate corollary. Nagata then found a clever way to unify the Borges–Okuyama Theorem and the Filippov Theorem. An open cover $\mathcal{S}$ of X is a **separating cover** if given distinct points x and y of X, there exists $S \in \mathcal{S}$ such that $x \in S$ and $y \notin S$. Clearly every point-countable base is a point-countable separating open cover; moreover, it is easy to check that every paracompact space with a G_δ-diagonal also has such an open cover. Theorem 15(4) on M-spaces can be derived from Nagata's Theorem and the following extension of Miščenko's Theorem due to Arhangel'skiĭ and Proizvolov: a countably compact space with a point-countable separating open cover is compact. A useful result in this context is known as **Miščenko's lemma**: a point-countable collection of subsets of a set X has only countably many finite irreducible subcovers. For more on the metrizability of M-spaces, see p. 291 of Tanaka's survey paper in [MN].

The results in Theorems 9 and 15 on point-countable bases extend to $\delta\theta$-bases as follows.

THEOREM 16. *Let X have a $\delta\theta$-base. The following are equivalent*:

(1) *X is metrizable*;
(2) *X is collectionwise normal and semistratifiable* (*Aull*);
(3) *X is a collectionwise normal Σ-space* (*Chaber*);
(4) *X is a paracompact β-space* (*Chaber*; *Bennett and Berney for θ-base*).

Finally, we give two metrization theorems that seem to lie in the intersection of metrization theory and cardinal invariants.

THEOREM 17 (Balogh). *A space X is metrizable if and only if it is a perfectly normal space and every subspace of X is a paracompact p-space.*

The proof uses Nagata's result: metrizable $\Longleftrightarrow$ paracompact p-space + point-countable separating open cover. Balogh's Theorem generalizes an earlier result of Arhangel'skiĭ, which states that a space X has a countable base if and only if every subspace of X is a Lindelöf p-space.

THEOREM 18 (Dow). *Let X be a countably compact space. If every subspace of X of cardinality $\leqslant \omega_1$ is metrizable* (*more generally, has a point-countable base*), *then X has a countable base and is metrizable.*

Note that Dow's Theorem can be viewed as an extension of Miščenko's Theorem. Dow's proof uses the countable version of a reflection theorem of Hajnal and Juhász for the cardinal function w (weight). This countable version states that a space X has a countable base if and only if every subspace of X of cardinality $\leqslant \omega_1$ has a countable base.

For detailed references, the reader is referred to Gruenhage's article in [KV, Chapter 10]; also see [E], [N], [MN, Chapters 7, 8, 9 and 10], and the two survey papers [11] and [13].

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e-4 Special Metrics

We shall consider ***metrizable spaces***. For a metrizable space X, there are many ***metrics*** which induce the original ***topology*** of X. Some of them may determine topological properties of X. For example, it is well known that a metrizable space X is ***separable*** if and only if X admits a ***totally bounded*** metric, and X is ***compact*** if and only if X admits a ***complete*** totally bounded metric. We shall discuss relations between metric and topological properties.

We begin with special metrics characterizing topological dimension. J. de Groot [4] proved that a metrizable space X admits an ***ultrametric*** if and only if X is ***strongly zero-dimensional*** (i.e., $\dim X = 0$, where $\dim X$ denotes the ***covering dimension*** of X). J. Nagata [11] and P. Ostrand [13] independently generalized the theorem to n-dimensional spaces: A metrizable space has $\dim X \leqslant n$ if and only if X admits a metric ρ on X satisfying; $(1)_n$ for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ of X there is a pair of distinct indices i and j such that $\rho(y_i, y_j) \leqslant \rho(x, y_i)$. J. de Groot also proved that a compact metrizable space has $\dim X \leqslant n$ if and only if X admits a metric ρ on X satisfying; $(2)_n$ for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ of X there is a triplet of indices i, j and k such that $i \neq j$ and $\rho(y_i, y_j) \leqslant \rho(x, y_k)$. It is clear that $(2)_n$ is weaker than $(1)_n$. The following question posed by J. de Groot still remains open: If a (general) metrizable space X admits a metric ρ satisfying the condition $(2)_n$, is it true that $\dim X \leqslant n$? On the other hand, every metrizable space X admits the following metrics ρ_1, ρ_2, ρ_3 and ρ_4 (see [HM, Chapter 10, Section 2]): (a) For every $x \in X$ and each sequence $\{y_1, y_2, \ldots\}$ in X there is a triplet of indices i, j and k such that $i \neq j$ and $\rho_1(y_i, y_j) \leqslant \rho_1(x, y_k)$. (b) $\{S_\varepsilon(x): x \in X\}$ is ***closure-preserving*** for each $\varepsilon > 0$, where $S_\varepsilon(x) = \{y \in X: \rho_2(x, y) < \varepsilon\}$ is an ***ε-ball*** with respect to ρ_2. (c) For every $x \in X$ and every sequence $\{y_1, y_2, \ldots\}$ in X with $\rho_3(x, y_i) \geqslant \delta$ for infinitely many y_i for some $\delta > 0$, there is a pair of distinct indices i and j such that $\rho_3(y_i, y_j) \leqslant \rho_3(x, y_i)$. We do not know whether if the condition "$\rho(x, y_i) \geqslant \delta$ for infinitely many y_i for some $\delta > 0$" can be dropped in (c). (d) X has a ***σ-locally finite open base*** consisting of open balls with respect to ρ_4. The property (a) seems to be a generalization of $(2)_n$ and (c) is a partial generalization of $(1)_n$. By use of the same technique as (d), we have a characterization of ***strongly metrizable spaces***: A metrizable space X is strongly metrizable if and only if X admits a metric ρ such that for every $\varepsilon > 0$, every $x \in X$ and every sequence $\{y_1, y_2, \ldots\}$ in X with $\rho(S_\varepsilon(x), y_i) < \varepsilon$ for all i, there is a pair of distinct indices i and j such that $\rho(y_i, y_j) < \varepsilon$. In connection with properties (b) and (d) above, it seems to be interesting whether every metrizable space X admits a metric ρ such that $\{S_\varepsilon(x): x \in X\}$ is ***hereditarily closure-preserving*** for each $\varepsilon > 0$ [12]. We note that every hereditarily closure-preserving collection in a metrizable space (even in a first-countable space) is locally finite. Y. Ziqiu and H. Junnila [14] proved that the ***hedgehog*** of the weight 2^{c^+} does not have such metric, and further, Z. Balogh and G. Gruenhage [1] proved that a metrizable space X admits such metric if and only if X is strongly metrizable. Hence, the hedgehog of weight ω_1 does not have such metric.

Let (X, ρ) be a metric space and $y \neq z$ be distinct points of X. A set of the form $M(y, z) = \{x \in X: \rho(x, y) = \rho(x, z)\}$ is called a **midset** or a **bisector**. Midsets are a geometrically intuitive concept, and several topological properties can be approached through the use of midsets. The covering dimension of a separable metrizable space X can be characterized by midsets: A separable metrizable space X has $\dim X \leqslant n$ if and only if X admits a totally bounded metric ρ such that $\dim M \leqslant n - 1$ for every midset M in X (Janos–Martin, see [HM, Theorem 2.12]). Furthermore, metrics which have special midsets may determine the topological structures of spaces. For a natural number n, a metric space (X, ρ) is said to have the **n-points midset property**, abbreviated as the **n-MP**, if every midset in X has exactly n points. (A metrizable space that admits a metric with n-points midset property is also said to be have the n-points midset property.) The 1-MP, 2-MP and 3-MP are sometimes called the **unique midset property**, **double midset property** and **triple midset property**, abbreviated as the **UMP**, **DMP** and **TMP**, respectively. The real line with the usual metric is the example of the space having the UMP, and the circle in the two-dimensional Euclidean plane is the example of the space having the DMP. Nadler [10] proved that every non-degenerate ***component*** of a metrizable space with the UMP is homeomorphic to an interval and that a separable, ***locally compact*** metrizable space with the UMP is homeomorphic to a subspace of the real line. Metrizable spaces that are homeomorphic to a subspace of the real line are also characterized in terms of another metric property [5]: A separable metrizable space X is homeomorphic to a subspace of the real line if and only if X admits a metric satisfying both of the following conditions: (i) The cardinality of any subset consisting of points which are equidistant from two distinct points is at most 1. (ii) The cardinality of any subset consisting of points which are equidistant from a point is at most 2. Furthermore, if X is locally compact, then X is homeomorphic to a subspace of the real line if and only if X admits a metric satisfying the condition (i) only.

Several spaces are known to have the UMP, or do not have the UMP (Ohta–Ono, 2000, and [6]): (a) Let I and J be separated intervals in the real line $\mathbb{R}$. Then $I \cup J$ has the UMP if and only if at least one of I and J is not compact.

(b) The union of odd numbers of disjoint closed intervals in $\mathbb{R}$ has the UMP. (c) The subsets $[0, 1] \cup \mathbb{Z}$ and $[0, 1] \cup \mathbb{Q}$ of $\mathbb{R}$ do not have the UMP, where $\mathbb{Z}$ and $\mathbb{Q}$ denote the sets of the integers and rational numbers, respectively. (d) Let X be the union of at most countably many subsets X_n of $\mathbb{R}$. If each X_n is either an interval or ***totally disconnected*** and if at least one of X_n is a noncompact interval, then X has the UMP. (e) A discrete space D has the UMP if and only if $|D| \neq 2, 4$ and $|D| \leqslant \mathfrak{c}$, where $|D|$ denotes the cardinality of D and $\mathfrak{c}$ is the cardinality of the continuum. (f) Let D be the discrete space with $|D| \leqslant \mathfrak{c}$. Then the product of countably many copies of D has the UMP. In particular, the ***Cantor set*** and the space of irrationals have the UMP. The UMP of a finite discrete space can be considered in terms of graph theory. Let G be a simple graph (i.e., a graph which does not contain neither multiple edges nor loops). By a **coloring** of G we mean a map defined on the set of edges of G. A coloring φ of G is said to have the **unique midset property** if for every pair of distinct vertices x and y there is a unique vertex p such that xp and yp are edges of G and $\varphi(xp) = \varphi(yp)$. Let K_n be the **complete graph** (i.e., each vertex of K_n is adjacent to every other vertices) with n vertices. Then, a finite discrete space with n points has the UMP if and only if there is a coloring φ of the complete graph K_n with the UMP [6].

The **double midset conjecture** seems to be very interesting: A ***continuum*** (i.e., non-degenerate ***connected*** compact metric space) having the DMP must be a ***simple closed curve***. The conjecture still remains open. However, L.D. Loveland proved several partial results about the conjecture. A continuum X with the DMP satisfying either of the following conditions is a simple closed curve: (a) X contains a continuum with no ***cut points*** (L.D. Loveland and S.G. Wayment, 1974). (b) The midset function $M: \{(x, y) \in X \times X: x \neq y\} \to 2^X$ is continuous (L.D. Loveland, 1976). (c) $X \subset \mathbb{R}^2$ with the Euclidean metric [7]. Furthermore, L.D. Loveland and S.M. Loveland [8] proved that every continuum in the Euclidean plane with the n-MP for $n \geqslant 1$ must either be a simple closed curve or an ***arc***.

We shall consider a more general setting of the double midset conjecture: A nondegenerate compact metric space such that all of its midsets are homeomorphic to an $(n-1)$-sphere S^{n-1} is homeomorphic to an n-sphere S^n. The double midset conjecture is the special case, where $n = 1$, of this conjecture. A space is said to have the **k-sphere midset property** if each midset of X is homeomorphic to a k-sphere S^k. We have a few results in this direction: (a) If X is a metric space with 1-sphere midset property, and if X contains a subset homeomorphic to 2-sphere, then X is a 2-sphere (L.D. Loveland, 1977). (b) If X is a nondegenerate compact metric space with 1-sphere midset property and every simple closed curve separates X, then X is a 2-sphere (L.D. Loveland, 1977). (c) Let $n \geqslant 3$ and X be a nondegenerate compact subset of the n-dimensional Euclidean space such that each of its midsets is a convex $(n-2)$-sphere (= the boundary of a ***convex*** $(n-1)$-***cell***). Then X is a convex $(n-1)$-sphere (W. Dębski, K. Kawamura and K. Yamada, 1994).

Two subsets A and B of a metric space (X, ρ) are said to be **congruent** if there is an ***isometry*** between them. Then we have the following theorems which relate to certain dimensional properties (see [HM, Chapter 10, Section 2]):

(a) A separable metrizable space X is strongly zero-dimensional if and only if X admits a metric relative to which no two distinct sets of cardinality two are congruent.
(b) The real line $\mathbb{R}$ admits a metric relative to which no two distinct sets of cardinality three are congruent.
(c) If a locally compact separable metrizable space X admits a metric relative to which no two distinct sets of cardinality three are congruent, then X is homeomorphic to a subspace of the real line (and hence $\dim X \leqslant 1$).
(d) If a metrizable space X admits a metric relative to which no two distinct sets of cardinality three are congruent, then $\operatorname{ind} X \leqslant 1$, where $\operatorname{ind} X$ denotes the ***small inductive dimension*** of X.

Let (X, ρ) be a metric space and n a natural number. A point $x \in X$ is said to be of **metric order of** n if for each positive number r, $|\{y \in X: \rho(x, y) = r\}| = n$. A metric space (X, ρ) is said to have **metric order** n provided that every point of X is of metric order n. Then M.R. Currie proved the following: (a) A separable metrizable space X is homeomorphic to the space of irrationals if and only if X admits a complete metric of order 1 [HM, Chapter 10, Theorem 4.2]. (b) If a separable ***locally connected*** metrizable space X admits a complete metric of order 2, then X is homeomorphic to the real line $\mathbb{R}$ or $S^1 \times \omega$ [HM, Chapter 10, Theorem 4.3]. Y. Hattori also proved that if a metrizable space X admits a metric ρ such that the metric order of every point of X at most 1, then $\operatorname{ind} X \leqslant 0$ [HM, Chapter 10, Theorem 2.17(ii)].

The notion of the metric order seems to have originated from a similar, but topological, notion introduced in [Ku II, §51]. Let (X, ρ) be a metric space and $\mathfrak{n}$ a cardinal $\leqslant \mathfrak{c}$. A point $x \in X$ is said to be of **order** $\leqslant \mathfrak{n}$ if for every $\varepsilon > 0$ there is an open neighbourhood U of x such that the diameter of U is less than ε and $|\operatorname{Bd} U| \leqslant \mathfrak{n}$. We give some examples. It is easy to see that every point of the ***Cantor set*** with the usual metric has order 0. The second example is the **Sierpiński curve** (or the **Sierpiński Carpet**), which is defined as follows. Let $F_0 = I^2$ be the unit square. Divide F_0 into nine congruent squares and delete the interior of the central square. Let F_1 be the subset of F_0 which is the union of remaining eight squares. Divide each of eight squares included in F_1 into nine congruent squares and delete the interior of the central square. Let F_2 be the subset of F_1 which is the union of remaining 8^2 squares. Continue in this manner step by step. Then we have a decreasing sequence F_0, F_1, $F_2, \ldots$ of closed subsets of the unit square I^2. The intersection $S = \bigcap_{i=0}^{\infty} F_i$ is the Sierpiński curve (or the Sierpiński carpet). All points of the Sierpiński curve have order $\mathfrak{c}$. The third example is the **triangular Sierpiński curve** (or the **Sierpiński Gasket**), which is constructed similar to the construction of the Sierpiński carpet, but here we use equilateral triangles instead of squares. In the Sierpiński gasket, only the three vertices of the original triangle have of order 2. The vertices of the countably many divided triangles have order 4 and all of other points of the Sierpiński gasket have order 3. The order of spaces may determine certain topological properties. For example, if all of points of a continuum (X, ρ) have order 2, then X is a simple closed curve. Furthermore, if all of points of a continuum (X, ρ) have the same order n $(n > 0)$, then X is a simple closed curve. Other results related to the order of (connected) spaces are found in [KI, § 51].

We shall discuss infinite-valued metrics. By an **infinite-valued metric** we mean a function $\sigma : X \times X \to [0, \infty]$ satisfying the usual metric axioms. A pair (X, σ) of a set X and an infinite-valued metric σ is called an **extended metric space**. Every extended metric space (X, σ) can be topologized in the same way as a metric space. The resulting topology coincides with the topology induced by the metric $\rho(x, y) = \min\{\sigma(x, y), 1\}$. Let X be a set and (Y, ρ) a metric space. Let $F(X, Y)$ and $F^*(X, Y)$ be the set of maps, and the set of bounded maps of X to Y, respectively. If X is a topological space, then we denote the subsets of $F(X, Y)$ and $F^*(X, Y)$, respectively, which consist of all continuous maps by $C(X, Y)$ and $C^*(X, Y)$. The function $d : F(X, Y) \times F(X, Y) \to [0, \infty]$ defined by $d(f, g) = \sup_{x \in X} \rho(f(x), g(x))$ is an infinite-valued metric on $F(X, Y)$. The restriction d' of d to $F^*(X, Y)$ is a metric on $F^*(X, Y)$. The infinite-valued metric d and the metric d' above are called **uniform metrics**. The topology induced by a uniform metric is called the **topology of uniform convergence**. If a metric space (Y, ρ) is complete, then $(F^*(X, Y), d')$ and $(C^*(X, Y), d')$ are complete metric spaces.

There are other well-known examples of infinite-valued metrics, e.g., infinite-valued metrics on families of closed subsets of metric spaces. Let (X, ρ) be a metric space and $CL(X)$ be the family of all non-empty closed subsets of X. For $A, B \in CL(X)$, we define $\mathrm{Hd}_\rho(A, B) = \max\{\sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(b, A)\}$. Then, Hd_ρ is an infinite-valued metric on $CL(X)$ and is called the **Hausdorff metric** on $CL(X)$. It is easy to show $\mathrm{Hd}_\rho(A, B) = \inf\{\varepsilon > 0: S_\varepsilon(A) \subset B \text{ and } S_\varepsilon(B) \subset A\} = \sup_{x \in X} |\rho(x, A) - \rho(x, B)|$. Hence, the extended metric space $(CL(X), \mathrm{Hd}_\rho)$ is isometric to a subspace of $C(X, \mathbb{R})$ with the uniform metric by the isometry defined by $\varphi(A) = \rho(\cdot, A)$ for each $A \in CL(X)$. The Hausdorff metric was introduced at the origin of topology and it is one of the most important concepts in topology as well as functional analysis. Some natural relations between the Hausdorff metric and the underlying metric are known: (a) The metric space (X, ρ) is totally bounded if and only if the space $(CL(X), \mathrm{Hd}_\rho)$ is totally bounded. (b) The metric space (X, ρ) is complete if and only if the space $(CL(X), \mathrm{Hd}_\rho)$ is complete. (c) The metric space (X, ρ) is compact if and only if the space $(CL(X), \mathrm{Hd}_\rho)$ is compact. Let ρ and σ be compatible metrics on a metrizable space X. Then the topologies induced by the Hausdorff metrics Hd_ρ and Hd_σ on $CL(X)$ coincide if and only if ρ and σ induce the same ***uniformity***.

We turn our attention to ***normed linear spaces***. For a normed linear space $(X, \|\cdot\|)$, let $d_{\|\cdot\|}$ be the metric determined by the ***norm*** $\|\cdot\|$. Then, the Hausdorff metric $\mathrm{Hd}_{\|\cdot\|}$ induced by the metric $d_{\|\cdot\|}$ on non-empty closed bounded convex sets can be expressed as follows: $\mathrm{Hd}_{\|\cdot\|}(A, B) = \sup\{|s(f, A) - s(f, B)|: f \text{ is a } \textit{linear functional} \text{ with norm} \leqslant 1\}$, where $s(f, A) = \sup\{f(a): a \in A\}$ is the ***support functional***. This representation leads the Hörmander's Theorem [2, Theorem 3.2.9]: Let $(X, \|\cdot\|)$ be a normed linear space. Then the space of non-empty closed bounded convex subsets of X with the Hausdorff metric $\mathrm{Hd}_{\|\cdot\|}$ can be algebraically and isometrically embedded in the ***Banach space*** of bounded continuous real functions on the closed unit ball of the ***dual space*** X^* of X equipped with the norm of uniform metric $\|\varphi\|^* = \sup\{\varphi(f): f \text{ is a linear functional with norm} \leqslant 1\}$.

The following theorem due to D. Curtis and R. Schori is one of the most important results in the theory of hyperspace topologies [3]: If (X, ρ) is a compact metric space, then $(CL(X), \mathrm{Hd}_\rho)$ is homeomorphic to the ***Hilbert cube*** if and only if X is a non-degenerate ***Peano continuum*** (i.e., connected and locally connected metric compact space). The theory of hyperspaces (i.e., spaces of (non-empty) closed subsets) of continua with the Hausdorff metrics have been successfully developed and the details are mentioned in another part of the Encyclopedia.

Hyperspaces of metrizable spaces may have many different topologies. We shall consider certain relations between the Hausdorff metric topology and other hyperspace topologies. Let $\mathcal{T}_\rho$ and $\mathcal{T}_{\mathrm{Hd}_\rho}$ be topologies induced by the metric ρ and the Hausdorff metric Hd_ρ, respectively. The Vietoris topology is another well-known hyperspace topology and the Hausdorff metric topology is closely related to the Vietoris topology. The **Vietoris topology** on $CL(X)$ is defined as the topology generated by all sets of the form $V^- = \{A \in CL(X): A \cap V \neq \emptyset\}$ and $W^+ = \{A \in CL(X): A \subset W\}$, where V and W are open sets of X. The Vietoris topology is stronger than the Hausdorff metric topology if and only if (X, ρ) is totally bounded. Conversely, the Hausdorff metric topology contains the Vietoris topology if and only if (X, ρ) is u-normal space (E. Michael, [9]), where a metric space (X, ρ) is called a **u-normal space** if $\rho(A, B) > 0$ for every disjoint closed subsets A and B of X. A u-normal space is sometimes called a **UC-space**. Since every u-normal space is complete, the Hausdorff metric topology coincides with the Vietoris topology if and only if (X, ρ) is compact. The details of u-normal spaces are mentioned below. We shall make a quick review of other hyperspace topologies. Let X be a metrizable space and $\mathcal{V}$ be a locally finite family of non-empty open sets of X. We put $\mathcal{V}^- = \{A \in CL(X): V \cap A \neq \emptyset \text{ for each } V \in \mathcal{V}\}$. The **locally finite topology** on $CL(X)$ is defined as the topology which has the subbase consisting of all of the sets of the form $\mathcal{V}^-$, where $\mathcal{V}$ be a locally finite family of non-empty open sets of X and sets of the form V^+, where V is an open set of X. The locally finite topology is the supremum of all of Hausdorff metric topologies induced by compatible metrics on X

(G. Beer, C. Himmelberg, C. Prikry and F. van Vleck (see [HM, Chapter 10, Theorem 3.6] or [2, Theorem 3.3.12]). Furthermore, they proved that the following conditions are equivalent for a metrizable space X [HM, Chapter 10, Theorem 3.7]:

(a) There is a compatible metric ρ of X such that (X,ρ) is a u-normal space.
(b) The locally finite topology on $CL(X)$ is metrizable.
(c) The locally finite topology on $CL(X)$ is first-countable.
(d) There is a compatible metric ρ of X such that the Hausdorff metric topology $\mathcal{T}_{\mathrm{Hd}_\rho}$ coincides with the locally finite topology.

We refer the reader to the book [2] for the details of hyperspace topologies (e.g., **Wijsman topology**, **Attouch–Wets topology**, locally finite topology, ***finite topology***, ***Fell topology***, **Mosco topology**, etc.) on the space of (non-empty) closed sets of metric spaces and (non-empty) closed convex subsets of normed linear spaces.

Finally, we shall consider u-normal spaces. The u-normality is a metric property. A closed subspace of a u-normal space is u-normal. A locally compact u-normal space can be represented as a union of a compact subset and a uniformly discrete set, where a subset A of a metric space (X,ρ) is **uniformly discrete** if there is an $\varepsilon > 0$ such that $\rho(a,b) \geqslant \varepsilon$ for each distinct points $a, b \in A$. The following conditions are equivalent for a metric space (X,ρ) (see [HM, Chapter 10, Theorem 3.4] and [2, Section 2.3]): (a) (X,ρ) is a u-normal space. (b) The set X' of all ***accumulation point***s of X is compact, and $X \setminus S_\varepsilon(X')$ is uniformly discrete for each $\varepsilon > 0$. (c) Every real-valued continuous function is ***uniformly continuous***. (d) Every continuous map from X to any metric space Y is uniformly continuous. (e) Each open cover of X has a ***Lebesgue number***. (f) For every sequence $\{x_n: n = 1,2,\ldots\}$ of X with $\lim_{n\to\infty} \rho(x_n, X\setminus\{x_n\}) = 0$, $\{x_n: n = 1,2,\ldots\}$ has a ***cluster point***. (g) Each pseudo-Cauchy sequence with distinct term has a cluster point, where a sequence $\{x_n: n = 1,2,\ldots\}$ is said to be **pseudo-Cauchy** if for every $\varepsilon > 0$ and every natural number n there are distinct integers $k > n$ and $m > n$ such that $\rho(x_m, x_k) < \varepsilon$. On the other hand, we have the topological characterizations of a metrizable space X that admits a u-normal metric (see [HM, Chapter 10, Theorem 3.2]): (i) X admits a u-normal metric ρ. (ii) The ***fine uniformity*** of X is metrizable. (iii) The set X' of all accumulation points of X is compact. (iv) All T_1-***quotient*** spaces of X are metrizable. (v) All closed continuous images of X are metrizable. (vi) Every closed subset of X has a compact boundary. (vii) Every closed subset of X has a countable ***neighborhood base***. (viii) For every ***zero set*** F of X, the closure of F in the ***Stone–Čech compactification*** βX of X is a zero set of βX. (ix) There is a sequence $\{\mathcal{U}_n: n = 1,2,\ldots\}$ of open covers of X such that for any ***discrete*** closed subset D of X there is a natural number n such that for any distinct points $x, y \in D$, $\mathrm{St}(x,\mathcal{U}_n) \cap \mathrm{St}(y,\mathcal{U}_n) = \emptyset$, where $\mathrm{St}(x,\mathcal{U}_n) = \bigcup\{U \in \mathcal{U}_n: x \in U\}$. (x) There is a sequence $\{\mathcal{U}_n: n = 1,2,\ldots\}$ of open covers of X such that for any disjoint discrete closed subsets D and E of X there is a natural number n such that $\mathrm{St}(D,\mathcal{U}_n) \cap \mathrm{St}(E,\mathcal{U}_n) = \emptyset$, where $\mathrm{St}(D,\mathcal{U}_n) = \bigcup\{U \in \mathcal{U}_n: U \cap D \neq \emptyset\}$.

The reader may find other results on special metrics in [HM, Chapter 10], but some of results mentioned here are also included in [HM, Chapter 10].

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e-5 Completeness

A metric space (X, ρ) is a **complete metric space** (and ρ is a **complete metric**) if any decreasing sequence $A_1 \supset A_2 \supset A_3 \supset \cdots$ of nonempty closed subsets of X with the diameters converging to zero has nonempty intersection. Equivalently, (X, ρ) is complete if every Cauchy sequence in the space converges to a point in X, where $(x_n)_{n=1}^{\infty}$ is a **Cauchy sequence** if the diameters of the sets $A_m = \{x_n : n \geqslant m\}$ converge to zero.

Compactness implies completeness and a complete metric space (X, ρ) is compact if and only if for each $\varepsilon > 0$ it can be covered by finitely many sets with diameters less than ε, i.e., the metric ρ is **totally bounded**.

The completeness of the real line $\mathbb{R}$ with the Euclidean metric is vital for Analysis. Actually, one constructs the real line from the rationals to guarantee completeness. The first constructions go back to Ch. Méray and K. Weierstrass who considered Cauchy series with rational terms, i.e., the series whose partial sums form Cauchy sequences of rational numbers. Exact constructions of the real line were provided by G. Cantor and R. Dedekind in 1872. Cantor determined the reals by equivalence classes of Cauchy sequences of rationals, where the distances between the corresponding terms of the equivalent sequences converge to zero. Dedekind identified the reals with cuts in the ordered set of the rationals, cf. [2]. In full generality, complete metric spaces were introduced by M. Fréchet in 1906. Each metric space (X, ρ) can be extended to a complete metric space (X^*, ρ^*) with X being dense in X^*, and all **completion**s (X^*, ρ^*) are equivalent in a natural way. A completion (X^*, ρ^*) can be obtained by a generalization of the Cantor process of completing the rationals, cf. Hausdorff [8].

The process of completing a **normed space** $(E, \|\cdot\|)$ – a linear space with the metric $\rho(x, y) = \|x - y\|$ determined by the ***norm*** $\|\cdot\|$, yields a complete linear normed space $(E^*, \|\cdot\|^*)$, i.e., a **Banach space**. Useful completions of metric spaces are provided by isometric embeddings into Banach function spaces. Any metric space (X, ρ) embeds isometrically into the Banach space of bounded continuous real-valued functions on X with the supremum norm (K. Kuratowski, K. Kunugui); (X, ρ) can be also embedded isometrically onto a linearly independent subset, closed in its linear span, of the Banach space $l^{\infty}(S)$ of bounded real-valued functions on a set S, with the supremum norm (M. Wojdysławski, R. Arens and J. Eells).

Important examples of completions are the fields $\mathbb{Q}_p$ of p-adic numbers. For each prime p one considers the valuation $|\ |_p$ on the rationals $\mathbb{Q}$ defined by $|p^r m/n|_p = p^{-r}$, where r, m, n are integers with m, n not divisible by p, and the associated metric $\rho_p(s, t) = |s - t|_p$, $s \neq t$. The algebraic operations extend continuously from $\mathbb{Q}$ to the completion $(\mathbb{Q}_p, \rho_p^*)$ of $(\mathbb{Q}, \rho_p)$, cf. [9].

The completion of the normed space $(C([0, 1]), \|\cdot\|)$ of continuous real-valued functions on $[0, 1]$ with $\|f\| = \int_0^1 |f(t)|\, dt$ can be identified with the Banach space L^1 obtained from the space of Lebesgue integrable functions on $[0, 1]$ upon the identification of the functions equal almost everywhere in the sense of Lebesgue measure, cf. [11, Chapter 10].

We refer the reader to [15, Chapter I,10] for some other instances of completions of great significance for Analysis.

Yet another example concerns Carathéodory's "prime ends" of ***simply connected*** bounded regions in the complex plane, obtained as a result of the completion of the region with respect to Mazurkiewicz's metric, cf. [4, Chapter 9]. Assume in addition that the region G contains the closed unit disc D centered at 0, and let for $x, y \in G \setminus D$, the distance $\rho_0(x, y)$ be the infimum of Euclidean diameters of closed ***connected*** sets in G contained in $G \setminus D$ which separate in G the point x from 0, but not from y. The metric ρ_0 extends to a metric ρ on G generating the Euclidean topology. In the completion (G^*, ρ^*) of (G, ρ) the adjoined points in $G^* \setminus G$ are the prime ends of G. Each conformal map from G onto the interior of D extends to a homeomorphism of G^* onto D (C. Carathéodory).

Many applications of complete metric spaces deal with fixed points of nonexpansive maps, cf. [5]. Among the most useful results in this area is the **Banach Contraction Principle** asserting that any map $T : X \to X$ of a complete metric space (X, ρ) into itself, satisfying $\rho(T(x), T(y)) \leqslant c\rho(x, y)$, $0 \leqslant c < 1$, i.e., a **contraction**, has a unique fixed point. Closely related is the fact that for any finite collection $\mathcal{T}$ of contractions of a complete metric space (X, ρ) there is a unique nonempty compact set K such that $K = \bigcup\{T(K) : T \in \mathcal{T}\}$ (J.E. Hutchinson). For certain finite collections $\mathcal{T}$ of contractions of Euclidean spaces taking sets to similar ones, the invariant sets K provide striking examples of self-similar sets, cf. [7, Chapter 8.3].

A space X is **completely metrizable** if there is a metric ρ on X generating the topology of X such that the metric space (X, ρ) is complete. For metrizable X, complete metrizability of X is equivalent to the property that for some (equivalently, for any) metric ρ inducing the topology of X, X is a ***G_δ-set*** in the completion X^* of (X, ρ). In particular, all G_δ-subsets of completely metrizable spaces are completely metrizable (F. Hausdorff; P.S. Aleksandrov – for separable spaces). Furthermore, this is equivalent to the statement that for any embedding $e : X \to Y$ into a ***completely regular*** space Y, $e(X)$ is a G_δ-set in its closure in Y. The last property, coupled with complete regularity, defines the class of **Čech-complete space**s. A completely regular space X is Čech-complete if and only if X is a G_δ-set in the ***Čech–Stone compactification*** βX (equivalently, in any compactification of X).

Locally compact spaces are Čech-complete and Čech-complete spaces are ***Baire space***s. Čech-completeness is preserved by closed subsets, G_δ-subsets, countable ***products*** and, in both directions, by ***perfect maps***. The class of perfect preimages of completely metrizable spaces coincides with the class of ***paracompact*** Čech-complete spaces (Z. Frolík).

An essential feature of complete metrizability is that it allows extensions of maps over G_δ-sets, preserving some vital properties of the maps. Let $A \subset X$, $B \subset Y$ be subspaces of completely metrizable spaces X, Y and let $f : A \to B$ be a homeomorphism. Then there are G_δ-sets, $A^* \supset A$ in X and $B^* \supset B$ in Y, and a homeomorphism $f^* : A^* \to B^*$ extending f (the **Lavrentieff Theorem**). If f is a ***closed*** surjection, a similar assertion is true with f^* being a closed map onto B^* (I.A. Vaĭnšteĭn). If f is an ***open*** surjection and A is separable, an analogous fact is true where f^* is an open map onto B^* (S. Mazurkiewicz). The extension of closed maps to closed maps over G_δ-sets yields immediately that closed maps between metrizable spaces preserve complete metrizability. If the range is nonmetrizable it need not be Čech-complete, but it always contains a dense completely metrizable subspace (K.R. Van Doren).

Although the corresponding extension theorem for open maps fails outside the class of separable spaces [E, 4.5.14(c)], nevertheless, open maps between metrizable spaces also preserve complete metrizability (F. Hausdorff; for separable spaces – W. Sierpiński). For any open map $f : X \to Y$ from a Čech-complete space onto a paracompact space Y there is a closed G_δ-set E in X such that the restriction $f|E : E \to Y$ is a perfect surjection, and in particular, Y is Čech-complete (B.A. Pasynkov). If both X and Y are metrizable, the existence of the set E follows from a weaker assumption that all ***fibers*** of f are complete in a fixed metric on X (E. Michael).

There is, however, an open map with finite fibers $f : X \to Y$ from a completely metrizable X onto a completely regular Y that is not Čech-complete (J. Chaber, M.M. Čoban, K. Nagami); the space Y is ***metacompact*** and no perfect restriction of f covers Y.

Any open map $f : X \to Y$ of a Čech-complete space onto a ***Hausdorff space*** is **compact-covering**, i.e., each compact set in Y is an image of a compact set in X (N. Bourbaki). Compact-covering maps with separable fibers between metrizable spaces preserve complete metrizability (A.V. Ostrovskiĭ, E. Michael; for separable spaces – J.P.R. Christensen). This yields a remarkable characterization of completely metrizable separable spaces X in terms of the space $\mathcal{K}(X)$ of compact subsets of X with the ***Vietoris topology***: a metrizable separable X is completely metrizable if and only if $\mathcal{K}(X)$ is a continuous image of the irrationals $\mathbb{P}$, i.e., it is an ***analytic set*** (J.P.R. Christensen). Indeed, if $\varphi : \mathbb{P} \to \mathcal{K}(X)$ is a continuous surjection, the space $E = \{(t, x): x \in \varphi(t)\}$ is completely metrizable and the projection of E onto X is compact-covering. A particular instance of this fact is a classical theorem of Hurewicz that for the rationals $\mathbb{Q}$, $\mathcal{K}(\mathbb{Q})$ is not analytic.

Given a completely metrizable space X and a metrizable space T with $\dim T = 0$, any lower-semi-continuous map $\varphi : T \to \mathcal{F}(X)$ to the collection of nonempty closed subsets of X has a continuous selection (E. Michael). The **lower-semi-continuity** of the set-valued map φ means that $\{t: \varphi(t) \cap U \neq \emptyset\}$ is open, whenever U is open, and $s : T \to X$ is a selection for φ if $s(t) \in \varphi(t)$, for $t \in T$. This property of metrizable spaces X characterizes the completely metrizable ones (J. van Mill, J. Pelant, R. Pol). Closely related to this topic are selections $s : \mathcal{F}(X) \to X$, continuous with respect to the Vietoris topology on $\mathcal{F}(X)$. If X is a completely metrizable space with $\dim X = 0$, there is a linear order on X such that each $F \in \mathcal{F}(X)$ has the first element in this order, and the choice of the first elements is a continuous selection on $\mathcal{F}(X)$ (R. Engelking, R.W. Heath, E. Michael). On the other hand, for any metrizable X, the existence of a continuous selection $s : \mathcal{F}(X) \to X$ yields complete metrizability of X.

The Lavrentieff theorem on extending homeomorphisms over G_δ-sets shows that being a ***Borel set*** in a completely metrizable space is an absolute property: if A is a Borel set in a completely metrizable space X and $e : A \to Y$ embeds A in a metrizable space Y then A is a Borel set in Y. Up to **Borel isomorphism**s (i.e., bijections preserving Borel sets in both directions) there are only two types of infinite Borel sets in separable completely metrizable spaces: the countable ones and the sets Borel isomorphic to the unit interval. The much more complex classification of Borel sets in arbitrary completely metrizable spaces was accomplished by A.H. Stone [13].

Čech-completeness of completely regular spaces is characterized by the existence of a complete sequence of open covers (A.V. Archangelskiĭ, Z. Frolík). The sequence $(\mathcal{E}_n)_{n=1}^\infty$ of (not necessarily open) covers of X is a **complete sequence of covers** if every sequence $(E_n)_{n=1}^\infty$ with $E_n \in \mathcal{E}_n$, $n = 1, 2, \dots,$ is a **complete sequence**, i.e., for each collection $\mathcal{F}$ of nonempty subsets of X containing all E_n and closed under finite intersections, the intersection $\bigcap\{\overline{F}: F \in \mathcal{F}\}$ is nonempty.

For metrizable spaces complete metrizability is equivalent to the existence of a complete sequence of **exhaustive** covers (E. Michael), where a cover $\mathcal{E}$ of X is exhaustive if for any nonempty subspace $S \subset X$ there is $E \in \mathcal{E}$ with $E \cap S$ nonempty and relatively open in S. The complete sequences of exhaustive covers provide a very efficient tool. A noteworthy application is the following result: a metrizable space Y which is the image of a complete metric space (X, ρ) under a continuous map taking metrically discrete sets in X to ***scattered*** sets in Y is completely metrizable (E. Michael; for separable spaces – N. Ghoussoub and B. Maurey).

A **cover-complete** (or **partition-complete**) space is a ***regular*** space admitting a complete sequence of exhaustive covers. Čech-complete spaces are cover-complete and all cover-complete spaces contain a dense Čech-complete subspace, i.e., are **almost complete**, cf. [10]. In particular, cover-complete spaces are Baire. All regular scattered spaces are cover-complete. Cover-completeness is preserved by closed subsets, G_δ-subsets, countable products, open maps onto regular spaces and, in both directions, by perfect maps between regular spaces (R. Telgársky and H.H. Wicke).

The invariance of cover-completeness can be used to show that each cover-complete topological group is Čech-complete and paracompact. More specifically, any cover-complete topological group G contains a compact subgroup K such that the coset space G/K is metrizable and the projection $\pi : G \to G/K$ is open and perfect (B.A. Pasynkov). In effect, the metrizable space G/K is cover-complete, hence completely metrizable, and its perfect preimage G is paracompact and Čech-complete. Actually, if G is a cover-complete space which carries a group structure such that the multiplication is separately continuous, then G is a topological group, i.e., the inversion is continuous and the multiplication is jointly continuous (P.S. Kenderov, I.S. Kortezov, W.B. Moors; R. Ellis – for locally compact G, and A. Bouziad – for Čech-complete G).

The Banach spaces E whose closed ***unit ball*** with the ***weak topology*** $(B_E, weak)$ is cover-complete or Čech-complete have remarkable structural properties, cf. [3, Chapter 4]. The ball $(B_E, weak)$ is cover-complete if and only if each nonempty closed set A in B_E admits a point of continuity from the weak to norm topologies on A, and it is Čech-complete if and only if E splits into a direct sum of Banach spaces $E = X \oplus Y$, where $(B_X, weak)$ is compact (i.e., X is reflexive) and $(B_Y, weak)$ is completely metrizable separable, with the dual of Y also separable (G.A. Edgar and R.W. Wheeler). For the Banach space l^1 of summable sequences of reals, $(B_{l^1}, weak)$ is cover-complete, but not Čech-complete.

It is worth noticing that any continuous bijection $f : X \to Y$ with regular almost complete domain and Hausdorff range, which takes open in X sets U to sets $f(U)$ contained in the interior of the closure $\overline{f(U)}$, is a homeomorphism (D. Noll; J.D. Weston – for completely metrizable spaces). This minimality property underlines some interesting generalizations of the **Banach Open Mapping Theorem** asserting that any continuous linear surjection between Banach spaces is open, cf. [12].

An important topic involving completeness is the existence of ***arc***s in topological spaces. To this area belong the following three classical results, the first concerned with convexity in metric spaces, the second dealing with Riemannian manifolds, and the last one being the celebrated **Mazurkiewicz–Moore theorem** on ***arcwise connectedness***.

Let (X, ρ) be a complete metric space such that for any $x, y \in X$ there is $z \in X \setminus \{x, y\}$ with $\rho(x, z) + \rho(z, y) = \rho(x, y)$. Then for any $x, y \in X$ there is an isometric embedding of the Euclidean interval $e : [0, a] \to X$, $a = \rho(x, y)$, with $e(0) = x$ and $e(a) = y$ (K. Menger). Let M be a Riemannian manifold and let $\rho(x, y)$ be the infimum of the lengths of piecewise smooth arcs from x to y in M; the metric generates the manifold topology in M. Then completeness of (M, ρ) implies that for any $x, y \in M$ there is an isometric embedding $e : [0, a] \to M$, $a = \rho(x, y)$, with $e(0) = x$ and $e(a) = y$ (H. Hopf and W. Rinow). Finally, the Mazurkiewicz–Moore theorem asserts that for any two points x, y in a completely metrizable, connected and ***locally connected*** space X, there is an arc joining x and y, i.e., a homeomorphic embedding $e : [0, 1] \to X$ with $e(0) = x$ and $e(1) = y$ (the theorem was proved by Mazurkiewicz and Moore for compact spaces, and extended to the completely metrizable ones by K. Menger). The assertion is true for any regular space X with a sequence $(\mathcal{B}_n)_{n=1}^{\infty}$ of bases such that each decreasing sequence $(B_n)_{n=1}^{\infty}$, where $B_n \in \mathcal{B}_n$, $n = 1, 2, \ldots$, converges to a point $x \in X$, i.e., any neighbourhood of x contains almost all B_n (N. Aronszajn). Among regular spaces, this property characterizes cover-complete spaces with a ***base of countable order***, or equivalently, open images of completely metrizable spaces (H.H. Wicke).

A convenient instrument in the investigation of open images of complete spaces are **open sieves**, i.e., collections of open subsets of X indexed by a ***tree*** T of countable height, such that the sets corresponding to the indices at the first level of T cover X, and the element indexed by $t \in T$ is the union of the elements indexed by the immediate successors of t, cf. [KV, Chapter 10, 6].

Any open image of a Čech-complete space has a **complete open sieve**, i.e., an open sieve such that for each branch of the indexing tree the sequence of the sets corresponding to this branch is complete. Regular spaces having a complete open sieve are called **sieve-complete** (or **λ_b-space**s, in the terminology of H.H. Wicke and J.M. Worrell Jr). For regular spaces, the class of sieve-complete spaces coincides with the class of open images of paracompact Čech-complete spaces (H.H. Wicke), and therefore paracompact sieve-complete spaces are Čech-complete (J.M. Worrell Jr). Any sieve-complete unit ball $(B_E, weak)$ of a Banach space E, equipped with the weak topology, is Čech-complete, cf. [6, 4.3]

Sieve-complete spaces are cover-complete. The stability properties with respect to maps and product operations, which have been indicated for cover-completeness, are also true for sieve-completeness. Any countable scattered regular space which is not ***first-countable*** is cover-complete, but not sieve-complete. In the class of spaces with a ***base of countable order***, in particular, in the class of Moore spaces, the two properties are equivalent (H.H. Wicke). There exists, however, a completely regular ***Moore space*** which is sieve-complete, but not Čech-complete (M.E. Rudin), cf. [1, III].

Relaxing the definition of complete sequences by assuming that the collections $\mathcal{F}$ are countable, one gets wider classes of spaces which include ***countably compact*** ones. There are also some other useful completeness-type properties which are equivalent to complete metrizability in the realm of metrizable spaces, cf. [1, 14]. Among them is the existence of a winning strategy for player II in the **strong Choquet game**. The game in a space X is played by two players I and II who choose subsequently $(U_0, x_0), V_0, (U_1, x_1), V_1, \ldots$, where $x_n \in U_n$, the sets U_n, V_n are open and the moves of player II are subject to the condition $x_n \in V_n \subset U_n$, while I has to play $U_{n+1} \subset V_n$. Player II wins the game if the intersection $\bigcap_n V_n$ is nonempty, and the winning strategy tells player II the next move, depending on the previous moves, leading eventually to the success in the game. The existence of such strategies defines the class of **strong Choquet space**s. The existence of a winning strategy that always provides player II with a complete sequence $(V_n)_n$ characterizes, among regular spaces, the sieve-complete ones (F. Topsøe). Some characterizations of cover-completeness and almost completeness in terms of games are given in [10].

Strong Choquet spaces are ***Choquet spaces***, hence Baire; however, they need not be ***Namioka spaces*** (M. Talagrand). Although closed subspaces of strong Choquet spaces are not necessarily Baire, any product of strong Choquet spaces is strong Choquet. In contrast, the conditions defined in terms of complete sequences are inherited by closed subsets, but no uncountable product of the natural numbers with the discrete topology is almost complete, and therefore, such products are not cover-complete. The ***Sorgenfrey line*** shows that even first-countability and ***Lindelöf*** property of a strong Choquet space do not ensure almost completeness.

The spaces homeomorphic to closed subsets of products of completely metrizable spaces are called **Dieudonné complete**. The "completeness" refers to the existence of a ***complete uniformity*** generating the topology, and this property is of quite different character than the completeness properties discussed hitherto – all paracompact spaces are Dieudonné complete. Each completely regular space embeds onto a dense subset of a Dieudonné complete space X^* such that any continuous map $f : X \to E$ with metrizable range can be extended continuously over X^*. The space X^* is called a Dieudonné completion of X and any two Dieudonné completions of X are equivalent in a natural way. The intersection of all paracompact subspaces of the Čech–Stone compactification βX of X containing X is a Dieudonné completion of X. Countably compact Dieudonné complete spaces are paracompact. A Dieudonné complete space X is ***realcompact***, i.e., it embeds onto a closed subspace of a product of the real line, if and only if every discrete closed subset of X is of non-measurable cardinality (T. Shirota).

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e-6 Baire Spaces

1. General properties

A topological space X is a **Baire space** if every intersection of countably many dense open sets in X is dense in X. A **meager set** (or **set of first category**) is one that can be covered by countably many closed sets with empty interior. The **comeager** (or **residual**) sets in X are the complements of meager sets. For any non-meager A in X there is a nonempty open U in X such that $U \cap A$ is a non-meager Baire subspace of X (the **Banach localization principle**).

First category sets in Euclidean spaces were introduced by R. Baire [1], and independently, for the real line, by W. Osgood [12], in the process of investigation of different aspects of pointwise convergence of sequences of continuous functions. Baire considered also the set of points of joint-continuity of separately continuous functions on $\mathbb{R}^n$. Both authors established that the complement of any first category set in $\mathbb{R}$ is dense in $\mathbb{R}$ [1, p. 65], [12, p. 173] (the terminology is from [1]). It is the fundamental **Baire Category Theorem** that the assertion is true in any ***completely metrizable*** space. The theorem appeared in full generality in Hausdorff's monograph [7] and became gradually a standard tool in mathematics. Some essential steps in this process are indicated in the next section.

The class of Baire spaces includes ***G_δ-subsets*** of ***countably compact*** spaces. Open subspaces and dense G_δ-subspaces of Baire spaces are Baire. Also, every space containing a dense Baire subspace is Baire.

A set A in X is **open modulo meager sets** (or A has the **Baire property** in X) if $A = (U \setminus C) \cup D$ with U open and C, D meager in X. The **Souslin sets** in X, i.e., the projections parallel to the irrationals $\mathbb{N}^{\mathbb{N}}$ of closed sets in $X \times \mathbb{N}^{\mathbb{N}}$, are open modulo meager sets in X (**Nikodým's Theorem**). Therefore, any non-meager set S in X such that either S or $X \setminus S$ is a Souslin set in X, contains a non-meager G_δ-set in X.

If A is meager in the ***product*** $X \times Y$, where Y has a countable base, the set of points $x \in X$ for which the vertical section of A at x is meager in Y, is comeager in X (the **Kuratowski–Ulam Theorem**). This remains true if Y has a countable **pseudo-base**, a countable collection of nonempty open sets such that any nonempty open set in Y contains a member of this collection. Any product of Baire spaces with countable pseudo-bases is Baire (J.C. Oxtoby). In such products **Oxtoby's zero-one law** holds true: for any **tail set** T in the product (i.e., if $x \in T$, and y differs from x in at most finitely many coordinates, then $y \in T$) which is open modulo meager sets, T is either meager, or comeager in the product. However, the product of two ***metrizable*** Baire spaces, and even the product of two ***normed linear*** Baire spaces, need not be Baire, cf. [4, 14]. Also, Oxtoby's zero-one law fails for countable products of arbitrary metrizable Baire spaces, cf. [13, Notes to Chapter 21, p. 99].

Given a ***filter*** $\mathcal{F}$ on a set S, with $\bigcap \mathcal{F} = \emptyset$, the set $T(\mathcal{F})$ of characteristic functions of elements of $\mathcal{F}$ is a tail set in the ***Cantor cube*** $\{0,1\}^S$. The set $T(\mathcal{F})$ is meager if and only if there are pairwise disjoint finite sets $S_1, S_2, \ldots$ in S such that each $F \in \mathcal{F}$ meets all but finitely many S_i (M. Talagrand). For any ***free ultrafilter*** $\mathcal{F}$ on $\mathbb{N}$, both $T(\mathcal{F})$ and its complement are non-meager in the Cantor set $\{0,1\}^{\mathbb{N}}$, hence not open modulo meager sets (W. Sierpiński).

An ***open*** continuous function $f: X \to Y$ with Baire fibers between metrizable spaces can map a non-Baire X onto a Baire space Y (even if the ***fibers*** are discrete). This is impossible if the open map has either separable fibers or all fibers are complete with respect to a fixed metric on X. For information concerning closed maps the reader is referred to the article by D. Burke in this volume.

The meager sets are "negligible" from the point of view of Baire category. One should note however, that meager sets in the unit interval can have Lebesgue measure 1. Let τ be the **density topology** on $\mathbb{R}$ consisting of those sets A that have Lebesgue density one at each of its points, i.e., for every $x \in A$, $\lim_{h \to 0} \frac{1}{2h} \lambda(A \cap (x-h, x+h)) = 1$ (where λ denotes Lebesgue measure). Then $(\mathbb{R}, \tau)$ is a ***completely regular*** Baire space (in fact all subspaces of $(\mathbb{R}, \tau)$ are Baire) and the meager sets in this space coincide with the sets of Lebesgue measure null in the real line. In particular, all meager sets in $(\mathbb{R}, \tau)$ are nowhere dense.

The spaces all of whose closed subspaces are Baire are called **hereditarily Baire**. A metrizable space X is hereditarily Baire exactly when X contains no closed copy of the rationals, and the complement of a Souslin set in a completely metrizable space is completely metrizable provided it is hereditarily Baire (W. Hurewicz). It is independent of the axioms for ZFC whether for Souslin sets in the real line being hereditarily Baire implies complete metrizability (V.G. Kanoveĭ, A.V. Ostrovskiĭ). The product of two hereditarily Baire subspaces of the real line may not be hereditarily Baire, but arbitrary products of metrizable hereditarily Baire spaces are Baire. There are always ultrafilters $\mathcal{F}$ on $\mathbb{N}$ such that the Baire space $T(\mathcal{F})$ is not hereditarily Baire, and it is independent of the axioms for ZFC if all free ultrafilters on $\mathbb{N}$ have this property (S.P. Gul'ko and G.A. Sokolov).

Concerning the cardinality of Baire spaces, there are models of ZFC in which there are no regular Baire spaces of cardinality $\aleph_1$ without isolated points (S. Shelah, S. Todorcevic). The minimal cardinality of a collection of meager sets covering the real line is an important cardinal whose properties depend on the axioms for ZFC. The cardinal has always uncountable cofinality (A.W. Miller).

Each metrizable space X all of whose nonempty open sets are non-separable is the union of an increasing transfinite sequence of type ω_1 of closed sets with empty interiors (P. Štěpánek, P. Vopěnka). However, if $\mathcal{E}$ is any collection of meager sets in a completely metrizable space X such that each point in X is in at most countably many elements of $\mathcal{E}$, and the union of any subcollection of $\mathcal{E}$ is a ***Borel set***, then the union of $\mathcal{E}$ is meager in X. This topic is discussed in depth by Fremlin [5, Section 8].

The last fact in this section is a representation theorem for ***Boolean algebra***s involving Baire category: every σ-complete Boolean algebra is isomorphic to the quotient algebra $\mathcal{F}/\mathcal{M}$ of the σ-field $\mathcal{F}$ generated by open ***σ-compact*** sets modulo the ***σ-ideal*** $\mathcal{M}$ of meager sets in a compact space (L.H. Loomis, R. Sikorski).

Basic sources for the subject of this section are [10], [KI], [13], and also [8].

2. The Baire Category Method

In many important cases, the objects under consideration form a comeager set in a suitable Baire space. If so, these objects can be viewed as "typical", or "generic", even if their individual construction may be fairly complicated. We shall present a sample of some significant instances of this phenomenon. Numerous applications of this approach in the real function theory are discussed in the article by A. Bruckner in this volume.

(A) The condensation of singularities
S. Banach and H. Steinhaus published in 1927 the following result: for any double sequence T_{nm} of bounded linear operators on a complete ***normed linear space*** X, if for any n there is x_n with $\overline{\lim}_m \|T_{nm}(x_n)\| = \infty$ then the set of x with $\overline{\lim}_m \|T_{nm}(x)\| = \infty$ for all n simultaneously, is comeager in X (the **Banach–Steinhaus Theorem** [2]).

This was a far-reaching extension of "principles of condensation of singularities" going back to H. Hankel. The paper was a decisive step toward the systematic use of the Baire Category Theorem in functional analysis, and a contribution of S. Saks in that respect is acknowledged in [2]. Numerous classical applications of the method in functional analysis are gathered in a separate chapter of Yoshida's monograph [15].

(B) Typical continua in the square are hereditarily indecomposable
More specifically, the collection of ***continua*** in the square with the ***Hausdorff metric*** is a ***compact*** space, and S. Mazurkiewicz showed in 1930 that the continua C having the following striking property form a comeager set in this space: for any two intersecting continua A, B in C, either $A \subset B$ or $B \subset A$. The existence of such continua was demonstrated earlier by B. Knaster, cf. the article by Tymchatyn in this volume. Mazurkiewicz's result was apparently the first substantial application of the Baire category method in topology, beyond the real function theory.

(C) Typical maps of n-dimensional compacta into Euclidean spaces
The following theorem, published by W. Hurewicz in 1931, opened up numerous applications of the Baire category method in dimension theory, cf. [11, §45]. Let X be an n-dimensional compact metric space, $m > 0$, and let $C(X, \mathbb{R}^{n+m})$ be the complete space of continuous maps of X into the Euclidean space, with the ***supremum norm***. Then for all maps, apart from a meager set in the function space, the set of points in the image with at least k-point fibers has dimension not greater than $n - m(k-1)$ (the set is empty if the number is negative).

For $m = n+1$ and $k = 2$ this yields the existence of ***embeddings*** of X in $\mathbb{R}^{2n+1}$, a celebrated theorem proved earlier by Lefschetz, Nöbeling and Pontriagin and Tolstova.

Typical embeddings $e : X \to \mathbb{R}^{2n+1}$ of n-dimensional compact metric spaces X have the following remarkable property: for each continuous $f : \mathbb{R}^{2n+1} \to \mathbb{R}$ there exist continuous $g_i : \mathbb{R} \to \mathbb{R}$ such that for any $y = (y_1, \ldots, y_{2n+1}) \in e(X)$, $f(y) = \sum_{i=1}^{2n+1} g_i(y_i)$, and if a compact space X admits any such embedding in $\mathbb{R}^{2n+1}$, $n \geqslant 1$, then $\dim X \leqslant n$ (Y. Sternfeld). The existence of embeddings with this property was proved for $[0,1]^n$ by A.N. Kolmogorov and, in general case, by P.A. Ostrand, cf. the section "Representation of functions" in Bruckner's article in this volume.

(D) Typical Cantor sets in the real line are Kronecker sets
A compact set K in the real line is a **Kronecker set** if each continuous function from K to the unit circle in the complex plane can be uniformly approximated on K by the exponents e^{int}. Let $\mathbb{K}$ be the standard Cantor set. Then a typical continuous function $f : \mathbb{K} \to \mathbb{R}$ embeds $\mathbb{K}$ onto a Kronecker set. It follows that in the complete metric space of all compact subsets of $\mathbb{R}$ with the Hausdorff metric Kronecker sets homeomorphic with $\mathbb{K}$ form a comeager set. This is a special case of a functional method of R. Kaufman, discussed extensively by Kahane [9, Chapter VII].

3. The Banach–Mazur game and Choquet spaces

Let us consider a ***topological game*** in the nonempty space X played by I and II choosing alternatively nonempty open sets $U_0 \supset V_0 \supset U_1 \supset V_1 \supset \cdots$. Let us declare that II wins provided the intersection $\bigcap_n V_n$ is nonempty. Then X is a Baire space if and only if I has no winning strategy in this game (the strategy tells the player, knowing all previous moves of the opponent in the game, what his next move should be). The spaces X where II has a winning strategy in this game are called **Choquet space**s [10] (or **weakly α-favorable** [5]). Choquet spaces are Baire and an arbitrary product of Choquet spaces is Choquet. Also the product of a Choquet space and a Baire space is Baire. Among metrizable spaces, the Choquet spaces are the ones that contain dense completely metrizable subspaces (H.E. White). In particular, the Baire space $T(\mathcal{F})$ associated with a free ultrafilter on $\mathbb{N}$ is not Choquet.

A Choquet space playing an important role in the study of the Borel structure of the real line is the space $(\mathbb{N}^{\mathbb{N}}, \tau)$ of the irrationals with the **Gandy–Harrington topology** τ. The topology τ (which is ***Hausdorff***, but not ***regular***) has a countable base whose elements are the projections of "effectively closed" sets F in the product $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ (i.e., the sets F whose complements are unions of recursively enumerable subcollections of the family of standard basic neighbourhoods in $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ endowed with a fixed enumeration), cf. [6].

Another noteworthy non-metrizable Choquet space is the space $\mathcal{P}_\infty(\mathbb{N})$ of all infinite sets of natural numbers with the ***Vietoris topology***. In this space, the sets open modulo meager sets are exactly the sets $\mathcal{S} \subset \mathcal{P}_\infty(\mathbb{N})$ that are **completely Ramsey**: for any $K \subset L \subset \mathbb{N}$ with K finite and L infinite, there is an infinite J between K and L such that either $\mathcal{S}$ or its complement contains all infinite sets between K and J (**Ellentuck's Theorem**), cf. [10, 5].

Also, the real line with the density topology is a Choquet space.

If, in the game played by I and II in X, player II wins whenever $\bigcap_n V_n \subset A$, where $A \subset X$ is fixed, one gets the original game considered by Banach and Mazur for the real line. The comeager sets in X are exactly the sets for which player II has a winning strategy in this game.

4. Separately continuous functions and Namioka spaces

Let $f : X \times K \to \mathbb{R}$ be a **separately continuous** real-valued function defined on the product of the ***Čech-complete*** space X and the compact space K, i.e., for every x the function $f(x, \cdot)$ is continuous and likewise $f(\cdot, k)$ for every $k \in K$. Then there is a comeager set $A \subset X$ such that f is jointly continuous at each point of $A \times K$ (**Namioka's Theorem**).

For X and K being the unit interval this was already proved by Baire [1].

Equivalently, using the formula $F(x)(y) = f(x, y)$, one can consider continuous maps $F : X \to C_p(K)$ from X to the space of continuous real-valued functions on the compact space K with the ***pointwise topology***. In this setting the assertion of Namioka's Theorem means that the set A of continuity points of F with respect to the ***uniform topology*** in the function space is comeager in the space X.

The completely regular spaces X for which the assertion of this theorem holds true for any compact space K and any separately continuous $f : X \times K \to \mathbb{R}$ (or any continuous $F : X \to C_p(K)$) are called **Namioka space**s.

Each Namioka space is Baire and each Baire space that is either metrizable or separable is Namioka (J. Saint Raymond). However, a Choquet space need not be Namioka (M. Talagrand), cf. [HvM, Chapter 16, 2.6]. There is also a Baire subspace X of a certain $C_p(K)$ such that the inclusion $X \to C_p(K)$ has no points of continuity with respect to the uniform topology in the function space (R. Haydon), cf. [3, Chapter VII, 7.6]. Hereditarily Baire subspaces X of $C_p(K)$ spaces are Namioka; indeed, for each such X the points of continuity from the pointwise to uniform topologies on X are dense in X (G.A. Edgar and R.W. Wheeler).

5. Category transforms and selections

Let Z be a compact metric space and let $\mathcal{K}(Z)$ be the compact space of closed subsets of Z with the Hausdorff metric. For each $X \subset Z$ the **category transforms** $X^\#$, $X^* \subset \mathcal{K}(Z)$ are defined as follows: $X^\# = \{K \in \mathcal{K}(Z)\colon K \cap X$ is non-meager in $K\}$, $X^* = \{K \in \mathcal{K}(Z)\colon K \cap X$ is comeager in $K\}$. Then, if X is a Borel or Souslin set in Z, the transforms $X^\#$, X^* are respectively Borel or Souslin in $\mathcal{K}(Z)$ (**Vaught's Theorem**). If $C \subset Z$ is the complement of a Souslin set in Z, and $\mathcal{E} \subset C^\#$ is a Souslin set in $\mathcal{K}(Z)$, then $\mathcal{E} \subset B^\#$ for some Borel set $B \subset C$, and there is a Borel ***selection*** $f : B^\# \to B$, $f(K) \in K$, for $K \in B^\#$ (J. Burgess). It follows that the complement of a Souslin set in the square all of whose vertical sections are non-meager contains the graph of a Borel function (D. Cenzer and R.D. Mauldin). The category transforms, introduced by R. Vaught, play an important role in the study of Borel theory of group actions, cf. [10, 16.B].

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e-7 Uniform Spaces, I

1. Definitions

Uniform spaces can be defined in various equivalent ways. We shall discuss the two principal methods and show how they are used to put uniform structures on ***metric spaces*** and ***topological groups***. Below the metric space referred to will be (X, d) and the topological group will be G and $\mathcal{N}$ denotes the ***neighbourhood filter*** of the neutral element e. A word on terminology: usually the uniform structure is simply called a **uniformity** – for expository purposes we use adjectives (diagonal, covering, pseudometric) to distinguish the various approaches.

Entourages
A **diagonal uniformity** on a set X is a ***filter*** $\mathcal{U}$ of subsets of $X \times X$ with the following properties.

(E1) $\Delta \subseteq U$ for all $U \in \mathcal{U}$;
(E2) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$; and
(E3) for every $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$.

Here $\Delta = \{(x, x) \colon x \in X\}$, the **diagonal** of X; $U^{-1} = \{(x, y) \colon (y, x) \in U\}$, the **inverse** of U; and $V^2 = \{(x, y) \colon (\exists z)((x, z), (z, y) \in V)\}$, the **composition** of V with itself. Each member of $\mathcal{U}$ is called an **entourage** of the diagonal.

For a metric space its **metric uniformity** is the filter generated by the sets $U_r = \{(x, y) \colon d(x, y) < r\}$.

For a topological group we have four natural filters. The **left uniformity** $\mathcal{U}_l$ is the filter generated by the sets $L_N = \{(x, y) \colon x^{-1}y \in N\}$, the **right uniformity** $\mathcal{U}_r$ is generated by the sets $R_N = \{(x, y) \colon yx^{-1} \in N\}$, the **two-sided uniformity** $\mathcal{U}_t$ generated by the sets $T_N = L_N \cap R_N$ and the fourth (nameless) uniformity $\mathcal{U}_s$ generated by the sets $S_N = L_N \cup R_N$.

Note that $\mathcal{U}_t$ is the filter generated by $\mathcal{U}_l \cup \mathcal{U}_r$ and that $\mathcal{U}_s = \mathcal{U}_l \cap \mathcal{U}_r$; also observe that if G is Abelian all four filters coincide.

A **base for a uniformity** $\mathcal{U}$ is nothing but a base for the filter $\mathcal{U}$. All the uniformities above were described by specifying bases for them. This explains why ***filter bases*** satisfying (E1)–(E3) above are sometimes called **uniformity bases**.

Uniform covers
A **covering uniformity** on a set X is a family $\mathfrak{U}$ of covers with the following properties.

(C1) If $\mathcal{A}, \mathcal{B} \in \mathfrak{U}$ then there is a $\mathcal{C} \in \mathfrak{U}$ that is a ***star refinement*** of both $\mathcal{A}$ and $\mathcal{B}$;
(C2) if a cover has a refinement that is in $\mathfrak{U}$ then the cover itself is in $\mathfrak{U}$.

These conditions say, in effect, that $\mathcal{U}$ is a ***filter*** in the partially ordered set of all covers, where the order is by star refinement. Each member of $\mathcal{U}$ is called a **uniform cover**.

A cover of a metric space is uniform if it is refined by $\{B_d(x, r) \colon x \in X\}$ for some $r > 0$.

In a topological group we get, as before, four types of uniform covers, each one defined by the requirement of having a refinement of the form $\mathcal{L}_N = \{xN \colon x \in G\}$, $\mathcal{R}_N = \{Nx \colon x \in G\}$, $\mathcal{T}_N = \{xN \cap Nx \colon x \in G\}$ or $\mathcal{S}_N = \{xN \cup Nx \colon x \in G\}$, respectively. Again, in an Abelian group, for each N, the covers $\mathcal{L}_N$, $\mathcal{R}_N$, $\mathcal{T}_N$ and $\mathcal{S}_N$ are identical.

A subfamily $\mathfrak{B}$ of a uniformity $\mathfrak{U}$ is a base if every element of $\mathfrak{U}$ has a refinement that is in $\mathfrak{B}$. Any family of covers that satisfies (C1) can serve as a base for some uniformity.

Equivalence of the approaches
From a diagonal uniformity $\mathcal{U}$ one defines a base for a covering uniformity: all covers of the form $\{U[x] \colon x \in X\}$ for some $U \in \mathcal{U}$.

Conversely from a covering uniformity $\mathfrak{U}$ one defines a base for a diagonal uniformity: all sets of the form $\bigcup\{A \times A \colon A \in \mathcal{A}\}$, where $\mathcal{A} \in \mathfrak{U}$.

These operations are each others inverses and establish an order-preserving bijection between the families of both kinds of uniformities.

Pseudometrics (Gauges)
Yet another way of introducing uniform structures is via ***pseudometrics*** or **gauges** as they are often called in this context.

If above, instead of a metric space, we had used a ***pseudometric space*** nothing would have changed. In fact, one can start with any family P of pseudometrics and define $U_d(r) = \{(x, y) \colon d(x, y) < r\}$ for $d \in P$ and $r > 0$. The resulting family $\{U_d(r) \colon d \in P,\ r > 0\}$ of entourages is a **subbase for a uniformity** in that the family of finite intersections is a base for a uniformity, denoted $\mathcal{U}_P$.

It is a remarkable fact that every uniformity has a subbase, even a base, of this form. Given a sequence $\{V_n \colon n \in \mathbb{N}\}$ of entourages on a set X such that $V_0 = X^2$ and $V_{n+1}^3 \subseteq V_n$ for all n one can find a pseudometric d on X such that $U_d(2^{-n}) \subseteq V_n \subseteq \{(x, y) \colon d(y, x) \leqslant 2^{-n}\}$ for all n [E, 8.1.10] (there is a similar theorem for ***normal sequences*** of covers [E, 5.4.H]). Thus every uniform structure can be defined by a family of pseudometrics. The family of all pseudometrics d that satisfy $(\forall r > 0)(U_d(r) \in \mathcal{U})$ is denoted $P_{\mathcal{U}}$; it is the largest family of pseudometrics that generate $\mathcal{U}$. The family $P_{\mathcal{U}}$ satisfies the following two properties.

(P1) if $d, e \in D$ then $d \vee e \in D$;

(P2) if e is a pseudometric and for every $\varepsilon > 0$ there are $d \in P$ and $\delta > 0$ such that always $d(p,q) < \delta$ implies $e(p,q) < \varepsilon$ then $e \in P$.

A family P of pseudometrics with these properties is called a **pseudometric uniformity**; it satisfies the equality $P_{\mathcal{U}_P} = P$.

Uniform topology
Every uniform space carries a natural topology $\tau_{\mathcal{U}}$, its **uniform topology**. It is defined using ***neighbourhood bases***. Given a diagonal uniformity $\mathcal{U}$ one uses $\{U[x]: U \in \mathcal{U}\}$ for each x, in the case of a covering uniformity $\mathfrak{U}$ one uses $\{\mathrm{St}(x,\mathcal{A}): \mathcal{A} \in \mathfrak{U}\}$ and a pseudometric uniformity P provides $\{B_d(x,r): d \in P,\ r > 0\}$.

Alternatively one could have used a ***closure operator***, for in the uniform topology one has $\mathrm{cl}\, D = \bigcap\{U[D]: U \in \mathcal{U}\} = \bigcap\{\mathrm{St}(D,\mathcal{A}): \mathcal{A} \in \mathfrak{U}\}$ for all subsets of X.

In general not all entourages are open, nor do all uniform covers consist of open sets but the open entourages and the open uniform covers do form bases for the uniform structures – this implies that every uniform cover is a ***normal cover***. Also, if U is an entourage then $\mathrm{cl}\, U \subseteq U^2$ so that the closed entourages form a base as well.

It is readily seen that for every pseudometric d in the family $P_{\mathcal{U}}$ and every x the map $y \mapsto d(x,y)$ is continuous with respect to $\tau_{\mathcal{U}}$; this establishes that the uniform topology is ***completely regular*** (possibly not ***Hausdorff***). The uniform topology is Hausdorff iff it is T_0 and this is the case if the uniformity is **separated** or **Hausdorff**, which means that $\bigcap \mathcal{U} = \Delta$ or, equivalently, $\{x\} = \bigcap\{\mathrm{St}(x,\mathcal{A}): \mathcal{A} \in \mathfrak{U}\}$ for all x.

The intersection $\equiv\, = \bigcap \mathcal{U}$ is an equivalence relation on the set X and the uniformity $\mathcal{U}$ can be transferred to a uniformity $\widehat{\mathcal{U}}$ on the set $\widehat{X} = X/\equiv$ to yield a separated uniform space, that to most intents and purposes is interchangeable with $(X,\mathcal{U})$.

A topological term applied to a uniform space usually refers to a property of the uniform topology, although ambiguity has crept in, see, e.g., the term ***weight*** below. Certain topological terms take the modifier ‘uniformly’; it usually means that one entourage works for all points simultaneously. A **uniformly locally compact space** for instance has an entourage such that $U[x]$ is compact for all x; the locally compact ***ordinal space*** ω_1 is locally compact but not uniformly so.

A topological space $(X,\mathcal{T})$ is **uniformizable** if there is a (**compatible**) uniformity on X whose uniform topology is $\mathcal{T}$. Thus, a uniformizable topological space is completely regular; the converse is also true: associate to every real-valued function $f: X \to \mathbb{R}$ the pseudometric d_f defined by $d_f(x,y) = |f(x) - f(y)|$; the resulting family of pseudometrics generates a compatible uniformity.

Some natural questions
It makes topological sense to ask whether the open sets in a space can generate a uniformity.

The property that the family of all neighbourhoods of the diagonal forms a uniformity is called **divisibility**. It is a property shared by ***paracompact*** Hausdorff spaces and ***generalized ordered spaces*** and it implies ***collectionwise normality***.

The related property that the family of all open covers is a base for a uniformity characterizes ***fully normal spaces***.

2. Uniform properties

When trying to generalize metric concepts to wider classes of spaces one encounters the countability barrier: almost no non-trivial uncountable construction preserves metrizability. The category of uniform spaces and uniformly continuous maps provides a convenient place to carry out these generalizations.

Below we invariably let X be our uniform space, with $\mathcal{U}$ its family of entourages and $\mathfrak{U}$ the family of uniform covers.

Uniform continuity
A map $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$ between uniform spaces is **uniformly continuous** if $(f \times f)^{-1}[V] \in \mathcal{U}$ whenever $V \in \mathcal{V}$, equivalently, if $\{f^{-1}[A]: A \in \mathcal{A}\} \in \mathfrak{U}$ whenever $\mathcal{A} \in \mathfrak{V}$. A uniformly continuous map is also continuous with respect to the uniform topologies and the converse is, as in the metric case, true for compact Hausdorff spaces.

A bijection that is uniformly continuous both ways is a **uniform isomorphism**. A **uniform property** then is a property of uniform spaces that is preserved by uniform isomorphisms.

Products and subspaces
It is straightforward to define a uniform structure on a subset Y of a uniform space X: simply intersect the entourages with $Y \times Y$ (or trace the uniform covers on Y). To define a **product uniformity** one may follow the construction of the ***product topology*** and define a subbase for it. Given a family $\{(X_i,\mathcal{U}_i)\}_{i \in I}$ of uniform spaces define a family of entourages in the square of $\prod_i X_i$ using the projections π_i: $\{(\pi_i \times \pi_i)^{-1}[U]: U \in \mathcal{U}_i,\ i \in I\}$.

These constructions have the right categorical properties, so that we obtain subobjects and products in the category of uniform spaces and uniformly continuous maps. The uniform topology derived from the product and subspace uniformities are the product and subspace topologies derived from the original uniform topologies, respectively.

Uniform quotients
A map $q: X \to Q$ between uniform spaces is a **uniform quotient map** if it is onto and has the following universality property: whenever $f: Q \to Y$ is a map to a uniform space Y then f is uniformly continuous if $f \circ q$ is. Every uniformly continuous map $f: X \to Y$ admits a factorization $f = f_0 \circ q$ with q a uniform quotient map and f_0 a (uniformly continuous) injective map.

In analogy with the topological situation one can, given a surjection f from a uniform space X onto a set Y, define the **quotient uniformity** on Y to be the finest uniformity that makes f uniformly continuous. The resulting map is uniformly quotient and all uniform quotient maps arise in this way.

The uniform topology of a quotient uniformity is not always the quotient topology of the original uniform topology: if X is completely regular but not normal, as witnessed by the disjoint closed sets F and G, then identifying F to one point results in a space which is Hausdorff but not (completely) regular, hence the quotient uniformity from the fine uniformity (see below) does not generate the quotient topology. There is even a canonical construction that associates to every separated uniform space X a uniform space Y with a uniform quotient map $q : X \to Y$ and such that the uniform topology of Y is discrete, see [6, Exercise III.3].

Uniform quotient maps behave different from topological quotient maps in other respects as well: every product of uniform quotient maps is again a uniform quotient map [5].

Completeness
We say that X is **complete** (and $\mathcal{U}$ or $\mathfrak{U}$ a **complete uniformity**) if every **Cauchy filter** converges. A filter $\mathcal{F}$ is Cauchy if for every $V \in \mathcal{U}$ there is $F \in \mathcal{F}$ with $F \times F \subseteq V$ or, equivalently, if $\mathcal{F} \cap \mathcal{A} \neq \emptyset$ for all $\mathcal{A} \in \mathfrak{U}$. Closed subspaces and products of complete spaces are again complete.

Every uniform space has a **completion**, this is a complete uniform space that contains a dense and uniformly isomorphic copy of X. As underlying set of a completion one can take the set $\widetilde{X}$ of *minimal* Cauchy filters. Every entourage U of $\mathcal{U}$ is extended to $\widetilde{U} = \{(\mathcal{F}, \mathcal{G})\colon (\exists F \in \mathcal{F})(\exists G \in \mathcal{G})(F \times G \subseteq U)\}$; the family $\{\widetilde{U}\colon U \in \mathcal{U}\}$ generates a complete uniformity on $\widetilde{X}$. If $x \in X$ then its neighbourhood filter $\mathcal{F}_x$ is a minimal Cauchy filter and $x \mapsto \mathcal{F}_x$ is a uniform embedding.

As in the case of ***metric completion*** the completion of a separated uniform space is unique up to uniform isomorphism.

Using the canonical correspondence between nets and filters (see the article on *Convergence*) one can define a **Cauchy net** to be a ***net*** whose associated filter is Cauchy. This is equivalent to a definition more akin to that of a ***Cauchy sequence***: A net $(t_\alpha)_{\alpha \in D}$ is Cauchy if for every entourage U there is an α such that $(t_\beta, t_\gamma) \in U$ whenever $\beta, \gamma \geqslant \alpha$.

Total boundedness
We say X is **totally bounded** or **precompact** if for every entourage U (or uniform cover $\mathcal{A}$) there is a finite set F such that $U[F] = X$ (or $\mathrm{St}(F, \mathcal{A}) = X$). Subspaces and products of precompact spaces are again precompact.

A metrizable space has a compatible totally bounded metric iff it is ***separable***. A uniformizable space always has a compatible totally bounded uniformity; indeed, for any uniform space $(X, \mathcal{U})$ the family of all finite uniform covers is a base for a (totally bounded) uniformity $p\mathcal{U}$ with the same uniform topology, this uniformity is the **precompact reflection** of $\mathcal{U}$.

The metric theorem that equates compactness with completeness plus total boundedness remains valid in the uniform setting; likewise a Tychonoff space is compact if every compatible uniformity is complete. It is not true that a Tychonoff space is compact iff every compatible uniformity is totally bounded. The ordinal space ω_1 provides a counterexample: it is not compact and it has only one compatible uniformity (the family of all neighbourhoods of the diagonal), which necessarily is totally bounded.

Uniform weight
The **weight**, $w(X, \mathcal{U})$, is the minimum cardinality of a base. A uniformity $\mathcal{U}$ can be generated by κ pseudometrics iff $w(X, \mathcal{U}) \leqslant \kappa \cdot \aleph_0$ iff the separated quotient $\widehat{X}$ admits a uniform embedding into a product of κ many (pseudo)metric spaces (with its product uniformity). In particular: a uniformity is generated by a single pseudometric iff its weight is countable.

The **uniform weight** $u(X)$ of a Tychonoff space X is the minimum weight of a compatible uniformity. This is related to other cardinal functions by the inequalities $u(x) \leqslant w(X) \leqslant u(X) \cdot c(X)$. The first follows by considering a compactification of the same weight as X, the second from the fact that each pseudometric contributes a ***σ-discrete*** family of open sets to a base for the open sets. The uniform weight is related to the **metrizability degree**: $m(X)$ is the minimum κ such that X has an open base that is the union of κ many discrete families, whereas $u(X)$ is the minimum κ such that X has an open base that is the union of κ many discrete families of ***cozero sets***. Thus $m(X) \leqslant u(X)$; equality holds for normal spaces and is still an open problem for Tychonoff spaces.

3. Further topics

Fine uniformities
Every family $\{\mathcal{U}_i\}_i$ of uniformities has a supremum $\bigvee_i \mathcal{U}_i$. In terms of entourages it is generated by the family of all finite intersections of elements of $\bigcup_i \mathcal{U}_i$, i.e., $\bigcup_i \mathcal{U}_i$ is used as a subbase. If all the $\mathcal{U}_i$ are compatible with a fixed topology $\mathcal{T}$ then so is $\bigvee_i \mathcal{U}_i$. This implies that every Tychonoff space admits a finest uniformity, the **fine uniformity** or **universal uniformity**, it is the one generated by the family of *all* normal covers or by the family of all pseudometrics d_f defined above. The fine uniformity is denoted $\mathcal{U}_f$.

One says that a uniformity $\mathcal{U}$ itself is **fine** (or a **topologically fine uniformity**) if it is the fine uniformity of its uniform topology $\tau_{\mathcal{U}}$.

The equivalence of full normality and paracompactness combined with the constructions of pseudometrics described above yield various characterizations of the covers that belong to the fine uniformity: they are the covers that have ***locally finite*** (or ***σ-locally finite*** or ***σ-discrete***) refinements consisting of cozero sets. From this it follows that the precompact reflection of the fine uniformity is generated by the finite cozero covers.

Continuity versus uniform continuity
Every continuous map from a fine uniform space to a uniform space (or pseudometric space) is uniformly continu-

ous; this property characterizes fine uniform spaces. Uniform spaces on which every continuous real-valued function is uniformly continuous are called ***UC-spaces***. A *metric* UC-space is also called an **Atsuji space**.

The precompact reflection of a fine uniformity yields a space where all *bounded* continuous real-valued functions are uniformly continuous, these are also called ***BU-spaces***.

Compactifications

There is a one-to-one correspondence between the families of ***compactifications*** of a Tychonoff space and the compatible totally bounded uniformities. If γX is a compactification of X then the uniformity that X inherits from γX is compatible and totally bounded. Conversely, if $\mathcal{U}$ is a compatible totally bounded uniformity on X then its completion is a compactification of X, the **Samuel compactification** of $(X, \mathcal{U})$. The correspondence is order-preserving: the finer the uniformity the larger the compactification. Consequently the compactification that corresponds to the precompact reflection $\mathcal{U}_{\mathrm{fin}}$ of the fine uniformity is exactly the ***Čech–Stone compactification***. It also follows that a space has exactly one compatible uniformity iff it is ***almost compact***.

Proximities

There is also a one-to-one correspondence between the ***proximities*** and precompact uniformities.

Indeed, a uniformity $\mathcal{U}$ determines a proximity $\delta_{\mathcal{U}}$ by $A \; \delta_{\mathcal{U}} \; B$ iff $U[A] \cap U[B] \neq \emptyset$ for every entourage U (intuitively: proximal sets have distance zero).

Conversely, a proximity δ determines a uniformity $\mathcal{U}_\delta$: the family of sets $X^2 \setminus (A \times B)$ with $A \not\delta B$ forms a subbase for $\mathcal{U}_\delta$. This uniformity is always precompact and, in fact, $\mathcal{U}_{\delta_{\mathcal{U}}}$ is the precompact reflection of $\mathcal{U}$.

The Samuel compactification of $(X, \mathfrak{U}_\delta)$ is also known as the **Smirnov compactification** of (X, δ).

Function spaces

Uniformities also allow one to formulate and prove theorems on uniform convergence and continuity in a general setting. Thus, given a uniform space $(Y, \mathcal{V})$ and a set (or space) X one can define various uniformities on the set Y^X of all maps from X to Y. Let $\mathcal{A}$ be a family of subsets of X. For $V \in \mathcal{V}$ and $A \in \mathcal{A}$ one defines the entourage $E_{A,V}$ to be the set $\{(f, g)\colon (\forall x \in A)((f(x), g(x)) \in V)\}$. The family of sets $E_{A,V}$ serves as a subbase for a uniformity. The corresponding uniform topology is called the **topology of uniform convergence** on members of $\mathcal{A}$.

If $\mathcal{A} = \{X\}$ then we obtain the topology of uniform convergence: a net $(f_\alpha)_\alpha$ converges with respect to this topology iff it **converges uniformly**: $f_\alpha \to f$ uniformly if for every $V \in \mathcal{V}$ there is an α_0 such that $(f_\alpha(x), f_{\alpha_0}(x)) \in V$ for all $x \in X$ and all $\alpha \geqslant \alpha_0$. One proves that uniform limits of (uniformly) continuous maps are again (uniformly) continuous, thus freeing these theorems from the bonds of countability.

If $\mathcal{A}$ is the family of finite subsets of X then one recovers the product uniformity and the ***topology of pointwise convergence***. If X is a topological space and $\mathcal{A}$ is the family of compact sets then the uniform topology, when restricted to the set $C(X, Y)$ of all continuous maps, is the ***compact-open topology***.

The combinatorics of uniform covers

It follows from the proof of the theorem that fully normal spaces are paracompact (see the article *Paracompact spaces*) that every uniform cover has a locally finite open (even cozero) refinement. The natural question whether this refinement may be chosen to be a uniform cover has a negative answer [8, 10]. Indeed, the metric uniformity of the Banach space $\ell_\infty(\aleph_1)$ provides a counterexample. A more general theorem can be formulated using some additional terminology. The **degree of a family** $\mathcal{A}$ is the minimum cardinal κ with the property that $|\mathcal{B}| < \kappa$ whenever $\mathcal{B} \subseteq \mathcal{A}$ and $\bigcap \mathcal{B} \neq \emptyset$. The **point-character of a uniform space** $(X, \mathfrak{U})$ is the minimum κ such that $\mathfrak{U}$ has a base consisting of covers of degree less than κ, it is denoted $pc(X, \mathfrak{U})$. Thus $pc(\ell_\infty(\aleph_1)) > \aleph_0$ and, in general, $pc(\ell_\infty(\lambda)) > \kappa$, whenever $\kappa < \lambda$ is regular, see [8].

In a uniform space $(X, \mathfrak{U})$ the finite uniform covers form a base for a uniformity, as do the countable uniform covers. The corresponding statement for higher cardinals is consistent with and independent of ZFC: for example, the ***Continuum Hypothesis*** implies that the uniform covers of cardinality $\aleph_1$ (or less) form a base for a uniformity, whereas ***Martin's Axiom*** implies that for $\ell_\infty(\aleph_1)$ this is not the case, see [8].

4. Completeness and completions

A Tychonoff space is **Dieudonné complete** or **topologically complete** or **completely uniformizable** if it has a compatible complete uniformity, equivalently, if the fine uniformity is complete.

Paracompact spaces are Dieudonné complete, indeed if a filter $\mathcal{F}$ does not converge then $\{X \setminus \operatorname{cl} F\colon F \in \mathcal{F}\}$ is an open cover, which belongs to the fine uniformity, so that $\mathcal{F}$ is not Cauchy.

Realcompact spaces are Dieudonné complete – the countable cozero covers generate a complete uniformity $\mathfrak{U}_\omega$ – and the converse is true provided the cardinality of the space (or better of its closed discrete subspaces) is not ***Ulam measurable*** – this is Shirota's theorem. The role of ***measurable cardinals*** is plain from the fact that a non-trivial countably complete ***ultrafilter*** is a Cauchy filter with respect to $\mathfrak{U}_\omega$ but not with respect to the fine uniformity, where we consider the measurable cardinal with its discrete topology.

Many books in General Topology provide introductions to uniform spaces; we mention Chapter 8 of [E], Chapter 7 of [3] and Chapter 15 of [4]; the latter deserves mention because it uses pseudometric exclusively. Isbell's book [6] is more comprehensive and spurred a lot of research in the years after its publication.

Page's book [7] concerns the workings of uniform spaces in topological groups and (Functional) Analysis; the monograph [9] by Roelcke and Dierolf treats topological groups from a uniform viewpoint; and Benyamini and Lindenstrauss' [2] offers more applications in the geometry of Banach spaces.

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e-8 Uniform Spaces, II

1. Uniformities from Walman bases

Wallman–Shanin bases can be used to construct compatible ***totally bounded uniformities*** on ***completely regular spaces***. This is apparent from the theory of ***Wallman–Shanin compactifications*** but a direct description is as follows. Let $\mathcal{B}$ be a Wallman–Shanin base (for the closed sets) of the space X and set $\mathcal{C} = \{X \setminus B\colon B \in \mathcal{B}\}$. So the families $\mathcal{B}$ and $\mathcal{C}$ are both closed under finite unions and intersections, every member of $\mathcal{C}$ is a union of members of $\mathcal{B}$ and disjoint members of $\mathcal{B}$ can be separated by (disjoint) members of $\mathcal{C}$. The family of all finite covers of X by members of $\mathcal{C}$ can serve as a base for a uniformity $\mathfrak{U}_{\mathcal{B}}$ on X. The family $\{(X \times X) \setminus (B_1 \times B_2)\colon B_1, B_2 \in \mathcal{B},\ B_1 \cap B_2 = \emptyset\}$ is a subbase for the corresponding filter of entourages. The ***Samuel compactification*** of $(X, \mathfrak{U}_{\mathcal{B}})$ is simply the Wallman–Shanin compactification $\omega(X, \mathcal{B})$.

For example, the family of $\mathcal{Z}(X)$ of ***zero sets*** of a completely regular space X is a Wallman–Shanin base of X; the corresponding uniformity is the ***precompact reflection*** of the ***fine uniformity*** on X.

2. Some special normal covers

As every uniform cover is a ***normal cover*** it is of interest to consider sufficient conditions for a cover to be normal. Standard examples of normal covers are: any countable cozero cover of any topological space, every finite open cover of a ***normal*** topological space, and every open cover of a ***paracompact*** topological space.

For a family of normal covers to generate a uniformity it is necessary (and sufficient) that its elements have star refinements that again belong to the family. Every ***star-countable*** cozero cover (of cardinality κ) has a star refinement that is a ***star-finite*** cozero cover (of cardinality at most $\kappa \cdot \aleph_0$). As every star-finite family is star-countable, this shows that the families of star-finite and star-countable cozero covers respectively generate one and the same uniformity, denoted $\mathfrak{U}_{\delta}$.

Likewise, every ***locally finite*** cozero cover (of cardinality κ) has a star refinement that is a locally finite cozero cover (of cardinality $\kappa \cdot \aleph_0$), which may be constructed to have further properties in common with the first cover, such as star-finiteness, star-boundedness and being special. A family is **star-bounded** if there is a natural number n such that every member meets at most n other members. A **special cover** is a cozero cover that is the union of finitely many ***discrete*** families.

The fine uniformity is generated by each of the families of normal, ***σ-discrete***, locally finite and ***σ-locally finite*** cozero covers, respectively.

The family of countable cozero covers generates a uniformity $\mathfrak{U}_{\omega}$, which is also generated by the countable star-finite cozero covers.

3. Continuity and uniform continuity

A uniform space is **UC** if every continuous real-valued function on it is uniformly continuous (where $\mathbb{R}$ carries its natural metric uniformity). A uniform space is **BU** if every *bounded* continuous real-valued function on it is uniformly continuous, or equivalently every continuous function $f: X \to [0, 1]$ is uniformly continuous.

To see the difference between these notions, we mention that a Hausdorff uniform space is BU if and only if every finite cozero cover is uniform, whereas it is UC if and only if every countable star bounded cozero cover is uniform.

A uniform space $(X, \mathfrak{U})$ is BU if and only if for every zero set $H \subseteq X$ and every cozero set U containing H there exists a cover $\mathcal{U} \in \mathfrak{U}$ such that $\mathrm{St}(H, \mathcal{U}) \subseteq U$.

A uniform space $(X, \mathfrak{U})$ is UC if and only if for every discrete sequence $U_1, U_2, \ldots$ of cozero sets of X and every sequence of zero sets $H_1, H_2, \ldots$ of X with $H_n \subseteq U_n$ for every n, there exists a cover $\mathcal{U} \in \mathfrak{U}$ such that $\mathrm{St}(H_n, \mathcal{U}) \subseteq U_n$ for every n.

A subset A of a space X is called **C-discrete** provided for every $a \in A$ there exists a cozero set $U_a \subseteq X$ such that $a \in U_a$ and the family $\{U_a\colon a \in A\}$ is discrete in X. In a BU space of ***uniform weight*** κ every C-discrete set of non-isolated points has cardinality less than κ. Hence in a BU-space of countable uniform weight the set of non-isolated points is compact and the space itself is UC.

4. Some completions

A completely regular space X is ***realcompact*** iff its uniformity $\mathfrak{U}_{\omega}$ is complete; in general the completion of $(X, \mathfrak{U}_{\omega})$ is υX, the ***Hewitt–Nachbin realcompactification***. We let δX and μX denote the completions with respect to $\mathfrak{U}_{\delta}$ and $\mathfrak{U}_f$ respectively, where $\mathfrak{U}_f$ denotes the fine uniformity. The inclusions $\mathfrak{U}_{\omega} \subseteq \mathfrak{U}_{\delta} \subseteq \mathfrak{U}_f$ translate into the inclusions $X \subseteq \mu X \subseteq \delta X \subseteq \upsilon X \subseteq \beta X$. and into the implications realcompact $\Rightarrow$ δ-complete $\Rightarrow$ ***topologically complete*** (**δ-complete** means that $\mathfrak{U}_{\delta}$ is complete).

Lindelöf spaces are realcompact; ***locally compact paracompact Hausdorff*** spaces are δ-complete and paracompact Hausdorff spaces are topologically complete.

Realcompactness of all C-discrete subsets ensures equivalence of realcompactness and topological completeness as well as the equality $\upsilon X = \mu X$.

The three classes of realcompact, δ-complete and topologically complete spaces are invariant under the taking of arbitrary products and closed subspaces.

A space is realcompact if and only if it is homeomorphic to a closed subspace of a product of real lines; it is δ-complete if and only if it is homeomorphic to a closed subspace of a product of real lines and discrete spaces and it is topologically complete if and only if it is homeomorphic to a closed subspace of a product of complete metric spaces (or of a power of some ***hedgehog*** $J(\kappa)$).

One can give concrete descriptions inside βX of the completions μX, δX and υX. The completion μX is the intersection of all paracompact G_δ-sets in βX that contain X, δX is the intersection of all paracompact open sets in βX that contain X, and νX is the intersection of all σ-compact open sets in βX that contain X.

One can also consider the uniformities $\mathfrak{U}_s$ and $\mathfrak{U}_{sb}$, generated by the special and star-bounded cozero covers, respectively. Though these may differ from the three uniformities above, the completion with respect to $\mathfrak{U}_s$ is μX and the completion with respect to $\mathfrak{U}_{sb}$ is δX.

Pseudocompact spaces can be characterized in terms of uniformities: a space is pseudocompact iff its fine uniformity is totally bounded iff $\mu X = \beta X$.

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e-9 Quasi-Uniform Spaces

A **quasi-uniformity** on a set X is a ***filter*** $\mathcal{U}$ on $X \times X$ satisfying the conditions: Each $U \in \mathcal{U}$ is reflexive, and for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 := V \circ V \subseteq U$. The pair $(X,\mathcal{U})$ is called a **quasi-uniform space**. Here $V \circ V := \{(x,z) \in X \times X$: there is $y \in X$ such that $(x,y) \in V$ and $(y,z) \in V\}$. For each quasi-uniformity $\mathcal{U}$ the filter $\mathcal{U}^{-1}$ consisting of the inverse relations $U^{-1} = \{(y,x) \in X \times X\colon (x,y) \in U\}$ where $U \in \mathcal{U}$ is called the **conjugate quasi-uniformity** of $\mathcal{U}$. A quasi-uniformity $\mathcal{U}$ is said to be a **uniformity** if $\mathcal{U}^{-1} = \mathcal{U}$ (see [E, Chapter 8]). If $\mathcal{U}_1$ and $\mathcal{U}_2$ are quasi-uniformities on X such that $\mathcal{U}_1 \subseteq \mathcal{U}_2$, then $\mathcal{U}_1$ is called **coarser** than $\mathcal{U}_2$. The coarsest uniformity $\mathcal{U}^s$ finer than $\mathcal{U}$ is generated by the ***subbase*** $\mathcal{U} \cup \mathcal{U}^{-1}$. A map $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ between quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is called **quasi-uniformly continuous** if for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(f \times f)U \subseteq V$. Basic facts about the ***category*** **Quu** of quasi-uniform spaces and quasi-uniformly continuous maps can be found in [3]. The theory of quasi-uniform spaces represents a generalization of the theories of ***metric spaces*** and ***partial orders***, which allows one to unify common features of these theories like ***fixed point*** theorems and completions. The survey article [4] provides a comprehensive exposition (without proofs) of the area. Omitting many interesting topics like fuzzy quasi-uniformities, quasi-uniformities on frames and approach quasi-uniformities and not discussing applications of quasi-uniform spaces outside topology we shall concentrate in the following on some major aspects of the theory. Each quasi-uniformity $\mathcal{U}$ on a set X induces a topology $\tau(\mathcal{U})$ determined by the neighbourhood filters $\mathcal{U}(x) = \{U(x)\colon U \in \mathcal{U}\}$ $(x \in X)$, where $U(x) = \{y \in X\colon (x,y) \in U\}$. Each quasi-uniformly continuous map between quasi-uniform spaces is continuous. A quasi-uniformity $\mathcal{U}$ is called **small-set symmetric** if $\tau(\mathcal{U}^{-1}) \subseteq \tau(\mathcal{U})$; its conjugate is then called **point-symmetric** [3, Definition 2.22]. J. Marín and S. Romaguera proved that every continuous map from a Lebesgue quasi-uniform space [3, Definition 5.1] to a small-set symmetric quasi-uniform space is quasi-uniformly continuous. Similarly, each continuous ***open map*** from a ***compact*** *uniform* space to any quasi-uniform space is quasi-uniformly open. Each topological space (X,τ) is **quasi-uniformizable**, since τ coincides with the topology induced by the **Pervin quasi-uniformity** generated by the subbase $\{[(X \setminus G) \times X] \cup [X \times G]\colon G \in \tau\}$, and admits a finest compatible quasi-uniformity called its **fine quasi-uniformity**. H. Künzi proved that a topological space admits a unique quasi-uniformity if and only if all its ***interior-preserving*** open collections are finite. The condition implies hereditary compactness and is equivalent to this property in quasi-sober spaces. A topological space X of ***network weight*** $nw(X)$ admits at most $2^{2^{nw(X)}}$ quasi-uniformities. H. Künzi also characterized internally the topological spaces admitting a **coarsest quasi-uniformity**. His condition is implied by ***local compactness*** (i.e., each point has a ***neighbourhood base*** consisting of compact sets) and is equivalent to it in spaces in which the ***limit set*** of each ***convergent ultrafilter*** is the ***closure*** of some unique singleton (i.e., **supersober** spaces).

Let $[0,\infty)$ denote the set of the nonnegative reals. A **quasi-pseudometric** d on a set X is a function $d\colon X \times X \to [0,\infty)$ that satisfies $d(x,x) = 0$ and $d(x,z) \leqslant d(x,y) + d(y,z)$ whenever $x,y,z \in X$. Then d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ whenever $x,y \in X$ is the **conjugate quasi-pseudometric** of d. A quasi-pseudometric d on X is called a **quasi-metric** if $x,y \in X$ and $d(x,y) = 0$ imply $x = y$; it is called **non-Archimedean** if $d(x,z) \leqslant \max\{d(x,y), d(y,z)\}$ whenever $x,y,z \in X$. Each quasi-pseudometric d on X generates a quasi-uniformity $\mathcal{U}_d$ with ***base*** $\{\{(x,y) \in X \times X\colon d(x,y) < \varepsilon\}\colon \varepsilon > 0\}$. For each quasi-uniformity $\mathcal{U}$ possessing a countable base there is a quasi-pseudometric d such that $\mathcal{U} = \mathcal{U}_d$. Every quasi-uniformity can be represented as the supremum of a family of quasi-pseudometric quasi-uniformities (see, e.g., I. Reilly's work cited in [4]). Other approaches to quasi-uniformities use **pair-covers**, or concepts of quasi-pseudometrics taking values in structures more general than the reals (see, e.g., R. Kopperman's **continuity spaces**). Many classical counterexamples in topology like the ***Sorgenfrey plane***, the ***Niemytzki Plane*** or the ***Michael line*** are **quasi-metrizable**, i.e., their topology is equal to $\tau(\mathcal{U}_d)$ for some quasi-metric d. A γ**-space** is a topological T_1***-space*** admitting a **local quasi-uniformity** [3, Definition 7.15] with a countable base. R. Fox and J. Kofner constructed γ-spaces that are not quasi-metrizable. A topological space is non-archimedeanly quasi-pseudometrizable if and only if it possesses a σ-interior-preserving base; it is quasi-pseudometrizable if and only if it admits a local quasi-uniformity with a countable base so that the conjugate filter is also a local quasi-uniformity. No completely satisfactory characterization of quasi-pseudometrizability in topological terms seems to be known. H. Junnila established that each ***developable*** γ-space is quasi-metrizable. Several kinds of γ-spaces are known to be non-archimedeanly quasi-metrizable, e.g., the ***suborderable*** γ-spaces and the γ-spaces having an **orthobase** (this is a base $\mathcal{B}$ with the following property: for each $\mathcal{B}' \subseteq \mathcal{B}$ then either $\bigcap \mathcal{B}'$ is an open set, or it is a one-point set $\{p\}$ and $\mathcal{B}'$ is a local base at p). J. Kofner showed that a ***first-countable*** image of a quasi-metrizable space under a continuous ***closed map*** is quasi-metrizable; hence ***perfect*** surjections preserve quasi-metrizability. He also proved that a ***Moore space*** that is the image of a quasi-metrizable space under a continuous open map with compact fibers need not be quasi-metrizable.

A quasi-uniform space $(X, \mathcal{U})$ is called **precompact** (respectively **totally bounded**) if for each $U \in \mathcal{U}$, $\{U(x): x \in X\}$ has a finite subcover (respectively $\mathcal{U}^s$ is precompact). The concept of a totally bounded quasi-uniform space is (in the categorical sense) equivalent to the notion of a **quasi-proximity space** (see [3]). For any topological space, the Pervin quasi-uniformity is its finest compatible totally bounded quasi-uniformity. Each quasi-uniformity $\mathcal{U}$ contains a finest totally bounded quasi-uniformity $\mathcal{U}_\omega$ coarser than $\mathcal{U}$, which induces the same topology as $\mathcal{U}$. If the **quasi-proximity class** $\{\mathcal{V}$ is quasi-uniformity on X: $\mathcal{V}_\omega = \mathcal{U}_\omega\}$ of a quasi-uniformity $\mathcal{U}$ on X possesses more than one member, then it contains at least 2^c quasi-uniformities. H. Künzi proved that a topological space (X, τ) admits a unique totally bounded quasi-uniformity if and only if its topology τ is the unique ***base*** of open sets for τ that is closed under finite unions and finite intersections. (Here the convention $\bigcap \emptyset = X$ is used.) The condition is strictly weaker than hereditary compactness, even in T_1-spaces, but is equivalent to this property in supersober compact spaces. For each nonzero cardinal κ there exists a T_0-***space*** admitting exactly κ totally bounded quasi-uniformities.

A quasi-uniformity is called **transitive** if it has a base consisting of transitive relations. Each topological space that does not admit a unique quasi-uniformity admits at least 2^c nontransitive and at least 2^c transitive quasi-uniformities. Each infinite ***Tychonoff*** space admits at least 2^c transitive totally bounded and at least 2^c nontransitive totally bounded quasi-uniformities. Let (X, τ) be a topological space and let $\mathcal{A}$ be a collection of interior-preserving open covers $\mathcal{C}$ of X such that $\bigcup \mathcal{A}$ is a subbase for τ. For any $\mathcal{C}$ set $U_\mathcal{C} = \bigcup_{x \in X}(\{x\} \times \bigcap\{D: x \in D \in \mathcal{C}\})$. Then $\{U_\mathcal{C}: \mathcal{C} \in \mathcal{A}\}$ is a subbase for a compatible transitive quasi-uniformity $\mathcal{U}_\mathcal{A}$ on X (the so-called **Fletcher construction**). If $\mathcal{A}$ is the collection of all finite (respectively ***locally finite***; interior-preserving) open covers of X, then $\mathcal{U}_\mathcal{A}$ is the Pervin quasi-uniformity (respectively the **locally finite quasi-uniformity**; the **fine transitive quasi-uniformity**) of X (see [3]). Similarly, if $\mathcal{A}$ is the collection of all **well-monotone open cover**s, i.e., covers well-ordered by set-theoretic inclusion, then $\mathcal{U}_\mathcal{A}$ is the **well-monotone quasi-uniformity** of X. Its conjugate is always hereditarily precompact. A quasi-uniform space $(X, \mathcal{U})$ is totally bounded if and only if it is **doubly hereditarily precompact**, i.e., both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are hereditarily precompact. The quasi-uniformity generated by a ***preorder*** $\leqslant$ is hereditarily precompact if and only if $\leqslant$ is a **well-quasi-order** (i.e., $\leqslant$ has neither infinite antichains nor infinite strictly descending chains). A topological space is called **transitive** if its fine quasi-uniformity is transitive. The property is neither finitely ***productive*** nor ***hereditary***. Countable unions of closed (respectively open) transitive topological spaces are transitive. Transitivity is preserved under continuous closed surjections. Kofner's plane [HvM, Chapter 20] is a nontransitive topological space, since it is quasi-metrizable, but not non-archimedeanly quasi-metrizable. Various classes of topological spaces are known to be transitive (compare [4]): e.g., the ***orthocompact semistratifiable***; the suborderable; the hereditarily ***metacompact*** locally compact ***regular***; and the T_1-spaces having an orthobase.

A quasi-uniform space $(X, \mathcal{U})$ is called **bicomplete** provided that $\mathcal{U}^s$ is a ***complete*** (see [E, Chapter 8.3]) uniformity. Each quasi-uniform T_0-space $(X, \mathcal{U})$ has an (up to quasi-uniform isomorphism) unique **bicompletion** $(\widetilde{X}, \widetilde{\mathcal{U}})$ in the sense that the space $(\widetilde{X}, \widetilde{\mathcal{U}})$ is a bicomplete T_0-extension of $(X, \mathcal{U})$ in which $(X, \mathcal{U})$ is $\tau(\widetilde{\mathcal{U}^s})$-dense. The uniformities $(\widetilde{\mathcal{U}})^s$ and $\widetilde{\mathcal{U}}^s$ coincide. Furthermore if $f: (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a quasi-uniformly continuous map into a bicomplete quasi-uniform T_0-space $(Y, \mathcal{V})$, then there exists a (unique) quasi-uniformly continuous extension $\tilde{f}: \widetilde{X} \to Y$ of f. Each quasi-pseudometric T_0-space possesses a *quasi-pseudometric* bicompletion. Much work was devoted to spaces admitting bicomplete quasi-pseudometrics (quasi-uniformities). We mention only two results: A metrizable space admits a bicomplete quasi-metric if and only if it is an absolute metric $F_{\sigma\delta}$; the fine quasi-uniformity of each quasi-pseudometrizable and each quasi-sober space is bicomplete. Bicompletions of totally bounded T_0-quasi-uniformities $\mathcal{U}$ yield **joincompactifications**, i.e., $\tau(\widetilde{\mathcal{U}^s})$ is compact. The first thorough investigations of this concept were conducted by Á. Császár and S. Salbany. The **Fell compactification** of a locally compact (T_0-)space can be constructed as the bicompletion of its coarsest compatible quasi-uniformity. The locally compact supersober compact (also called **skew compact**) spaces provide an analogue in the category of T_0-spaces to the class of compact spaces in the category of ***Hausdorff*** spaces. They are characterized as the T_0-spaces that admit a totally bounded bicomplete quasi-uniformity; it is their coarsest compatible quasi-uniformity.

Let T denote the (obvious) ***forgetful functor*** from the category **Quu** to the category **Top** of topological spaces and continuous maps. A **functorial quasi-uniformity** is a functor $F: \mathbf{Top} \to \mathbf{Quu}$ such that $TF = 1$, i.e., F is a T-***section***. (All the quasi-uniformities constructed above by Fletcher's method satisfy this criterion.) G. Brümmer showed that the coarsest functorial quasi-uniformity is the Pervin quasi-uniformity. Using the idea of initiality, he built functorial quasi-uniformities by the **spanning construction** (see [1]). In the following discussion about the **bicompletion functor** K, we restrict ourselves to T_0-spaces. G. Brümmer called a T-section F **lower** K**-true** (respectively **upper** K**-true**) if $KFX \leqslant FTKFX$ (respectively $KFX \geqslant FTKFX$) for any space X; sections that satisfy both conditions are called K**-true**. (Here $\leqslant$ corresponds to the coarser relation for quasi-uniformities.) If F is K-true, then TKF is a ***reflection***. G. Brümmer showed that a T-section F is lower K-true if and only if F is spanned by some class of bicomplete quasi-uniform spaces and H. Künzi noted that F is upper K-true if and only if F is finer than the well-monotone quasi-uniformity. The fine (respectively Pervin; locally finite) quasi-uniformity is K-true (respectively lower K-true, but not upper K-true; neither lower nor upper K-true); see [5].

The categories **Top** and **Quu** are related to two other categories, namely the category of ***bitopological spaces***

(= bispaces) and ***bicontinuous maps*** and the category of topological ordered spaces and continuous increasing maps. Many concepts can be transferred (often with some difficulties) between these four categories. We next give some examples: Each quasi-uniform space $(X,\mathcal{U})$ induces the bispace $(X,\tau(\mathcal{U}),\tau(\mathcal{U}^{-1}))$. Exactly the **pairwise completely regular** bispaces can be obtained in this way. L. Nachbin observed that for any quasi-uniform T_0-space $(X,\mathcal{U})$, the triple $(X,\mathcal{T}(\mathcal{U}^s),\bigcap\mathcal{U})$ determines a topological **T_2-ordered** space (i.e., the partial order $\bigcap\mathcal{U}$ is $\tau(\mathcal{U}^s)\times\tau(\mathcal{U}^s)$-closed); his construction determines exactly the completely regularly (T_2-)ordered spaces. The theory of the bicompletion allows one to build bicompactifications (respectively ordered compactifications) in these two categories. A completely regularly ordered space $(X,\mathcal{T},\leqslant)$ is said to be **strictly completely regularly ordered** (in the sense of J. Lawson) if the bispace $(X,\mathcal{T}^\sharp,\mathcal{T}^\flat)$ is pairwise completely regular. Here $\mathcal{T}^\sharp$ and $\mathcal{T}^\flat$ denote the ***upper topology*** and ***lower topology*** of X, respectively. H. Künzi and S. Watson constructed completely regularly ordered spaces that are not strictly completely regularly ordered; e.g., such spaces cannot be ***topological lattice***s. R. Fox characterized quasi-metrizable bispaces as the pairwise stratifiable doubly γ-bispaces, but pointed out that there are pairwise stratifiable, pairwise developable bispaces which are *not* quasi-metrizable.

The search continues for completions based on a less symmetric idea than the bicompletion. Originally the interest was directed towards Pervin–Sieber completeness. A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called a **PS-filter** if for each $U\in\mathcal{U}$ there is $x\in X$ such that $U(x)\in\mathcal{F}$. A quasi-uniform space $(X,\mathcal{U})$ is called **PS-complete** provided that each PS-filter has a ***cluster point***. For **locally symmetric** [3, Definition 2.24] quasi-uniform spaces the property is equivalent to **convergence completeness**, i.e., each PS-filter ***converges***. P. Fletcher and H. Künzi showed that the fine quasi-uniformity of any nontrivial Tychonoff ***Σ-product*** is not PS-complete. To date no regular quasi-metrizable space is known whose fine quasi-uniformity is not PS-complete. Each Tychonoff sequentially PS-complete quasi-metric space is ***Čech-complete***. Moreover each Tychonoff ***orthocompact*** Čech-complete space with a ***G_δ-diagonal*** admits a convergence complete (non-Archimedean) quasi-metric. Unfortunately, PS-filters are often difficult to handle and no satisfactory theory of PS-completions exists. In recent investigations PS-completeness was often abandoned in favour of S- or left K-completeness. The latter concepts discussed below allow involved combinatorial arguments and with their help many classical results could be generalized satisfactorily.

Any ***net*** $(x_d)_{d\in D}$ in a quasi-uniform space $(X,\mathcal{U})$ is called a **left K-Cauchy net** (respectively a **right K-Cauchy net**) if for any $U\in\mathcal{U}$ there is $d\in D$ such that $d_2,d_1\in D$ and $d_2\geqslant d_1\geqslant d$ imply that $(x_{d_1},x_{d_2})\in U$ (respectively $(x_{d_2},x_{d_1})\in U$). A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is said to be left (respectively right) K-Cauchy if for each $U\in\mathcal{U}$ there is $F\in\mathcal{F}$ such that $U(x)\in\mathcal{F}$ (respectively $U^{-1}(x)\in\mathcal{F}$) whenever $x\in F$. A **left K-complete quasi-uniformity** is one for which each left K-Cauchy filter converges; A **right K-complete quasi-uniformity** is defined similarly. A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called **stable** if $\bigcap_{F\in\mathcal{F}}U(F)\in\mathcal{F}$ whenever $U\in\mathcal{U}$. We next mention some observations (mainly due to H. Künzi and S. Romaguera) about the introduced concepts that extend classical results, whose proofs however often need nontrivial modifications: PS-completeness implies left K-completeness. Any right K-Cauchy filter is stable; the converse holds for ultrafilters. A filter on a quasi-uniform space $(X,\mathcal{U})$ is both stable and left K-Cauchy if and only if it is a ***Cauchy filter*** in $(X,\mathcal{U}^s)$. Left (respectively right) K-completeness is equivalent to left (respectively right) K-completeness defined with nets. For a quasi-pseudometric quasi-uniform space $(X,\mathcal{U}_d)$ left K-sequential completeness, left K-completeness and the condition that each co-stable filter has a $\tau(\mathcal{U}_d)$-cluster point are equivalent. Here a filter is called **co-stable** if it is stable in the conjugate space. A quasi-uniform space $(X,\mathcal{U})$ is hereditarily precompact if and only if each ultrafilter is left K-Cauchy (equivalently, each filter is co-stable); $\tau(\mathcal{U})$ is compact if and only if $\mathcal{U}$ is precompact and left K-complete. The property that each co-stable filter clusters is preserved under quasi-uniformly open continuous surjections between quasi-uniform spaces. This property implies left K-completeness and for uniform spaces is equivalent to **supercompleteness** (i.e., completeness of the Hausdorff uniformity). Each regular left K-complete quasi-pseudometric space is a ***Baire space***. A quasi-metrizable space admits a left K-complete quasi-metric if and only if it possesses a λ-base in the sense of H. Wicke and J. Worrell (see [5]). Although many classical results on completeness do not generalize to right K-completeness, that property behaves better than left K-completeness in function spaces and hyperspaces.

From the point of view of Category Theory besides the bicompletion at least two further satisfactory asymmetric completions for quasi-uniform spaces are known: The theory of the **Smyth-completion** of topological quasi-uniform spaces (in a quasi-pseudometric variant often called the **Yoneda-completion**) and the theory of the **Doitchinov-completion** of quiet quasi-uniform spaces (with its quasi-metric variant for **balanced quasi-metrics**). M. Smyth endowed a quasi-uniform space $(X,\mathcal{U})$ with a topology τ that is not necessarily its standard topology $\tau(\mathcal{U})$, but is linked to $\mathcal{U}$ by some additional axioms. He defined the (now called) Smyth- or S-completion for a **topological quasi-uniform T_0-space** $(X,\mathcal{U},\tau)$. P. Sünderhauf called a quasi-uniform T_0-space equipped with its standard topology **S-completable** if its Smyth-completion also carries the standard topology; e.g., each **weightable T_0-quasi-pseudometric** in the sense of S. Matthews induces an S-completable quasi-uniformity. For S-completable spaces the construction of the Smyth-completion coincides with the bicompletion. A quasi-uniform T_0-space $(X,\mathcal{U})$ is S-completable (S-complete) if and only if each left K-Cauchy filter is Cauchy in $(X,\mathcal{U}^s)$ ($\tau(\mathcal{U}^s)$-converges). Hence each

S-complete quasi-uniform space is left K-complete and bicomplete. The Smyth-completion can be interpreted as a kind of K-completion. Since convergent filters in a quasi-uniform space are not left K-Cauchy in general, a simple theory of left K-completion is unlikely to exist. N. Ferrario and H. Künzi showed that if we equip an arbitrary T_0-space with the well-monotone quasi-uniformity (and its standard topology), then we obtain the ***sobrification*** equipped with its well-monotone quasi-uniformity as its Smyth-completion (= bicompletion). Extending this result, R. Flagg, R. Kopperman and P. Sünderhauf introduced the concept of a well-monotone quasi-uniformity $\mathcal{W}_\tau$ for any topological quasi-uniform T_0-space and proved that its Smyth-completion can be obtained via the bicompletion of $\mathcal{W}_\tau$.

A kind of bitopological extension of the bicompletion for *a subclass* of quasi-uniform spaces was found by D. Doitchinov. Call a filter $\mathcal{G}$ on a quasi-uniform space $(X,\mathcal{U})$ a **D-Cauchy filter** if there exists a **co-filter** $\mathcal{F}$ on X (i.e., for each $U \in \mathcal{U}$ there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$). Then $(\mathcal{F},\mathcal{G})$ is said to be a **Cauchy filter pair** on $(X,\mathcal{U})$. The quasi-uniformity $\mathcal{U}$ is called **D-complete** if each D-Cauchy filter converges in $(X, \tau(\mathcal{U}))$. Of course, each convergence complete quasi-uniform space is D-complete. A quasi-uniform space $(X,\mathcal{U})$ is called **quiet** if for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that for each Cauchy filter pair $(\mathcal{F},\mathcal{G})$ on $(X,\mathcal{U})$ and all $x, y \in X$ with $V^{-1}(y) \in \mathcal{F}$ and $V(x) \in \mathcal{G}$ we have $(x, y) \in U$. Each quiet quasi-uniform T_0-space $(X,\mathcal{U})$ has a standard D-completion, now called its Doitchinov completion, i.e., there is an (up to quasi-uniform isomorphism) uniquely determined quiet D-complete quasi-uniform T_0-space $(X^+,\mathcal{U}^+)$ in which X is doubly dense; X^+ has the reflection property for quasi-uniformly continuous maps into quiet D-complete quasi-uniform T_0-spaces. The Doitchinov completion of the conjugate of a quiet quasi-uniform T_0-space can be identified with the conjugate of its Doitchinov completion. Unfortunately quietness is a rather restrictive property, since, e.g., each quiet totally bounded quasi-uniform space is uniform by an observation due to P. Fletcher and W. Hunsaker. Extension theories for (more) general quasi-uniformities (and quasi-pseudometrics) were mainly developed in Hungary, e.g., by J. Deák (see [2]). D. Doitchinov himself also investigated a D-complete extension for **stable** (i.e., each D-Cauchy filter is stable) quasi-uniform T_0-spaces. For quasi-pseudometric spaces stability is a strong property; e.g., each regular doubly stable quasi-pseudometric space is ***subparacompact*** and closed sets are G_δ-sets.

The completeness properties discussed above were studied extensively for functorial quasi-uniformities; e.g., the well-monotone quasi-uniformity of any topological space X is left K-complete; it is S-complete if and only if X is quasi-sober (see [5]). The fine quasi-uniformity need neither be right K- nor D-complete. We next deal with two further interesting classes of quasi-uniform spaces investigated in recent years. A quasi-uniformity $\mathcal{U}$ on a set X is called **uniformly regular** if for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $\mathrm{cl}_{\tau(\mathcal{U})} V(x) \subseteq U(x)$ whenever $x \in X$. Quiet quasi-uniform spaces are doubly uniformly regular. For a topological space the property of admitting a uniformly regular quasi-uniformity lies strictly between regularity and complete regularity. Each uniformly regular quasi-uniformity is bicomplete if D-complete, and S-complete if left K-complete. P. Fletcher and W. Hunsaker established that the fine transitive quasi-uniformity of a topological space is quiet and convergence complete if it is uniformly regular. The fine quasi-uniformity of each regular preorthocompact semistratifiable space and each regular T_1-space with an orthobase is uniformly regular (see [5]). A quasi-uniformity $\mathcal{U}$ is called **monotonic** if there exists an operator $M : \mathcal{U} \to \mathcal{U}$ such that $M(U) \subseteq M(V)$ whenever $U, V \in \mathcal{U}$ and $U \subseteq V$, and such that $M(U)^2 \subseteq U$ whenever $U \in \mathcal{U}$. H. Junnila and H. Künzi established that each T_1-space with an orthobase admits a monotonic quasi-uniformity.

Only few results about uniformities on function spaces and ***hyperspaces*** generalize without complications to quasi-uniformities, as, e.g., work of J. Cao, S. Naimpally, B. Papadopoulos and H. Render showed. The following results may serve as examples: For a topological space X and a quasi-uniform space $(Y, \mathcal{V})$ the **quasi-uniformity of quasi-uniform convergence** on the set Y^X of all functions from X to Y is right K-complete if and only if $\mathcal{V}$ is right K-complete; the corresponding result does not hold for left K-completeness. The subspace $C(X, Y)$ of continuous maps from X to Y is closed in Y^X if $\mathcal{V}$ is locally symmetric, but not in general. The quasi-uniformity of **quasi-uniform convergence on compacta** induces on $C(X, Y)$ the ***compact-open topology*** if $\mathcal{V}$ is small-set symmetric, but not for arbitrary $\mathcal{V}$. Let $(X,\mathcal{U})$ be a quasi-uniform space. On the set $\mathcal{P}_0(X)$ of nonempty subsets of X the **Hausdorff quasi-uniformity** $\mathcal{U}_H$ is generated by the base $\{U_H : U \in \mathcal{U}\}$ where $U_H = \{(A, B) : B \subseteq U(A) \text{ and } A \subseteq U^{-1}(B)\}$. The Hausdorff quasi-uniformity is precompact (respectively totally bounded; joincompact) if and only if the underlying space has the corresponding property. Bicompleteness and left K-completeness (even for quasi-pseudometric spaces) behave rather badly under the Hausdorff hyperspace construction. Generalizing a well-known result due to B. Burdick and J. Isbell on supercompleteness of uniform spaces, H. Künzi and C. Ryser established that $\mathcal{U}_H$ is right K-complete if and only if each stable filter on $(X,\mathcal{U})$ clusters. H. Künzi and S. Romaguera characterized those quasi-uniformities $\mathcal{U}$ for which $\tau(\mathcal{U}_H)$ is compact (respectively $\mathcal{U}_H$ is hereditarily precompact); both results rely on combinatorial facts about well-quasi-ordering. Studies were conducted about those quasi-uniformities for which $\mathcal{U}_H$ induces on the set $\mathcal{K}_0(X)$ of nonempty $\tau(\mathcal{U})$-compact sets the ***Vietoris topology*** and about quasi-uniform variants of K. Morita's result that $(\mathcal{K}_0(X), \mathcal{U}_H|\mathcal{K}_0(X))$ is complete for any complete uniform space $(X,\mathcal{U})$ (see [5]). Furthermore authors studied spaces of ***multifunctions*** (equipped with their natural Hausdorff quasi-uniformity) and considered asymmetric versions of the result that each almost uniformly open (multivalued) map with closed graph from a supercomplete uniform space into an arbitrary uniform space is uniformly open.

Quasi-uniformities were also investigated in topological algebraic structures. The study of paratopological groups and asymmetric normed (real) vector spaces and cones with the help of canonical quasi-uniformities was particularly fruitful. A **paratopological group** is a group equipped with a topology such that the group multiplication is continuous; it carries natural quasi-uniformities whose definition can be copied verbatim from the definition of the natural ***uniformities*** of ***topological groups***. An **asymmetric norm** $\|\cdot\|$ possibly satisfies $\|ax\| = a\|x\|$ only for nonnegative real a. We shall concentrate here on paratopological groups, but note in passing that asymmetric norms provide, e.g., an efficient approach to the **complexity spaces** introduced by M. Schellekens. Among other things, J. Marín and S. Romaguera showed that the ground set of the bicompletion of the two-sided quasi-uniformity of a paratopological (T_0-)group naturally carries the structure of a paratopological group; the quasi-uniformity of the bicompletion yields its two-sided quasi-uniformity. Furthermore they observed that each first-countable paratopological group admits a left-invariant quasi-pseudometric compatible with its left quasi-uniformity. Together with H. Künzi they generalized parts of their theory about paratopological groups to ***topological semigroups*** (with neutral element). H. Künzi, S. Romaguera and O. Sipacheva noted that the two-sided quasi-uniformity of a regular paratopological group G is quiet. They established that its Doitchinov-completion can be considered a paratopological group if G is Abelian; in general however the multiplication does not extend to the completion. S. Romaguera, M. Sanchis and M. Tkachenko proved that for any topological (respectively quasi-uniform) space X the free and the free Abelian paratopological group over X exist (see [5]). H. Künzi continued work of P. Fletcher and K. Porter by showing that the group of all quasi-uniform isomorphisms of a bicomplete quasi-uniform (T_0-) space equipped with the quasi-uniformity of quasi-uniform convergence yields a paratopological group whose two-sided quasi-uniformity is bicomplete.

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e-10 Proximity Spaces

A **proximity** δ on a nonempty set X is concerned with nearness between two sets. There are many proximities in the literature, some symmetric, some non-symmetric. One of the simplest and widely used symmetric proximity is called the **Lodato or LO-proximity** [15] and satisfies the conditions (P0)–(P4) below (where $A\,\delta\,B$ means "A is near B and its negation $A \not\delta B$ means "A is far from B") :

(P0) $A\,\delta\,B = B\,\delta\,A$ (symmetry);
(P1) $A\,\delta\,B$ implies $A \neq \emptyset$ and $B \neq \emptyset$;
(P2) $A \cap B \neq \emptyset$ implies $A\,\delta\,B$;
(P3) $A\,\delta\,(B \cup C)$ if and only if $A\,\delta\,B$ or $A\,\delta\,C$ (union axiom);
(P4) $A\,\delta\,B$ and $\{b\}\,\delta\,C$ for each $b \in B$ implies $A\,\delta\,C$.

Further δ is **separated** if it satisfies:

(P5) $\{x\}\,\delta\,\{y\}$ iff $x = y$ (separated).

The pair (X, δ) is referred to as a **proximity space**. See [10, 11] for other symmetric proximities. A motivation for the above conditions is provided by the fact that if we replace A by $\{x\}$ in (P1)–(P4) we get precisely the well known Kuratowski closure axioms [KI]. Hence every LO-proximity space (X, δ) has a compatible topology $T(\delta)$ (the **proximal topology**), which is R_0 in view of the symmetry condition (P0), i.e., it satisfies the condition:

$$x \in \mathrm{cl}\{y\} \quad \text{iff} \quad y \in \mathrm{cl}\{x\}.$$

Conversely, every R_0 topological space (X, T) has a compatible LO-proximity δ_0 given by: $A\,\delta_0\,B$ iff $\mathrm{cl}\,A \cap \mathrm{cl}\,B \neq \emptyset$. This LO-proximity δ_0 is the most important symmetric proximity in view of the following:

THEOREM 1 [13]. *Every separated LO-proximity space (X, δ) can be densely embedded in a* ***compact T_1-space*** *αX with LO-proximity δ_0 such that*

$$A\,\delta\,B \text{ in } X \quad \text{iff} \quad A\,\delta_0\,B \text{ in } \alpha X.$$

Thus every abstract LO-proximity δ given by the axioms (P0)–(P5) comes from the concrete LO-proximity δ_0. If we partially order proximities by defining $\delta \leqslant \delta'$ iff $A\,\delta'\,B$ implies $A\,\delta\,B$, then we find that δ_0 is the finest LO-proximity. So we call δ_0 the **fine LO-proximity**. Every T_1-space (X, T) has the coarsest compatible LO-proximity δ_c given by:

$$A\,\delta_c\,B \quad \text{iff} \quad \mathrm{cl}\,A \cap \mathrm{cl}\,B \neq \emptyset \text{ or both } A \text{ and } B \text{ are infinite.}$$

A function $f : (X, \delta) \to (Y, \delta')$ is called **p-continuous** iff $A\,\delta\,B$ implies $f(A)\,\delta'\,f(B)$. The concept of p-continuity is analogous to continuity in topological spaces and uniform continuity in uniform spaces.

A stronger proximity which was widely known long before the LO-proximity is the Efremovic **EF-proximity** [17] which satisfies the following additional condition

(EF): $A \not\delta B$ implies there is an E such that $A \not\delta E$ and $E^c \not\delta B$.

In case (X, T) is ***Tychonoff***, then it has a compatible separated EF-proximity δ_F: $A \not\delta_F B$ iff there is an $f \in C(X, [0, 1])$ such that $f(A) = 0$ and $f(B) = 1$. In fact, every EF-proximity satisfies the above condition, which is reminiscent of the well known Urysohn's Lemma. Consequently, many results of ***normal spaces*** are true in Tychonoff spaces if we replace disjoint closed sets by those that are far with respect to some compatible EF-proximity! It is clear that δ_F is the fine EF-proximity and equals δ_0 iff X is normal. In contrast to LO-proximities, a Tychonoff space does not always have a compatible coarsest EF-proximity. The coarsest EF-proximity δ_1 exists iff the space is ***locally compact*** and is induced by the ***Alexandroff one-point compactification*** viz.

$$A\,\delta_1\,B \quad \text{iff} \quad \mathrm{cl}\,A \cap \mathrm{cl}\,B \neq \emptyset \text{ or both } \mathrm{cl}\,A \text{ and } \mathrm{cl}\,B \text{ are non-compact.}$$

Every compact Hausdorff space has a unique compatible EF-proximity δ_0. Theorem 1 has the following analogue for EF-proximity spaces:

THEOREM 2 [17]. *Every separated EF-proximity space (X, δ) can be* ***densely embedded*** *in a compact Hausdorff space σX (the* ***Smirnov compactification****) with EF-proximity δ_0 such that $A\,\delta\,B$ in X iff $A\,\delta_0\,B$ in σX.*

Moreover, the map $(X, \delta) \to (\sigma X, \delta_0)$ is a bijective order preserving map from EF-proximities to Hausdorff compactifications of a Tychonoff space, i.e., the finer EF-proximity maps to the larger compactification. Thus the ***Čech–Stone compactification*** βX is the Smirnov compactification corresponding to the fine EF-proximity δ_F. Therefore, it is clear that the study of compactifications is facilitated by proximities. Every (separated) uniform space $(X, \mathcal{U})$ has a compatible (separated) EF-proximity δ

$$A\,\delta\,B \quad \text{iff} \quad \text{for every } U \in \mathcal{U},\ U[A] \cap B \neq \emptyset.$$

Just as every Tychonoff topology T has, in general, several compatible EF-proximities, every EF-proximity δ has, in general, several compatible (Weil) uniformities. There is always a unique totally bounded uniformity compatible with

δ but there need not be a finest one. However, in the case of a metric space (X, d), the metric proximity δ_m, given by

$$A\ \delta_m\ B \text{ in } X \quad \text{iff} \quad \inf\{d(a,b)\colon a \in A,\ b \in B\} = 0, \ (*)$$

has the metric uniformity, generated by d, as the finest compatible uniformity [17]. "So proximity is a structural layer distinct from, and between, topological structure and uniform structure. Thus all proximity invariants are topological invariants, but some uniform invariants, such as total boundedness and completeness, are not proximity invariants" [18]. As for LO-proximities, analogous results have been obtained by the use of generalized uniformities [15]. Since proximity is a finer structure than topology, it is useful in solving many topological problems. We give a few examples and refer the reader to the bibliography for further details. One of the most important problems concerns extension of continuous functions from dense subspaces. The following result includes almost all known results in this area:

THEOREM 3 [13]. *Suppose X is dense in a T_1-space αX with fine LO-proximity δ_0, (Y, δ) is a separated EF-space with Smirnov compactification σY. Then a continuous function $f: X \to Y$ has a continuous extension $F: \alpha X \to \sigma Y$ if and only if f is p-continuous.*

A **semi-metric** is a distance function satisfying all the conditions of a metric with the possible exception of the triangle inequality. If we define a proximity from a semi-metric as in $(*)$ we get a LO-proximity. Using LO-proximities and generalized uniformities [15], one can get the following simple characterizations of developable and metrizable spaces; they have proved to be useful in General Relativity [7]:

THEOREM 4 [14, 1].

(a) *X is developable if and only if it has an upper semi-continuous semi-metric.*
(b) *X is metrizable if and only if it has a compatible semi-metric which induces an EF-proximity.*

Recently, proximity has been used in the study of hyperspaces, see [16, 2–4, 9] and function spaces [5, 6, 8, 12]. See [17] for literature up to 1970.

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e-11 Generalized Metric Spaces, Part I

1. Introduction

Generalized metric properties can be loosely classified as the *structural* topological invariants of metrizable spaces. Generalized metric properties are studied on the one hand to better understand the topology of metrizable spaces, and on the other to provide classes of non-metrizable spaces with some of the desirable features of metrizable spaces. In their first rôle, generalized metric properties frequently occur as factors in ***metrization theorems*** (see "Modern metrization theorems" in this volume).

Here we consider one group of classes of **generalized metric spaces** all connected by the idea of a **network**. A family $\mathcal{N}$ of subsets of a space X is a network for X if whenever a point x of X is in an open set U, then there is an N in $\mathcal{N}$ such that $x \in N \subseteq U$.

It is clear that the image of a base is a network, and a secondary theme will be the characterization of images of metrizable spaces, under various types of maps, in terms of the existence of special networks.

All spaces will be assumed to be regular and Hausdorff.

2. Σ-spaces, semistratifiable spaces, σ-spaces, and cosmic spaces

Motivating many of the generalized metric classes introduced here are the Nagata–Smirnov Metrization Theorem: a space is metrizable if and only if it has a ***σ-locally finite*** base, and the Bing Metrization Theorem: a space is metrizable if and only if it has a ***σ-discrete space***.

A space with a σ-locally finite network is a **σ-space**. A space, X say, is **semistratifiable** if for every closed subset C of X there is a sequence of open subsets $(U(C,n))_n$ such that (i) $C = \bigcap_n U(C,n)$ and (ii) if $C \subseteq C'$ then for all n, $U(C,n) \subseteq U(C',n)$ [3]. A space X is a **Σ-space** if there is a σ-locally finite closed family $\mathcal{F}$ and a cover $\mathcal{C}$ of closed ***countably compact*** subsets such that if $C \in \mathcal{C}$, U is open and $C \subseteq U$, then $C \subseteq F \subseteq U$ for some $F \in \mathcal{F}$ [20]. If the elements of $\mathcal{C}$ are in fact ***compact***, then X is said to be a **strong Σ-space** [20].

In the definition of Σ- and σ- space above, *locally finite* can be replaced by *discrete*. For σ-spaces, one may also replace *locally finite* with *closure-preserving*.

Subspaces, closed images, and countable products of σ-spaces (respectively semistratifiable spaces) are again σ-spaces (respectively semistratifiable). A closed subspace of a (strong) Σ-space is again a (strong) Σ-space. But a closed image of a strong Σ-space may fail to be a Σ-space [KV, Chapter 10, Example 4.18]. A fundamentally important property of Σ-spaces is that a countable product of Lindelöf (respectively paracompact) Σ-spaces is again a Lindelöf (respectively paracompact) Σ-space [20].

Immediate from the definition of semistratifiability is the fact that every semistratifiable space is ***perfect***, and hence (by productivity) has a ***G_δ-diagonal***. Every σ-space is semistratifiable and a Σ-space. A Σ-space is a σ-space if and only if it has a G_δ-diagonal. Thus any compact space without a G_δ-diagonal, such as the ***double-arrow space***, is a strong Σ-space that is not semistratifiable. Not every semistratifiable space is a σ-space (or Σ-space) (see [KV, Chapter 3, Example 9.10], and [2] for a hereditarily Lindelöf example from CH), although the two concepts are very close.

Developable spaces are an important class of σ-spaces. Another source of σ-spaces are the stratifiable spaces of the next section [11].

A space is a **cosmic space** if it has a countable network. Cosmicity of a space is preserved by taking arbitrary subspaces, countable products, and continuous images. A σ-space is cosmic if it has countable ***extent*** (for example, if it is ***Lindelöf***), or is ***collectionwise normal*** and has the ***countable chain condition*** (as is the case if it is ***separable***). Cosmic spaces are (hereditarily) Lindelöf and separable, in all finite powers; hence paracompact (and collectionwise normal) in all finite powers. It is unknown if the converse is true: 'if a space has all finite powers hereditarily Lindelöf and separable, then it is cosmic', although there are consistent counterexamples ([18] assuming the ***Continuum Hypothesis***, [22] assuming $\mathfrak{b} \neq \omega_2$). However, if we call a space **cometrizable** if there is a weaker metric topology such that each point has a neighbourhood base of metric-closed sets, then, assuming the ***Proper Forcing Axiom***, Gruenhage [9] shows that every cometrizable space whose square hereditarily has the countable chain condition, is cosmic.

Lindelöf Σ-spaces play the same rôle among Σ-spaces as cosmic spaces do for σ-spaces. They are especially useful in the context of C_p-theory. For example, a space X is a ***Gul'ko compactum*** if and only if $C_p(X)$ is a Lindelöf Σ-space [21].

3. Stratifiable and M_1-spaces

A space is **stratifiable** if it is a σ-space and ***monotonically normal*** [4]. The class of stratifiable spaces is closed under taking countable products, closed images and arbitrary subspaces. Alternative descriptions of stratifiability include: a space X is stratifiable if it is semistratifiable and monotonically normal, or if for every closed subset C of X there is a sequence of open subsets $(U(C,n))_n$ such that (i) $C = \bigcap_n U(C,n) = \bigcap_n \overline{U(C,n)}$ and (ii) if $C \subseteq C'$ then for all n, $U(C,n) \subseteq U(C',n)$. From this last characterization it is clear that every stratifiable space is ***perfectly normal***. Further, stratifiable spaces are hereditarily paracompact.

Another useful feature of stratifiable spaces, are their powerful extension properties. Denote by $C(X,Y)$, the set of all continuous maps from a space X into another space Y, either with the ***pointwise topology***, or the ***compact-open topology***. Borges [1], extending a result of Dugundji [5] for metrizable spaces, proved: let X be stratifiable, A a closed subspace of X and let L be a ***locally convex topological vector space***, then there is a map $e : C(A,L) \to C(X,L)$ such that

(i) $e(f)|A = f$;
(ii) $e(f)(X)$ is contained in the closed convex hull of $f(A)$;
(iii) e is linear;
(iv) e is continuous when the function spaces are both given either the pointwise or compact-open topologies.

Among other important classes of space contained in the class of stratifiable spaces are: the class of ***CW-complex***es, $C_k(X)$ for X ***Polish*** [8], and closed images of metrizable spaces (see below). The importance of stratifiable spaces lies in the combination of stability, attractive properties, and the wide variety of spaces which are stratifiable.

A **quasi-base** for a space X is a family, $\mathcal{B}$, of subsets of X such that whenever $x \in U$, where U is open in X, there is a $B \in \mathcal{B}$ such that $x \in B^o \subseteq B \subseteq U$. Then a space is an M_1**-space** (a M_2**-space**) if it has a σ-closure-preserving base (quasi-base) [4]. Gruenhage [10] and, independently, Junnila [14] proved that a space is stratifiable if and only if it is M_2. It follows that every M_1 space is stratifiable. Whether the converse, 'every stratifiable space is M_1', is true is the infamous M_1–M_3 problem (*note*: stratifiable spaces were defined by Ceder under the name M_3**-space**s, Borges subsequently introduced the term stratifiable).

The definition of M_1-space seems so natural, and so close to M_2, yet even basic questions concerning M_1-spaces remain unresolved. It is unknown if closed subspaces of M_1-spaces are M_1, or if the M_1 property is preserved by ***perfect*** maps. In fact, a positive answer to either question is *equivalent* to the M_1–M_3 problem, due to the remarkable fact that every stratifiable space embeds as a perfect retract in an M_1 space [12]. (And if M_1 is indeed the same as stratifiable, then the M_1 property inherits the stability of stratifiability, so that arbitrary subspaces and general closed images of M_1 spaces would again be M_1.)

There are a variety of positive partial results on the M_1–M_3 problem. First countable stratifiable spaces (also known as **Nagata space**s) are M_1 [13]. Recently this result has been improved by relaxing 'first-countable' to 'k-space' [19]. Further, defining a space to be F_σ**-metrizable** if it is the countable union of closed metrizable subspaces, and a μ**-space** if it can be embedded in a countable product of paracompact F_σ-metrizable spaces, then every stratifiable μ-space is M_1. Interestingly it is unknown whether or not every stratifiable space is μ.

In the opposite direction, searching for a counter-example, a plausible candidate is the function space $C_k(\mathbb{P})$, where $\mathbb{P}$ is the space of irrationals. This space is stratifiable, but despite the efforts of experts in the field it is unclear if $C_k(\mathbb{P})$ is M_1.

4. ℵ-spaces, and images of metrizable spaces

A k**-network** for a space X is a family $\mathcal{K}$ of subsets of X such that whenever compact K is contained in open U, then there is a finite subset $\mathcal{K}_0 \subseteq \mathcal{K}$ such that $K \subseteq \bigcup \mathcal{K}_0 \subseteq U$. Now a space is an **ℵ-space** if it has a σ-locally finite k-network. In the definition, *locally finite* may be replaced by *discrete*, also the elements of the k-network may be assumed to be closed. Clearly a k-network is a network, and hence every ℵ-space is a σ-space.

The class of ℵ-spaces is closed under taking countable products, closed subspaces, and perfect images.

An $\aleph_0$-space is one with a countable k-network. Both ℵ and $\aleph_0$ spaces find application in the study of $C_k(X)$ (the space of continuous real valued functions on X, with the compact-open topology). For instance, $C_k(X)$ is cosmic if and only if it is $\aleph_0$, which occurs if and only if the space X itself is $\aleph_0$ [17].

Further, ℵ and $\aleph_0$ spaces play a starring rôle in the analysis of images of metrizable spaces under various types of map. Observe that the quotients of metrizable spaces are pecisely the sequential spaces, and Fréchet spaces are the images of metrizable spaces under pseudo-open maps. Ponomaraev showed that a space is first countable if and only if it is the open image of a metrizable space.

Starting with images of separable metrizable spaces, a space is a quotient space of a separable metrizable space precisely if it is an $\aleph_0$ k-space [17]. Weakening *quotient* to **compact-covering** (every compact subset of the range is the image of a compact subset of the domain) characterizes $\aleph_0$-spaces. A space is cosmic if and only if it is the **co**ntinuous image of a **s**eparable **m**etr**ic** space (hence the name); while a space is a Lindelöf Σ-space if and only if it is the continuous image of the perfect preimage of a separable metrizable space.

Closed images of metrizable spaces have attracted particular attention, and the name **Lašnev space** from N. Lašnev for his work on this class of spaces. Lašnev spaces are known to be M_1. They have been characterized both in terms of k-networks and standard networks. Both descriptions use the notion of **hereditarily closure-preserving** families: a collection $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ of subsets of X is hereditarily closure-preserving provided, for any choice of $G_\alpha \subseteq H_\alpha$ for each $\alpha \in A$, $\{G_\alpha : \alpha \in A\}$ is closure-preserving.

Foged's theorem states that a space is Lašnev if and only if it is Fréchet and has a σ-hereditarily closure-preserving k-network consisting of closed sets [6]. Lašnev's characterization is: a space, X, is Lašnev if and only if it is Fréchet, and has σ-hereditarily closure-preserving network, say $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$, such that each $\mathcal{N}_n$ is a cover, and, if

$x \in N_n \in \mathcal{N}_n$ for each n, then $\{N_n\}_n$ is either hereditarily closure-preserving or a network at x [15].

Spaces which are the image of a metrizable space under a closed map with separable fibers, have a particularly elegant description, they are the Fréchet $\aleph$-spaces [7, 16].

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e-12 Generalized Metric Spaces, Part II

A **development** for a space X is a sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of open covers of X such that for every $x \in X$ the collection $\{\mathrm{St}(x, \mathcal{G}_n)\colon n \in \omega\}$ is a local base. The notation $\mathrm{St}(x, \mathcal{G}_n)$, called the "star of $\mathcal{G}_n$ at x", denotes $\bigcup\{G\colon x \in G \in \mathcal{G}_n\}$. Alternatively, whenever $x \in U_n \in \mathcal{G}_n$, for every $n \in \omega$, then $\{U_n\colon n \in \omega\}$ is a local base at x. For convenience, it can be arranged that every $\mathcal{G}_{n+1}$ is a ***refinement*** of $\mathcal{G}_n$ or even that $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$. A **Moore space** space is a ***regular*** space with a development. It is clear that any ***metric space*** is developable (use the open cover of $1/n$-balls for $\mathcal{G}_n$) and so is a Moore space. By analogy with metric spaces one defines a Moore space to be **complete** if it has a development $\{\mathcal{G}_n\}_{n\in\omega}$ such that every decreasing sequence $\{F_n\}_{n\in\omega}$ of closed sets for which $F_n \subseteq G_n$, for some $G_n \in \mathcal{G}_n$, has a non-empty intersection. The classic example of a ***separable***, nonmetrizable Moore space is described in the next paragraph. All spaces considered in this article are Hausdorff topological spaces.

Let $\Gamma = \mathbb{R} \times [0, \infty)$. For $p \in \Gamma$, we define an open local base $\{U(p, n)\}_{n=1}^{\infty}$ at p as follows: If d is the usual Euclidean metric on $\mathbb{R}^2$ and $p = (p_1, p_2)$, with $p_2 > 0$, let $U(p, n) = \{x \in \mathbb{R} \times (0, \infty)\colon d(p, x) < 1/n\}$. If $p_2 = 0$, let $U(p, n) = \{p\} \cup \{x \in \Gamma\colon d((p_1, 1/n), x) < 1/n\}$. In this case we are describing a neighbourhood consisting of $\{p\}$ along with an open disc tangent to the x-axis at p. If $\mathcal{G}_n = \{U(p, n)\colon p \in \Gamma\}$ it is straight forward to verify that $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a development for the regular space Γ. We call this space Γ the **Moore Plane**. It is sometimes also called the **Niemytzki Plane** [E, 1.2.4].

It would be beneficial to observe a few properties of the Moore Plane. The set D of points in Γ with rational coordinates is countable and dense in Γ, so Γ is separable. The set $S = \mathbb{R} \times \{0\}$ is a uncountable closed discrete subset so Γ is not ***hereditarily separable*** and not ***Lindelöf***. The space cannot be metrizable since separable metric spaces are always Lindelöf (and hereditarily separable). Γ is a ***Tychonoff space*** but because $2^{|D|} < 2^{|S|}$, an application of ***Jones' Lemma*** [E, 1.7.12(c)] shows that the space is not ***normal***. Using the ***Baire Category Theorem*** one can specifically show that if A is the set of points from S having a rational x-coordinate and $B = S \setminus A$ then A, B are a pair of disjoint closed sets which cannot be separated. Γ is not ***locally compact*** since the elements of S do not have compact neighbourhoods. Γ is ***connected*** and ***locally connected***. The subspace $W = \Gamma \setminus S$ is a dense open completely metrizable subspace. This shows that Γ is a ***Baire space***. In fact, Γ is actually a ***Čech-complete space*** [E, 3.9.B(b)].

There are numerous examples of Moore spaces, with various properties, in the literature. G.M. Reed [11] has given a very versatile construction for designing Moore spaces with desired properties. Given any first-countable space the construction gives a Moore space as an output space.

Whether or not a given Moore space has a dense metrizable subspace can be of special interest. If X is a Moore space with a dense metrizable subspace Z then $d(X) = d(Z) = c(Z) = c(X) = L(Z) \leqslant L(X)$ (where $d(X)$ denotes the ***density***, $c(X)$ the ***cellularity***, and $L(X)$ the ***Lindelöf degree*** of X). A dense metrizable subspace Z has a ***σ-discrete*** open base which induces a ***σ-disjoint π-base*** for X. For the converse, H.E. White has shown that any Moore space (or any T_2 first-countable space), with a σ-disjoint π-base, has a dense metrizable subspace [14]. Every complete Moore space has a dense metrizable subspace; in fact, G.M. Reed has shown that every **completable Moore space** (i.e., embeddable in a complete Moore space) has a dense metrizable subspace. Without a dense metrizable subspace, there exist Moore spaces which satisfy the DCCC (every discrete open collection is countable) but not the CCC (***countable chain condition***) and there exist Moore spaces which satisfy the CCC but are not separable [11]. Any Moore space which is a Baire space has a σ-discrete π-base (so has a dense metrizable subspace). D.L. Fearnley [5] recently constructed an example of a Moore space with a σ-discrete π-base which cannot be densely embedded in any Moore space with the Baire property.

The ***Nagata–Smirnov Metrization Theorem*** shows that ***paracompact*** developable spaces are metrizable. R.H. Bing shows in [3] that collectionwise normal Moore spaces are metrizable. Bing defines a space X to be **collectionwise normal** if for every discrete collection $\mathcal{F}$ of closed subsets of X there exists a pairwise disjoint collection $\{V(F)\colon F \in \mathcal{F}\}$ of open sets such that, for all $F \in \mathcal{F}$, $F \subseteq V(F)$ and $V(F) \neq V(F')$ if $F \neq F'$. The **Moore Metrization Theorem** [E, 5.4.2] gives metrizability of a developable space if the space has a **strong development**: A ***T_1-space*** X is metrizable if and only if X has a development $\{\mathcal{H}_n\}_{n=1}^{\infty}$ such that for every $x \in X$ and every neighbourhood U of x there exists a neighbourhood V of x and a natural number n such that $\mathrm{St}(V, \mathcal{H}_n) \subseteq U$.

For more than 60 years, much of the work on the metrizability of Moore spaces has been related to the ***Normal Moore Space Conjecture***. (See [KV, Chapter 15] for the technical details of most of what we give in this paragraph.) In 1937 F.B. Jones [9] proved, under the assumption $2^{\aleph_0} < 2^{\aleph_1}$, that every separable normal Moore space is metrizable. Under $2^{\aleph_0} < 2^{\aleph_1}$, this can be strengthened to say that normal Moore spaces which satisfy the ***countable chain condition*** are metrizable. R.H. Bing [3] showed that if there is a Q-set then there is a separable non-metrizable normal Moore space and R.W. Heath [7] showed that if there is a separable non-metrizable normal Moore space then there exists a Q-set. A **Q-set** is an uncountable subset $E \subseteq \mathbb{R}$ such that every subset of E is a relative ***G_δ-set*** in E. A few years after Heath's result, J. Silver showed that $\mathsf{MA}+\neg\mathsf{CH}$ implies the existence of a Q-set. In fact, under $\mathsf{MA}+\neg\mathsf{CH}$ every uncountable subset of $\mathbb{R}$ with cardinality less than $\mathfrak{c}=2^{\aleph_0}$ is a Q-set. Assuming E is an uncountable Q-set, we can now easily describe a separable normal non-metrizable Moore subspace of the Moore Plane by simply replacing the "bottom axis" $\mathbb{R}\times\{0\}$ (in the Moore Plane Γ) with $E\times\{0\}$. The metrizability of *separable* normal Moore spaces is now seen to be independent of and consistent with ZFC.

Assuming the ***Product Measure Extension Axiom*** (PMEA), P. Nyikos [10] showed in 1978 that every normal Moore space is metrizable. This was labeled "A provisional solution" to the Normal Moore Space Problem because PMEA implies the existence of a ***measurable cardinal*** and so it would be impossible to prove the consistency of this axiom within ZFC. However, W.G. Fleissner has shown that if there is a model of ZFC in which every normal Moore space is metrizable then there is a model with a measurable cardinal [KV, Chapter 16]. This shows an inescapable link between the Normal Moore Space Problem and ***large cardinals***.

Results of W.G. Fleissner [KV, Chapter 15] show that, assuming the axiom V = L, locally compact normal Moore spaces are metrizable. G.M. Reed and P. Zenor [12] show that locally compact, locally connected normal Moore spaces are metrizable in ZFC. It is not known if locally compact, locally connected, ***countably paracompact*** Moore spaces are metrizable in ZFC. For many other results related to the Normal Moore Space Problem and other normal versus ***collectionwise Hausdorff*** problems in first countable spaces, a countably paracompact analogue was later shown to be true. For example, assuming CH, Fleissner has shown that separable countably paracompact Moore spaces are metrizable. Under PMEA, D. Burke showed in 1984 that countably paracompact Moore spaces are metrizable [HvM, Chapter 7].

Suppose X is a topological space and $d: X\times X\to[0,\infty)$ such that, for all $(x,y)\in X\times X$, $d(x,y)=d(y,x)$ and $d(x,y)=0 \iff x=y$. The function d is said to be a **symmetric** [2] for X provided: For all nonempty $A\subseteq X$, A is closed $\iff \inf\{d(x,z): x\in A\}>0$ for every $z\in X\setminus A$. In this case, we would say (X,d) (or X) is ***symmetrizable*** (with symmetric d). The function d is said to be a **semi-metric** for X provided: For all $A\subseteq X$, $z\in\bar{A} \iff \inf\{d(x,z): x\in A\}=0$. A symmetric d for X is a semi-metric for X if and only if X is first-countable. In fact: For $x\in X$ and $r>0$ let $B_d(x,r)=\{y: d(x,y)<r\}$; then d is a semi-metric for X if and only if it is always true that $x\in \operatorname{Int} B_d(x,r)$. In case d is a semi-metric it is clear that $\{B_d(x,\frac{1}{n})\}_{n=1}^{\infty}$ is a local neighbourhood base (not necessarily open) at x. In case d is a symmetric it is true that $\{B_d(x,\frac{1}{n})\}_{n=1}^{\infty}$ is a ***weak neighbourhood base*** at x. A symmetrizable space (X,d) is always ***sequential*** and d is a semi-metric if and only if X is actually a ***Fréchet*** space. Since hereditarily sequential spaces are Fréchet spaces, a symmetrizable non-semi-metrizable space must always have a subspace which is not symmetrizable. Symmetrizability is hereditary to open or closed subspaces. The property of semi-metrizable is hereditary to all subspaces.

Whether a space is semi-metrizable or not is often recognized by the nature of a given local base rather than by observing an actual semi-metric. Heath [6] shows that a T_1 space X is semi-metrizable if and only if every $x\in X$ has a decreasing local neighbourhood base $\{U_n(x)\}_{n=1}^{\infty}$ such that $x\in U_n(x_n)$ implies $x_n\to x$. Notice that the last condition would be satisfied if the local bases are symmetric in the sense that $y\in U_n(x) \iff x\in U_n(y)$. A similar result can be shown for symmetrizable spaces: A T_1 space X is symmetrizable if and only if every $x\in X$ has a decreasing weak neighbourhood base $\{W_n(x)\}_{n=1}^{\infty}$ such that $x\in W_n(x_n)$ implies $x_n\to x$. It is now easy to see that every T_1 developable space is semi-metrizable. If desired, a canonical semi-metric d related to a development $\{\mathcal{G}_n\}_{n\in\omega}$ (assuming $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$) can always be defined by: $d(x,y)=\sup\{\frac{1}{n}: x\notin \operatorname{St}(y,\mathcal{G}_n)\}$, if $x\neq y$, and $d(x,y)=0$, if $x=y$.

It would be useful to have a representative example (essentially due to Heath [6]) of a semi-metric space which is not developable. For $\mathbb{H}=\mathbb{R}^2$, $\mathbf{z}=(z_1,z_2)\in X$ and $n\in\mathbb{N}$, let $B(\mathbf{z},n)=\{(x,y)\in X: |y-z_2|<\frac{1}{n}|x-z_1| \text{ and } |x-z_1|<\frac{1}{n}\}$. Use $\{B(\mathbf{z},n)\}_{n=1}^{\infty}$ as a local base at $\mathbf{z}$. A quick sketch reveals that $B(\mathbf{z},n)$ is a "horizontal bow-tie", centered at $\mathbf{z}$, with the radius and angle decreasing as n increases. We have not given a semi-metric for $\mathbb{H}$ but the result from the previous paragraph says that $\mathbb{H}$ is semi-metrizable. Observe that $\mathbb{H}$ is separable, connected, locally connected and Tychonoff. $\mathbb{H}$ is not Lindelöf since every vertical line is an uncountable closed discrete subset. A standard Jones' Lemma argument shows this space is not normal.

If $f: M\to X$ is a quotient map from a ***metric space*** (M,ρ) such that $\rho(f^{-1}(x), M\setminus f^{-1}(W))>0$ whenever $x\in W\subseteq X$ and W is open, then $d(x,y)=\rho(f^{-1}(x), f^{-1}(y))$ always defines a compatible symmetric on X [2]. Let (X,d) be the specific example of this type obtained from the quotient space of the real line $\mathbb{R}$ by identifying n and $\frac{1}{n}$ for every $n\in\mathbb{N}$. If f is the quotient map then $z=f(0)$ is the only point in X not having a countable local base. A symmetrizable space which is not semi-metrizable must always have a subspace which is not symmetrizable. In this case, $Y=X\setminus f(N)$ is a non-symmetrizable subspace (because it is not sequential).

Every semi-metric space is ***subparacompact***, ***perfect*** (all closed subsets are G_δ-sets) and has a G_δ-diagonal. Every collectionwise normal semi-metric space is paracompact. Symmetrizable spaces do not necessarily have these properties. There is an example [KV, Chapter 10] of a regular symmetrizable space X which is not ***submetacompact***, not ***countably metacompact*** (hence, not perfect), and does not have a G_δ-diagonal. It is not known whether the singleton subsets of a regular symmetrizable space must be G_δ-sets. It is not known whether collectionwise normal symmetrizable spaces must be paracompact. The class of semi-metric spaces is closed under countable products. That is, if X_n is semi-metrizable for all $n\in\omega$ then $\prod_{n\in\omega} X_n$ is semi-metrizable. The class of symmetric spaces is not even closed under finite products. Lindelöf semi-metric or symmetric spaces are hereditarily Lindelöf [KV, Chapter 10]. Every semi-metric space has a σ-discrete dense subset, so in a Lindelöf semi-metric space this dense set must be countable; i.e., every Lindelöf semi-metric space is separable. Arhangel'skiĭ [2] asks whether every Lindelöf symmetrizable space is separable. That is, do there exist symmetrizable ***L-spaces***? D. Shakhmatov [13] has constructed a consistent example of a regular symmetrizable L-space. It is still unknown whether it is consistent (perhaps under $\mathsf{MA}+\neg\mathsf{CH}$) that there are no regular symmetrizable L-spaces. A Hausdorff example (due to Z. Balogh, D. Burke and S. Davis) of a non-separable Lindelöf symmetrizable space has been constructed in ZFC.

It is not obvious, but the semi-metric space $\mathbb{H}$ described earlier does not have a ***σ-discrete network*** [KV, Chapter 10]; that is, $\mathbb{H}$ is not a σ-space. E. Michael has shown under CH that $\mathbb{H}$ has a Lindelöf subspace Z which is not a σ-space. This is also known to exist under the weaker axiom $\mathfrak{b}=\mathfrak{c}$ [KV, Chapter 3] but there is no example known in ZFC of a Lindelöf semi-metrizable space without a countable network. Semi-metric spaces seem to be "close" to being σ-spaces. It is true that a σ-space X is semi-metrizable if and only if X is first-countable [2]. A σ-space X is symmetrizable if and only if X is ***weakly first-countable***. All Moore spaces are σ-spaces in a natural way. Since Moore spaces are subparacompact one can find a σ-discrete closed refinement of every level of a development and the union of these refinements will clearly form a σ-discrete network for the Moore space.

If (X,d) is a semi-metric space there are certain properties which depend on the semi-metric d itself. If $x\in X$ and $r>0$, the "spheres of radius r about x" $S_d(x,r)=\{z\colon d(x,z)<r\}$ must contain x in their interior but these spheres do not have to be open. In fact, R. Heath has given an example of a regular semi-metric space which has no compatible semi-metric under which all spheres are open. This geometry of the spheres is related to the continuity of the real-valued functions $d_x:X\to[0,\infty)$: $d_x(y)=d(x,y)$. The function d_x is ***upper semi-continuous*** if and only if $S_d(x,\varepsilon)=d_x^{-1}([0,\varepsilon))$ is open in X for every $\varepsilon>0$. The function d_x is ***lower semi-continuous*** if and only if $\overline{S_d(x,\varepsilon)}\subseteq\{z\colon d(x,z)\leqslant\varepsilon\}$ for every $\varepsilon>0$. In a semi-metric space (X,d), convergent sequences need not be Cauchy (with respect to d) or may not even have a Cauchy subsequence. It is known that a T_1 space X is developable if and only X if is semi-metrizable by a semi-metric where all convergent sequences are Cauchy. In a semi-metric space in which all spheres are open it is easy to see that every convergent sequence has a Cauchy subsequence. Burke has shown that every semi-metric space (X,d) has a compatible semi-metric ρ under which every convergent sequence has a Cauchy subsequence. There are examples which show the analogue is not true for symmetrizable spaces. A semi-metric d for X is a ***K*-semi-metric** if $d(H,K)>0$ for every disjoint pair H,K of nonempty compact subsets of X. It is trivial that every metric is a K-semi-metric and every **submetrizable** (with a coarser metric topology) semi-metric space has a K-semi-metric. However, there is an example of a separable Moore space which does not have a compatible K-semi-metric. This example also does not admit a coarser T_2 ***quasi-metric*** topology.

A Tychonoff space X is a **p-space** [2] if there exists, in the ***Stone–Čech compactification*** βX, a sequence of open collections $\{\gamma_n\}_{n\in\omega}$, each covering X, such that if $x\in X$ then $\bigcap_{n\in\omega}\mathrm{St}(x,\gamma_n)\subseteq X$. X is a **strict p-space** if there exists such a sequence $\{\gamma_n\}_{n\in\omega}$ with the additional property that $\bigcap_{n\in\omega}\mathrm{St}(x,\gamma_n)=\bigcap_{n\in\omega}\overline{\mathrm{St}(x,\gamma_n)}$ for every $x\in X$. It is clear from the definition that T_2 ***locally compact*** spaces and ***Čech-complete*** spaces are always p-spaces. Some questions about the nature of p-spaces or strict p-spaces can be more easily studied by internal characterizations found in [4]. A Tychonoff space X is a p-space if and only if there is a sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of open covers of X such that: If $x\in G_n\in\mathcal{G}_n$, for all $n\in\omega$, then $C_x=\bigcap_{n\in\omega}\overline{G_n}$ is compact and $\{\bigcap_{k\leqslant n}\overline{G_k}\}_{n\in\omega}$ is an ***outer network*** about C_x. A Tychonoff space X is a strict p-space if and only if there is a sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of open covers of X such that: If $x\in X$ then $P_x=\bigcap_{n\in\omega}\mathrm{St}(x,G_n)$ is compact and $\{\mathrm{St}(x,G_n)\}_{n\in\omega}$ is a local base about P_x. Since these characterizations do not require a Hausdorff compactification for X it is tempting to talk about p-spaces in the context of regular spaces rather than requiring Tychonoff.

It is now easy to see that a Moore space is a strict p-space. In the characterization above, the $\{\mathcal{G}_n\}_{n\in\omega}$ could be the development and each $P_x=\{x\}$. Even though the class of p-spaces is much more general than compactness, p-spaces retain some of the properties of compactness or completeness. For example, if the singleton $\{x\}$ is a G_δ-set in a p-space X then x has a countable local base. A p-space is a ***k-space*** and is of ***point-countable type***. The class of p-spaces is closed under the formation of countable products.

There are several classes of topological spaces related to p-spaces. Let us define two such and indicate the relationships. K. Morita has defined an **M-space** to be a Tychonoff space X for which there is a ***normal sequence of open covers*** $\{\mathcal{H}_n\}_{n\in\omega}$ such that if $x\in X$ and $x_n\in\mathrm{St}(x,\mathcal{H}_n)$ then $\{x_n\}_{n\in\omega}$ has a cluster point. (Here, "normal sequence" means that $\mathcal{H}_{n+1}$ is a ***star refinement*** of $\mathcal{H}_n$.) Given such a sequence, let $C_x=\bigcap_{n\in\omega}\mathrm{St}(x,\mathcal{H}_n)$; then it is easily seen that $\{C_x\colon x\in X\}$ is a partition of X into ***countably compact*** subsets. This is the beginning of the proof of the characterization by Morita that the class of M-spaces is exactly the class of ***quasi-perfect*** preimages of metric spaces [KV, Chapter 10]. Any Tychonoff countably compact space is an M-space. Since countably compact, paracompact spaces are compact it is clear that paracompact M-spaces are exactly the perfect preimages of metric spaces. This brings us back to p-spaces since A. Arhangel'skiĭ has shown that paracompact p-spaces are equivalent to the perfect preimages of metrics spaces [2]. That is, X is a paracompact M-space if and only if X is a paracompact p-space. It is important to note here that for an M-space X to be paracompact it is only necessary that all closed countably compact subspaces of X

are compact. There is one more structure characterization of paracompact p-spaces which should be mentioned here. It is known that if $f: X \to Y$ is a ***perfect map*** from a Tychonoff space X then X is homeomorphic to a closed subspace of $\beta X \times Y$. If follows that X is a paracompact p-space if and only if X is homeomorphic to a closed subspace of some product of a compact space Z with a metric space Y. Either of these last two characterizations makes it easy to see one of the important properties of the class of paracompact p-spaces. This class is closed under the formation of countable products. M-spaces are ***Morita P-spaces*** [MN, Chapter 3] – the product $X \times Y$ of a normal M-space X and a metric space Y is normal. The class of M-spaces is not closed under the formation of finite products. See [MN, Chapter 9] for an example of two countably compact spaces A, B where $A \times B$ is not even a $w\Delta$-space. If X is a closed subspace of a product $Z \times Y$ of a countably compact space Z with a metric space Y then X is an M-space. (The restriction of the projection map $\pi_2|_X : X \to Y$ is a ***quasi-perfect map***.) However, the converse is not true. There are examples (see [KV, Chapter 10] for references) of M-spaces which cannot be expressed as a closed subspace of a product of a countably compact space with a metric space.

A space X is said to be a $w\Delta$**-space** [KV, Chapter 10] if X has a sequence of open covers $\{\mathcal{H}_n\}_{n\in\omega}$ such that if $x_n \in \mathrm{St}(x, \mathcal{H}_n)$, for all $n \in \omega$, then $\{x_n\}_{n\in\omega}$ has a cluster point. Certainly, paracompact $w\Delta$-spaces are equivalent to paracompact M-spaces but the usefulness of $w\Delta$-spaces comes about with conditions weaker than paracompactness. Notice that countably compact spaces and strict p-spaces are $w\Delta$-spaces but locally compact spaces (hence p-spaces) need not be $w\Delta$-spaces and countably compact spaces (hence $w\Delta$-spaces) need not be p-spaces. Examples are given in [4]. A Tychonoff $w\Delta$-space, in which every closed countably compact subset is compact, is a p-space. In the class of Tychonoff submetacompact spaces, the conditions of being a p-space, a strict p-space or a $w\Delta$-space are all equivalent.

Properties similar to the p-space property and to the symmetrizable property seem to generalize Moore spaces in two different directions. Of course, this is useful for giving metrizability theorems or theorems for characterizing developability. The following (due to Burke, Creede, and Heath) can be found in [KV, Chapter 10]: If X is a Tychonoff space then X is a Moore space $\iff$ X is a p-space with a σ-discrete network $\iff$ X is a semi-metrizable p-space $\iff$ X is a symmetrizable p-space. This result is valid with "$w\Delta$-space" substituted for "p-space". If a paracompact condition (or collectionwise normality) is added then you have a metrizability theorem.

A space X is said to have a G_δ**-diagonal** if the diagonal in $X \times X$ is a G_δ-set. This is equivalent to the existence of a sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of open covers of X such that, for any $x \in X$, $\bigcap_{n\in\omega} \mathrm{St}(x, G_n) = \{x\}$. The result in the previous paragraph can be generalized as: If X is a Tychonoff space then X is a Moore space $\iff$ X is a submetacompact p-space with a G_δ-diagonal $\iff$ X is a submetacompact $w\Delta$-space with a G_δ-diagonal [KV, Chapter 10]. The submetacompact condition cannot be dropped from this result. There is an example of a T_2 locally compact space with a G_δ-diagonal which is not developable and under CH there is an example of a $w\Delta$-space with a G_δ-diagonal which is not developable [1]. J. Chaber has shown that countably compact spaces with a G_δ-diagonal are compact [KV, Chapter 10], so we see that M-spaces with a G_δ-diagonal are paracompact. This says that a space X is metrizable if and only if X is an M-space with a G_δ-diagonal.

A comparison of the notions of a development and a strict p-space sequence (given earlier) led to the natural question of whether a strict p-space with a G_δ-diagonal must be developable. This became known as the "Strict p-space Problem" and was eventually solved by S. Jiang [8]. He actually characterizes the class of strict p-spaces as the class of submetacompact p-spaces. This also solved the question of whether strict p-spaces are preserved under a perfect map as it was previously shown by J. Worrell and later by J. Chaber that submetacompact p-spaces are preserved under a perfect map. The class of p-spaces is not preserved under a perfect map. See the article on "Perfect Maps" by D. Burke in this volume for references.

An old result of Miščenko says that in countably compact spaces, a ***point-countable base*** is actually countable (see [KV, Chapter 10]). Hence a countably compact space with a point-countable base is compact and metrizable. If X is locally compact with a point-countable base then X is also metrizable. In this case, X can be expressed as a disjoint union of open subspaces, each with a countable base – hence, X would be metrizable. V.V. Filippov has shown that a paracompact p-space with a point-countable base is metrizable. It now follows that a space X is an M-space with a point-countable base if and only if X is metrizable. A Moore space does not necessarily have a point countable base but this base condition can be used to help give a development. If X does have a point-countable base then X is developable if and only if X is a submetacompact p-space (or submetacompact $w\Delta$-space). The submetacompact condition cannot be dropped from the p-space portion as S.W. Davis has given an example of a Čech-complete space with a point-countable base which is not developable. It is not known whether a $w\Delta$-space with a point countable base must be developable. The results above are all valid with the weaker condition of "point-countable T_1-separating open cover" substituted for "point-countable base". However, a Moore space need not have a point-countable T_1-separating open cover.

A **quasi-development** for X is a sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of open collections (not necessarily covers) such that the collection $\{\mathrm{St}(x, \mathcal{G}_n)\colon n \in \omega \wedge x \in \bigcup \mathcal{G}_n\}$ is a local base at x. It is straightforward to show that a space X is developable if and only if X is perfect (all closed sets G_δ) and has a quasi-development. Quasi-developable spaces are equivalent to spaces with a θ-base. A θ**-base** for X is an open collection $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ such that if $x \in W \subseteq X$, with W open, then there is some $m \in \omega$ such that $\mathrm{ord}(x, \mathcal{B}_m) = |\{B \in \mathcal{B}_n\colon x \in B\}| < \omega$ and there exists $B \in \mathcal{B}_m$ with $x \in B \subseteq W$. In this definition, if one substitutes "$\mathrm{ord}(x, \mathcal{B}_m) \leqslant \omega$" for "ord$(x, \mathcal{B}_m) < \omega$" then we say $\mathcal{B}$ is a **$\delta\theta$-base**. A θ-base or $\delta\theta$-base is a weakening of the point-countable base condition and every Moore space has a θ-base. The results in the previous paragraph are all valid with "$\delta\theta$-base" substituted for "point-countable base".

Another interesting class of generalized metric spaces is **MOBI**, defined by Arhangel'skiĭ [2] as the smallest class of spaces that contains all metric spaces and is closed under open compact images, where a map is called a **compact map** if its every ***fiber*** is compact. The class MOBI contains all metacompact Moore speces and every space in MOBI has a point-countable base. Although a complete characterization of the spaces in MOBI is open, various interesting results have been obtained. See G. Gruenhage's article in [HvM].

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e-13 Generalized Metric Spaces III: Linearly Stratifiable Spaces and Analogous Classes of Spaces

This article is concerned with generalizations of concepts like stratifiability and metrizability to arbitrary infinite cardinalities, in a way that uses linear orders in key places. This has resulted in theories which are remarkably faithful generalizations of the theories of stratifiable, metrizable, etc. spaces. For metrizable spaces, the generalization is to the class of (***Tychonoff***) spaces admitting separated uniformities with totally ordered bases; this class is usually referred to as the class of ω_μ-metrizable spaces of arbitrary cardinality ω_μ, but the term "linearly uniformizable spaces" will be mostly used here, under the convention that "spaces" refers to ***Hausdorff*** spaces. The class of linearly stratifiable spaces is a simultaneous generalization of linearly uniformizable spaces and of stratifiable spaces, and most of the theory of stratifiable spaces carries over, including the basic covering and separation properties of ***paracompactness*** and ***monotone normality***. There are generalizations, along the same lines, of σ-spaces and semistratifiable spaces, as well as classes in between the linearly uniformizable spaces and linearly stratifiable spaces, generalizing M_1 spaces and Nagata spaces. Other generalizations, such as the one of **quasi-metrizable** spaces (quasi-metrics are defined like metrics but without symmetry of the distance function), are less well developed in the literature, and will only be touched on here.

The usual definition of linear stratifiability is based on the definition of stratifiable spaces that says they are monotonically perfectly normal, so to speak; this definition is the case $\omega_\mu = \omega$ of the definition of ω_μ-stratifiable spaces, where ω_μ is an infinite cardinal number. A space (X, τ) is said to be **stratifiable over** ω_μ if it is a T_1 space for which there is a map $S : \omega_\mu \times \tau \to \tau$, called an ω_μ**-stratification** which satisfies the following conditions.

(1) $c\ell(S(\beta, U)) \subset U$ for all $\beta < \omega_\mu$ and all $U \in \tau$.
(2) $\bigcup\{S(\beta, U)\colon \beta < \omega_\mu\} = U$ for all $U \in \tau$.
(3) If $U \subset W$, then $S(\beta, U) \subset S(\beta, W)$ for all $\beta < \omega_\mu$.
(4) If $\gamma < \beta < \omega_\mu$, then $S(\gamma, U) \subset S(\beta, U)$ for all $U \in \tau$.

X is called ω_μ**-stratifiable** if ω_μ is the least cardinal for which X is stratifiable over ω_μ. A space is **linearly stratifiable** if it is ω_μ-stratifiable for some infinite ω_μ, and ***stratifiable*** if it is ω-stratifiable. An ω-stratification is called a **stratification**. If condition (1) is omitted, we get the definition of an ω_μ**-semistratification**. The terms **semistratifiable over** ω_μ, ω_μ**-semistratifiable**, **linearly semistratifiable**, **semistratifiable**, and **semistratification** have the obvious definitions. The key theorem that a space is stratifiable iff it is semistratifiable and monotone normal generalizes easily to arbitrary ω_μ. Condition (4) is unnecessary in the case $\omega_\mu = \omega$ but it is needed to make the theories of stratifiable and semistratifiable spaces generalize to higher cardinals. Similar additions make it possible to generalize two characterizations of (semi-)stratifiable spaces and to make them coincide. One is a pair of Heath–Hodel style characterizations in [27] and [17] with their addition of condition (b), which is unnecessary in case $\omega_\mu = \omega$: A T_1-space (X, τ) is stratifiable over ω_μ if, and only if, there exists a family $\{g_\beta\colon \beta < \omega_\mu\}$ of functions with domain X and range τ such the following hold:

(a) $x \in g_\beta(x)$ for all $\beta < \omega_\mu$;
(b) if $\beta < \gamma < \omega_\mu$, then $g_\beta(x) \supset g_\gamma(x)$ for all x;
(c) if, for every $\beta < \omega_\mu$, $x \in g_\beta(x_\beta)$, then the net $\langle x_\beta\colon \beta < \omega_\mu\rangle$ coverges to x; and
(d) for every $F \subset X$, if $y \in c\ell(\bigcup\{g_\beta\colon x \in F\})$ for all $\beta < \omega_\mu$, then $y \in c\ell(F)$.

If condition (d) is omitted, we get a condition equivalent to being semistratifiable over ω_μ.

In [27] there is also a definition of a linearly cushioned pair-base that generalizes that of a σ-cushioned pair-base used in defining M_3 spaces; moreover, the proof that the M_3 concept coincides with stratifiability generalizes in [27] to this more general setting. A collection $\mathcal{P}$ of pairs $P = (P_1, P_2)$ of subsets of a space (X, τ) is said to be a **pair-base** if the members of each pair are open and, for each point x of X and each neighbourhood U of x, there is a pair $(P_1, P_2) \in \mathcal{P}$ such that $x \in P_1$ and $P_2 \subset U$. A collection $\mathcal{C}$ of subsets of a space X is **linearly closure-preserving with respect to** $\leqslant$ if $\leqslant$ is a linear order on $\mathcal{C}$ such that $\bigcup\{c\ell C\colon C \in \mathcal{C}'\} = c\ell(\bigcup \mathcal{C}')$ for any subcollection of $\mathcal{C}' \subset \mathcal{C}$ which has an upper bound w.r.t. $\leqslant$. A collection of pairs $P = (P_1, P_2)$ is **linearly cushioned** with respect to a linear order $\leqslant$ if $c\ell(\bigcup\{P_1\colon P = (P_1, P_2) \in \mathcal{P}'\}) \subset \bigcup\{P_2\colon P = (P_1, P_2) \in \mathcal{P}'\}$ for every subset $\mathcal{P}'$ of $\mathcal{P}$ which has an upper bound with respect to $\leqslant$. Hence in particular, $\mathcal{C}$ is linearly closure-preserving w.r.t. $\leqslant$ if $\{(C, C)\colon C \in \mathcal{C}\}$ is linearly cushioned with respect to $\leqslant$. A regular space X is said to be M_1 **over** ω_μ (respectively M_2 **over** ω_μ) (respectively M_3 **over** ω_μ) if X has a linearly closure-preserving base (respectively a linearly closure-preserving ***quasi-base***) (respectively a linearly cushioned pair-base) with a cofinal set of order type ω_μ. X is **linearly** M_i if it is M_i over ω_μ for some infinite cardinal ω_μ. An ω_μ-M_i space is defined analogously to an ω_μ-stratifiable space.

Clearly, these concepts are numbered in order of increasing generality. More general yet is the concept of having a linearly closure-preserving network of cofinality ω_μ, consisting of closed sets. If $\omega_\mu = \omega$ this gives us the familiar class of σ-spaces. Harris [11], generalizing the Nagata–Siwiec theorem for $\omega_\mu = \omega$, showed that these spaces have a network that is the union of $\leqslant \omega_\mu$ discrete collections. The converse is true if the space is **ω_μ-additive**, meaning that the union of strictly fewer than ω_μ closed sets is closed: this implies that the union of fewer than ω_μ discrete collections is discrete, hence every union of ω_μ discrete collections is linearly closure-preserving with respect to a linear order of cofinality $\mathrm{cf}(\omega_\mu)$. The Heath–Hodel theorem that every stratifiable space is a σ-space [13] generalizes to the theorem that every ω_μ-stratifiable space has a network which is the union of $\leqslant \omega_\mu$ discrete collections, and a linearly closure-preserving network [27]. The theorem that σ-spaces are semistratifiable generalizes to the theorem that a space with a linearly closure-preserving network is linearly semistratifiable [11]. In fact, having a linearly closure-preserving network of cofinality ω_μ consisting of closed sets is equivalent to having a Heath–Hodel function g satisfying (a), (b), and (c) above along with the following condition (e): if $y \in g_\beta(x)$ then $g_\beta(y) \subset g_\beta(x)$. For (c) it is possible to substitute the stronger (c+): if, for every $\beta < \omega_\mu$, $x \in g_\beta(y_\beta)$ and $y_\beta \in g_\beta(x_\beta)$, then the net $\langle x_\beta \colon \beta < \omega_\mu \rangle$ converges to x [11]. Another generalization, that of **elastic spaces**, relaxes the linear order requirement to that of a preorder, but otherwise keeps the pair-base definition of linearly M_3 with the formal restriction that the pair-base is a function; that is, each subset of the space appears as the first element in at most one pair. M. Jeanne Harris showed that this restriction is a mere formality in [11] and [12]: every space with a linearly cushioned pair-base has one which is a function.

Linearly stratifiable spaces enjoy many of the nice properties of the subclass of stratifiable spaces; for example, they are ***monotonically normal*** and (hereditarily) ***paracompact***. There is a subtle hole in the proof of the latter fact in [26] and [27], which is repaired by Harris's theorem. It is also possible to show, more simply, that every open cover in a linearly stratifiable space has an open refinement which is linearly cushioned in it [28]. This refinement condition is equivalent to paracompactness, and "linearly cushioned" can be weakened to "elastic" [26]. Linearly stratifiable spaces have most of the nice preservation properties possessed by stratifiable spaces. For example, the class is closed under the taking of subspaces and closed images, and finite unions of closed subspaces. This also applies to the class of linearly M_2-spaces. The best known of the (much weaker) known preservation properties of M_1 spaces also carries over: if f is a closed irreducible continuous map from a space X that is M_1 over ω_μ, onto a space Y such that for every $y \in Y$, $f^{-1}(y)$ is ω_μ-compact, then Y is linearly M_1 [11]. Finite products of spaces that are ω_μ-stratifiable over the same ω_μ are also ω_μ-stratifiable, as are box products of fewer than ω_μ of them. Both of these results are generalized by the fact that if ω_μ is regular, then the ω_μ-box product of ω_μ or fewer ω_μ-stratifiable spaces is ω_μ-stratifiable: the ω_μ-box product is defined like the box product except that one restricts fewer than ω_μ-many coordinates [3]. (The restriction on agreement in ω_μ is important: $\omega + 1$ and the one-point Lindelöfization of a discrete space of cardinality ω_1 constitute a pair of spaces, one stratifiable and the other ω_1-stratifiable, whose product is not linearly stratifiable – it is not even hereditarily normal.) If a space X is dominated by a collection of closed subsets, each of which is stratifiable over ω_μ, then X is stratifiable over ω_μ. If X and Y are stratifiable over ω_μ and A is a closed subset of X and $f : A \to Y$ is continuous, then $X \cup_f Y$ (the adjunction space) is stratifiable over ω_μ [27].

The celebrated Gruenhage–Junnila theorem that all M_3 spaces are M_2 has been generalized within the class of **ω_μ-additive space**s (also known as **P_{ω_μ}-space**s); that is, spaces in which the intersection of strictly fewer than ω_μ open sets is open. The theorem is that every P_{ω_μ} space which is ω_μ-M_3 is also ω_μ-M_2. The problem of whether the P_{ω_μ} condition can be dropped is still open. The notorious problem of whether all three classes are the same also generalizes to linearly M_i spaces; in fact, it is open for all infinite cardinalities ω_μ, even for P_{ω_μ}-spaces. Moreover, where uncountable ω_μ are concerned, we even have a fourth class, the class of spaces M_0 over ω_μ, to add to this coincidence problem. Spaces that are M_0 over ω_μ are defined like spaces M_1 over ω_μ but with "open" replaced by "clopen". That is, a space is **M_0 over** ω_μ if it is a regular space with a linearly closure-preserving base $\mathcal{B}$ of clopen sets, where the linear order on $\mathcal{B}$ has cofinality ω_μ. As might be expected, **linearly** M_0" and "**ω_μ-M_0**" are defined analogously to the same concepts for higher subscripts.

A big advantage of linearly M_0-spaces over the more general linearly M_1-spaces is that they are easily seen to be hereditary; their perfect images are linearly M_1 [11], but not necessarily linearly M_0, at least not when the domain is simply M_0: the closed unit interval is a non-M_0 perfect image of the Cantor set, which is clearly M_0, as is any **strongly zero-dimensional** metrizable space. The strongly zero-dimensional spaces can be characterized as those Tychonoff spaces in which disjoint zero sets can be put into disjoint clopen sets [6, 16.17], [E, 6.2.4] or those which have totally disconnected Stone–Čech compactifications [E, 6.2.12]. All ω_μ-M_0-spaces are strongly zero-dimensional, even in the case $\omega_\mu = \omega$ [14]. Also, every Tychonoff space which is a ***P-space*** [that is, a P_{ω_1}-space] is strongly zero-dimensional; indeed, every zero set is clopen in such spaces since it is a G_δ-set. Remarkably enough, it is not known whether every strongly zero-dimensional ω_μ-stratifiable space is ω_μ-M_0, whatever the value of ω_μ; nor whether every ω_μ-stratifiable space (or every space stratifiable over ω_μ) is strongly zero-dimensional when ω_μ is uncountable. Since stratifiability over ω_μ is preserved on collapsing a closed set to a point, the latter problem is equivalent to whether all ω_μ-stratifiable spaces (or all spaces stratifiable over ω_μ) are **zero-dimensional**, i.e., have a base consisting sets that are both open and closed.

Various well-known equivalences of the M_2-M_1 problem also carry over, some with the addition of ω_μ-additivity. Two generalizations by Harris [11] of a well-known theorem of Heath and Junnila [14] account for several of them, including the problems of whether every closed subspace, or every closed image of an M_1 space is M_1. One generalization says that every linearly M_2-space is the image of a linearly M_1-space under a retraction. The other says that if ω_μ is regular, and if the P_{ω_μ}-space X is stratifiable over ω_μ, then X is the image of a linearly M_1 space under a closed retraction with ω_μ-compact fibers. Some quite general classes of linearly stratifiable spaces are linearly M_1. For instance, if ω_μ is a regular cardinal and X is an ω_μ-stratifiable P_{ω_μ}-space in which every closed subset of X has a linearly closure-preserving neighbourhood base of open sets in which ω_μ is cofinal, then X is linearly M_1 [11]. The condition that X is ω_μ-stratifiable can be formally relaxed to the condition that X is paracompact and has a network which is the union of $\leqslant \omega_\mu$ discrete collections [11]. This generalizes an old result [2] for the case $\omega_\mu = \omega$, while the following generalizes one of Ito [16]: if X is a P_{ω_μ}-space that is M_3 over ω_μ, and every point of X has a closure-preserving open base, then every closed subset of X has a closure-preserving base of open sets [11] (and hence X is linearly M_1).

An important class of linearly stratifiable spaces might be called **linearly Nagata**: these are the ω_μ-Nagata spaces as ω_μ varies over all infinite regular cardinals. The ω_μ-Nagata spaces can be simply characterized as the ω_μ-stratifiable spaces in which each point has a totally ordered neighbourhood base. Of necessity, this base will have cofinality ω_μ if the point is nonisolated. By the foregoing theorems, and the elementary fact that every ω_μ-Nagata space is a P_{ω_μ}-space, it follows every linearly Nagata space is linearly M_1. There are other characterizations of ω_μ-Nagata spaces, including one based on the Nagata general metrization theorem [10]: an ω_μ-Nagata space is a T_1 space with a system $\langle \mathfrak{U}, \mathfrak{S} \rangle$ where $\mathfrak{U}$ and $\mathfrak{S}$ are collections of functions U_β and S_β ($\beta < \omega_\mu$), each with domain X, and such that (1) for each $x \in X$, $\{U_\beta(x): \beta < \omega_\mu\}$ is a base for the neighbourhoods of x, and so is $\{S_\beta(x): \beta < \omega_\mu\}$; (2) for every $x, y \in X$, $S_\beta(x) \cap S_\beta(y) \neq \emptyset$ implies that $x \in U_\beta(y)$; and (3) If $\beta < \gamma < \omega_\mu$, then $S_\beta(x) \supset S_\gamma(x)$ for all x. As usual, (3) is superfluous if $\omega_\mu = \omega$, and we simply have the class of ***Nagata spaces*** then. Another characterization [27] dispenses with $\mathfrak{U}$, requires that each $S_\beta(x)$ be open, and substitutes for (2) the condition that if U is a neighbourhood of x, there exists $\beta < \omega_\mu$ such that $S_\beta(x) \cap S_\beta(y) \neq \emptyset$ implies that $y \in U$. Clearly, any subspace of an ω_μ-Nagata space is ω_μ-Nagata, and any ω_μ-box product of ω_μ-Nagata spaces over the same ω_μ is again ω_μ-Nagata. The closed continuous image X of an ω_μ-Nagata space is likewise an ω_μ-Nagata space provided that, for each point $x \in X$, there exists a totally ordered neighbourhood base. If X is ω_μ-Nagata over an uncountable regular ω_μ, then X a P_{ω_μ}-space and hence is strongly zero-dimensional. As is well known, a space X satisfies $\dim(X) = 0$ iff X is normal and strongly zero-dimensional, and X is **ultraparacompact** iff it is paracompact and strongly zero-dimensional. Since linearly stratifiable spaces are paracompact and hence normal, the ω_μ-Nagata spaces have both of these other properties if ω_μ is uncountable. (And so too, of course, do all linearly M_0 spaces and all linearly stratifiable P-spaces.) This gives the theory of these kinds of linearly Nagata a different flavor from that of Nagata spaces (the countable case $\omega_\mu = \omega$).

An easy example of a space that is M_0 over a regular cardinal ω_μ and is ω-M_0 at the same time is obtained by isolating all but the last point of $\omega_\mu + 1$, taking the product of the resulting space with $\omega + 1$, and removing every nonisolated point except $\langle \omega, \omega_\mu \rangle$. The set of all open sets containing this point is a closure-preserving clopen base for the point, and the isolated points can be grouped either horizontally or vertically, with initial segments being clopen in either case. This is also an example a space that is M_0 over ω_μ but is not linearly Nagata. The converse problem, whether an ω_μ-Nagata space is necessarily ω_μ-M_0 if ω_μ is regular uncountable, is unsolved.

Linearly uniformizable spaces have a long history, due to the fact that they can be characterized by distance functions that satisfy the usual definition of a metric, except that the distances are not necessarily real numbers, but rather take on their values in an ordered Abelian group (often the additive group of an ordered field). Hausdorff [8, p. 285] introduced the use of such distance functions to general topology, and it was shown that a space is linearly uniformizable iff it admits such a generalized metric. Important examples of such generalized metrics are valuations, which play an important role in algebraic number theory [24]. Many well-known metrization theorems have generalizations that say when a space is linearly uniformizable: The Urysohn Metrization Theorem [23]; the Nagata–Smirnov Theorem [29]; Frink's Metrization Theorem, Bing's Metrization Theorem, Nagata's Generalized Metrization Theorem (the one on which the definition of a Nagata space is based) and several others [20]. The Morita–Hanai–Stone Theorem generalizes to the theorem that a closed map from a ω_μ-metrizable space to another space has ω_μ-metrizable image iff the boundary of each point-inverse is ω_μ-compact [20].

Linearly uniformizable spaces with bases of uncountable cofinality (in other words, ω_μ-metrizable, nonmetrizable spaces) are both linearly Nagata and linearly M_0. In a uniform space, the intersection of every descending sequence of entourgages with no last element is an equivalence relation. Hence, any uniform space with a linearly ordered base of uncountable cofinality has a (linearly ordered) base of equivalence relations; these partition the space into clopen sets. Well-ordering the members of the partitions, with members of coarser partitions preceding the members of the finer partitions, gives a linearly closure-preserving base of clopen sets – the linearly M_0 property. Bases like these are well suited for showing that ω_μ-box product of ω_μ-many ω_μ-metrizable spaces is ω_μ-metrizable and that a space is ω_μ-metrizable for uncountable regular ω_μ iff it embeds in a ω_μ-box product of ω_μ-many discrete spaces. Monotone normality and ultraparacompactness of linearly uniformizable nonmetrizable spaces follow easily from the fact that the base given by these partitions is a tree by reverse inclusion. For ultraparacompactness, the $\supset$-minimal members of a tree base $\mathcal{B}$ which can be put in some member of the open cover $\mathcal{U}$ constitute a partition into clopen sets refining $\mathcal{U}$. For a point x and an open set U containing x, one can let U_x be any member B whatsoever of $\mathcal{B}$ that satisfies $x \in B \subset U$, and then the Borges definition of monotone normality follows from the fact that if U_x meets V_y, then either $U_x \subset V_y$ or $V_y \subset U_x$. Indeed, every tree base for a space is a **base of rank** 1, which means that any two members are either disjoint or related by $\subset$. Spaces with rank 1 bases are called **non-Archimedean spaces**, and actually coincide with spaces with tree bases [19]. The natural common generalization of non-Archimedean and metrizable spaces is that of **proto-metrizable** spaces. These are the spaces with rank 1 pair-bases [7]; $\mathcal{P}$ is a **pair-base of rank** 1 if whenever $\langle P_1, P_2\rangle$ and $\langle P_1', P_2'\rangle$ are in $\mathcal{P}$ and $P_1 \cap P_1' \neq \emptyset$, then either $P_1 \subset P_2'$ or $P_1' \subset P_2$. These spaces share many of the nice properties common to metrizable and non-Archimedean spaces, including paracompactness and monotone normality.

Non-Archimedean spaces are suborderable but not all orderable – the Michael line is a standard example [15, 21] of a non-orderable non-Archimedean space. There even exist examples of non-orderable ω_μ-metrizable spaces for all uncountable cofinality ω_μ. This is in contrast to the case of strongly zero-dimensional metrizable spaces (the cofinality $= \omega$ case), all of which are linearly orderable. In fact, a space is metrizable and strongly zero-dimensional iff it is metrizable, linearly orderable, and totally disconnected [9]. Another characterization is that these are the spaces that can be given a compatible **non-Archimedean metric**, one that satisfies the **strong triangle inequality**: given any three points x, y, z, one has $d(x,z) \leqslant \max\{d(x,y), d(x,z)\}$ [5]. If ω_μ is uncountable regular, then every ω_μ-metrizable space can be given a distance function satisfying this property, with values an ordered Abelian group.

There are a few aspects of the theory of metrizable spaces that do not carry over to linearly uniformizable spaces without modification. One is that, for a ω_μ-metric space to be ω_μ-compact (meaning: every open cover has a subcover of cardinality $< \omega_\mu$) it is not enough for it to be complete and totally bounded. For completeness one must substitute the stronger concept of supercompleteness [1]; the two concepts coincide for metric spaces. Sometimes one must use extra qualities of the cardinal ω to have a really satisfactory extension of some classical result. For example, the elementary fact that ${}^{\omega}2$ with the product topology is compact only generalizes to ***weakly compact cardinal***s ω_μ in place of ω when the ω_μ-box product topology is used. Classical characterizations of the Cantor set (the only totally disconnected, compact, dense-in-itself metrizable space) and the irrationals (the only zero-dimensional, ***nowhere locally compact***, ***completely metrizable***, ***separable*** space) only generalize for weakly compact cardinals and ***strongly inaccessible cardinal***s, respectively [19], and one must substitute spherical completeness for ordinary completeness.

In principle, almost every "generalized metric" property can be effectively generalized with judicious uses of total orderings. Sometimes, as with metrizable and non-Archimedeanly metrizable spaces, two or more distinct classes coalesce for uncountable regular ω_μ. One such example is that of quasi-metrizable and ***non-Archimedeanly quasi-metrizable*** spaces [22]. The argument in [22] can be easily modified to show that the uncountable analogues of ***γ-spaces*** also coincide with those of quasi-metrizable spaces.

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e-14 Monotone Normality

1. Introduction

A topological space X is **monotonically normal** provided X is T_1 and, for all open U in X and $x \in U$, there is an $H(x, U)$ such that:

(a) (normality) $H(x, U) \cap H(y, V) \neq \emptyset$ implies either $x \in V$ or $y \in U$, and
(b) (monotonicity) $U \subseteq W$ implies $H(x, U) \subseteq H(x, W)$.

The function H is called a **monotone normality operator** for X. In some sense the property is local, but it has global implications. It was used, unnamed by C.J.R. Borges in 1966; named by P.L. Zenor in 1971; and in 1973 we have an excellent analysis of it by its founders in [4].

Metric spaces (even ***stratifiable space***s) and linearly ordered spaces (as well as their subspaces) are all monotonically normal and their monotone normality is precisely the property standardly used to prove that these spaces are normal. Monotone normality is preserved in subspaces and closed continuous images and such spaces are not only Hausdorff, regular, and normal but also collectionwise normal and these, and other properties are, of course, hereditary for this class. Even these elementary facts kill many of the standard pathologies we study in non metrizable spaces.

During the following fifteen years there were many papers published proving theorems about monotone normality, making conjectures, asking questions, studying various strengthenings. Besides Borges, Heath, Lutzer and Zenor who continued to contribute some of the authors include S. Purisch, G. Gruenhage, P. Nyikos, J. Vaughan, A. Ostaszewski, D. Palenz, E. van Douwen, K.P. Hart, N. Kemoto, H. Tamano. Quite a number of the questions raised involved the refining of open covers and were answered in [1] by the following theorem which has continued to be useful in understanding the structure of monotonically normal spaces.

THEOREM 1. *Every open cover $\mathcal{U}$ of a monotonically normal space X has a σ-disjoint open partial refinement $\mathcal{V}$ such that $X \setminus \bigcup \mathcal{V}$ is the union of a discrete family of closed subspaces, each homeomorphic to some stationary subset of a regular uncountable cardinal.*

This theorem implies for instance that a monotonically normal space, like a linearly ordered one, fails to be paracompact if and only if it has a closed subset homeomorphic to a closed subset of a regular uncountable cardinal. Also every open cover can be shrunk and thus the space cannot be Dowker or κ-Dowker for any cardinal κ.

In the late 1980s and early 1990s several very active groups published a series of papers contributing to the understanding of monotone normality from different standpoints. A very excellent survey of the situation in 1992 can be found in section 3 of [3].

A group at Oxford in England, faculty members P.J. Collins, G.M. Reed and A.W. Roscoe, and especially students P. Moody, P. Gartside, I.S. Stares, R. Knight and D. McIntyre studied various generalizations of metrizability, in particular point-networks, which led them to monotone normality: metric spaces are both stratifiable and protometrizable. Both stratifiable and protometrizable spaces are elastic. Elastic spaces have a chain point-network. A space has a chain point-network if and only if the space is **acyclically monotonically normal**. By definition, a space is acyclically monotonically normal if it is T_1 and has a monotone normality operator without any cycles. In his thesis and elsewhere Phil Moody studied acyclic monotone normality in depth [5]. For instance, acyclically normal spaces are proved to be Kuratowski K_0. In [8] an example is given of a monotonically normal, non-K_0 (so non-acyclic) space. This example is Kuratowski K_1, separable, zero-dimensional, and quite simple; but every one of it monotone normality operators has a 3-cycle and it has no monotonically normal compactification. This raised the question of a *compact* nonacyclic monotonically normal space. Every closed continuous image of an acyclically monotonically normal spaces (for instance, a continuous image of a compact linearly ordered space) is acyclically monotonically normal [5].

In 1986 J. Nikiel [6] conjectured that a compact space is the continuous image of a compact, linearly ordered space if and only if it is monotonically normal.

Hao Zhou and Scott Williams, sometimes with the aid of Steve Purisch, wrote a series of papers in the late 1980s and 1990s analyzing the structure of monotonically spaces with special attention to compact ones. An especially illuminating construction, described in [10], is that of a tree T of open sets in the space with each level disjoint, closing down on an associated dense set D. (T is called a **Williams–Zhou tree** by Gartside in [2].) Actually it is a scheme for constructing many such trees so that one can build in desired properties in special trees and it is particularly helpful in closing in on the various bounds for cardinal functions of monotonically normal spaces. A number of consequences of [1] were arrived at independently and often more elegantly by theorems of Williams and Zhou (see [3]) and the scheme M.E. Rudin used to prove Nikiel's conjecture is indeed a modified Williams–Zhou type scheme.

The cardinal functions of a monotonically normal space X are now very well understood and are analyzed by Paul Gartside in [2]. Gartside gives the older results of Ostaszewski, Moody, Williams and Zhou, as well as his own more complex theorems for which Williams–Zhou tree was his principal topological tool. For instance, Ostaszewski's cellularity theorem proves X is not an S-space and the Williams–Zhou bound on the density proves X cannot be an L-space unless there is a Souslin line. But Gartside gives other bounds on the density, cellularity, tightness, extent, and many other cardinal functions and their hereditary properties using relationships between calibers as his set theoretic tool.

Another group of people coming from Continua Theory, concerned with monotone normality because they wished to generalize the Hahn–Mazurkiewicz theorem, include particularly J. Nikiel, B. Treybig, M. Tuncali, E. Tymchatyn and Treybig's student D. Daniel. They sought a characterization of those spaces which are the continuous images of compact *connected* linearly ordered spaces. Nikiel's conjecture implies that these spaces are precisely the compact, connected, locally connected, monotonically normal spaces, and it is these spaces which they primarily studied. The direction of many of these papers can be seen in [7] and Dale Daniel's 1998 thesis is titled *Concerning the Hahn–Mazurkiewicz theorem in monotonically normal spaces*, surveying the area.

After publishing a series partial results, in 1999 Rudin succeeded in proving Nikiel's conjecture [9]. Since continuous images of compact, linearly ordered spaces are relatively nice and easy to work with this theorem which basically says that a compact monotonically normal space has few pathologies. However, when not compact, monotonically normal spaces can exhibit many pathologies although not the ones we have so frequently studied in the past.

Presented here are only a small fraction of the work done on monotone normality. It comes up in many connections and we have not tried to present isolated results which were not part of some larger effort. Few of the stronger versions of monotone normality are mentioned here. Nor have monotonically normal topological groups been mentioned. One sees here only a limited view.

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e-15 Probabilistic Metric Spaces

The idea of a Probabilistic Metric Space (briefly PM space) was introduced by Menger in [8]. It generalizes that of a ***metric space***: a distribution function $F_{p,q}$ is associated with every pair of points p and q of a non-empty set S, rather than a non-negative number. By definition, a **distribution function** F maps the extended reals $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ into the closed interval $[0, 1]$, is left-continuous at every real point, non-decreasing and satisfies $F(-\infty) = 0$ and $F(+\infty) = 1$. The set of all distribution functions is denoted by Δ. The value $F_{p,q}(x)$ for $x \geqslant 0$ may then be interpreted as the probability that the distance between p and q is (strictly) less than x. Therefore only **distance distribution functions** are considered in the theory: they form the subset $\Delta_+ \subset \Delta$ of those distribution functions that satisfy $F(0) = 0$, $\Delta_+ := \{F \in \Delta\colon F(0) = 0\}$. The set Δ_+ can be partially ordered by the usual pointwise order, viz. $F \leqslant G$ if, and only if, $F(x) \leqslant G(x)$ for every x. The maximal and the minimal elements in Δ_+ are then the distribution functions ε_0 and ε_∞, where, for a in $\overline{\mathbb{R}}_+ := [0, +\infty]$, ε_a is the distribution function defined by

$$\varepsilon_a(x) := \begin{cases} 0, & x \leqslant a, \\ 1, & x > a, \end{cases}$$

if $a < +\infty$ and, if $a = +\infty$, by $\varepsilon_\infty(x) = 0$, for $x < +\infty$, and $\varepsilon_\infty(+\infty) = 1$.

A topology can be introduced in the space Δ of distribution functions through **Sibley's metric**, a modification of the classical Lévy metric. For $h > 0$, let $[F, G; h]$ denote the condition

$$\forall x \in \left]-\frac{1}{h}, \frac{1}{h}\right[, \quad F(x-h) - h \leqslant G(x) \leqslant F(x+h) + h.$$

Sibley's metric is defined by

$$d_S(F, G) := \inf\bigl\{h > 0\colon [F, G; h] \text{ and } [G, F; h] \text{ hold}\bigr\}, \tag{1}$$

it metrizes the **topology of weak convergence** of distribution functions, an essential concept in probability theory. The space (Δ, d_S) is compact, and hence complete; and Δ_+ is a closed subspace.

While translating the other properties of a ***metric*** was (and is) straightforward, it took some time before meaningful examples were given [11] and a viable ***triangle inequality*** was defined. For the historical details, as well as for the motivations behind the introduction of PM spaces, the reader should refer to the book by Schweizer and Sklar [12], where all the developments up to the early 80s are collected; thus, whenever an author is quoted without an explicit reference, this can be found in [12]. The book [4] is also useful, especially for the analytical applications, such as fixed points in PM spaces, a subject outside the scope of the present article.

In order to present the definition of a probabilistic metric space as we know it today, we need the notion of "triangle function" introduced by Šerstnev in [10].

DEFINITION 1. A **triangle function** τ is a binary operation on Δ_+ that is commutative, associative, non-decreasing in each of its variables and has ε_0 as identity. A binary operation on Δ that is commutative, associative, non-decreasing and whose restriction to Δ_+ is a triangle function is called a ***multiplication***.

A large class of triangle functions can be constructed through an earlier concept, which we now introduce.

DEFINITION 2. A **triangular norm** (briefly a ***t*-norm**) is a binary operation T on the unit interval $[0, 1]$ that is associative, commutative, non-decreasing in each of its variables and such that $T(x, 1) = x$ for every $x \in [0, 1]$.

The following examples of t-norms have all a probabilistic meaning, $W(x, y) := \max\{0, x + y - 1\}$, $\Pi(x, y) := x\,y$, and $M(x, y) := \min\{x, y\}$.

Along with a t-norm T, its t-conorm T^* defined by

$$T^*(x, y) := 1 - T(1 - x, 1 - y)$$

is also of interest.

If T is a left-continuous t-norm, then the function defined through

$$\tau_T(F, G)(x) := \sup\bigl\{T\bigl(F(u), G(v)\bigr)\colon u + v = x\bigr\} \tag{2}$$

is a triangle function (see [12, Section 7.2]).

Another important triangle function is given by the **convolution** $F * G$ of two distribution functions F and G in Δ_+:

$$(F * G)(x) := \int_{\mathbb{R}} F(x - t)\,\mathrm{d}G(t).$$

DEFINITION 3. A **Probabilistic Metric Space** (PM space) is a triple $(S, \mathcal{F}, \tau)$, where S is a non-empty set, $\mathcal{F}$ is a map (the **probabilistic distance**) from $S \times S$ into Δ_+, τ is a triangle function, and the following hold:

(a) $\mathcal{F}(p, q) = \varepsilon_0$ if, and only if, $p = q$;
(b) for all $p, q \in S$, $\mathcal{F}(p, q) = \mathcal{F}(q, p)$;
(c) for all $p, q, r \in S$, $\mathcal{F}(p, r) \geqslant \tau(\mathcal{F}(p, q), \mathcal{F}(q, r))$.

If $\tau = \tau_T$ as in (2), then $(S, \mathcal{F}, \tau)$ is said to be a **Menger space** under the t-norm T. If $\tau = *$, then $(S, \mathcal{F}, *)$ is said to be a **Wald space**.

The value at $(p, q) \in S \times S$ of the probabilistic distance $\mathcal{F}$ will be denoted by $F_{p,q} := \mathcal{F}(p, q)$. Inequality (c) is the triangle inequality in a PM space. Comparing Definition 3 to the usual definition of a metric space, one sees that the generalization from a metric to a probabilistic metric space has been achieved by (i) replacing the range space $\mathbb{R}_+$ of the metric by Δ_+ and (ii) by replacing the operation of addition in the ordinary triangle inequality by a triangle function τ. This latter point has important consequences for the proofs of the analogues in PM spaces of some results (e.g., completeness) in ordinary metric spaces, because, whereas $(\mathbb{R}, +)$ is a group, (Δ_+, τ) is only a semigroup. Therefore, whenever a proof valid in a metric space uses the group inverse of an element, a different argument is needed in the semigroup (Δ_+, τ) where such an inverse does not exist. Associativity is needed in order to consider polygonal inequalities.

The easiest example of PM space is constructed from a metric space (S, d) and a distribution function G, different from ε_0 and ε_∞. Setting $F_{p,q}(x) := G(x/d(p, q))$ one obtains the so-called **simple space** generated by (S, d) and G; this is a Menger space under the t-norm M.

The important class of E**-spaces** was introduced by Sherwood.

Let $(\Omega, \mathcal{A}, P)$ be a probability space (Ω is a non-empty set, $\mathcal{A}$ a σ-algebra of subsets of Ω and P a probability measure on $\mathcal{A}$) and let (M, d) be a metric space. Let S be a subset of functions from Ω to M and assume that, for all p and q in S, the set

$$\{d(p, q) < x\} := \{\omega \in \Omega: d\big(p(\omega), q(\omega)\big) < x\}$$

is measurable, i.e., it belongs to $\mathcal{A}$. When this condition is satisfied, the function $\omega \mapsto d(p(\omega), q(\omega))$ is a random variable and the distribution functions

$$F_{p,q}(x) := P\big(d(p, q) < x\big) \tag{3}$$

define a map $\mathcal{F}: S \times S \to \Delta_+$. The pair $(S, \mathcal{F})$ is said to be an E-space with base $(\Omega, \mathcal{A}, P)$ and target (M, d). For the map (3) one may well have $F_{p,q} = \varepsilon_0$ for distinct p and q. If (a) is satisfied, then $(S, \mathcal{F})$ is a Menger space under W.

In an E-space $(S, \mathcal{F})$, the function $d_\omega: S \times S \to \mathbb{R}_+$ defined, for every $\omega \in \Omega$, by $d_\omega(p, q) := d(p(\omega), q(\omega))$ is a pseudometric on S, but not necessarily a metric since one may have $d_\omega(p, q) = 0$ even when $p \neq q$. Now, let D be the set of all the pseudometrics thus constructed, $D := \{d_\omega: \omega \in \Omega\}$ and consider the σ-algebra $\mathcal{D}$ of the subsets of D of the form

$$B(A) := \{d_\omega \in D: \omega \in A,\ A \in \mathcal{A}\}.$$

A measure μ can then be defined on $(D, \mathcal{D})$ through $\mu(B(A)) := P(A)$. Thus the E-space $(S, \mathcal{F})$ is **pseudometrically generated** in the sense that there exists a probability space $(D, \mathcal{D}, \mu)$ such that: (i) D is a collection of pseudometrics on S; (ii) the sets $\{d \in D: d(p, q) < x\}$ $(p, q \in S,\ x \geqslant 0)$ are $\mathcal{D}$-measurable, and (iii) $F_{p,q}(x) = \mu(\{d \in D: d(p, q) < x\})$. Remarkably, Sherwood showed that the converse is also true: every pseudometrically generated space is isometric to an E-space and is therefore a Menger space under W. This fact allows one to study the probabilistic extension of metric properties like betweenness and convexity simply by looking at the measure of the set of all pseudometrics for the which property considered holds.

The **transformation generated spaces** form another class of PM spaces. Here, given a metric space (S, d) and a transformation $f: S \to S$, for any points p and q in S, the distance distribution function $F_{p,q}$ is (essentially) determined by the Cesaro means of the distances $d(f^{(n)}(p), f^{(n)}(q))$ between successive iterates of p and q under f. These spaces play a rôle in ergodic theory and, more significantly, in chaos theory [13].

Several topologies may be defined on a PM space. The one that has been most intensively studied is the "strong topology".

Let p be a point in a PM space $(S, \mathcal{F}, \tau)$; for $x > 0$, the **strong** x**-neighbourhood** of p is defined by

$$N_p(x) := \{q \in S: d_S(F_{p,q}, \varepsilon_0) < x\}. \tag{4}$$

The **strong neighbourhood system at** p is the family $\mathcal{N}_p := \{N_p(x): x > 0\}$ and the **strong neighbourhood system for** S is $\mathcal{N} := \bigcup_{p \in S} \mathcal{N}_p$. The function $\delta: S \times S \to [0, 1]$ defined by

$$\delta(p, q) := d_S(F_{p,q}, \varepsilon_0)$$

is a ***semi-metric***, but in general not a metric. However, the following weaker version of the triangle inequality

$$\delta(p, r) \leqslant d_S\big(\tau(F_{p,q}, F_{q,r}), \varepsilon_0\big)$$

holds for all $p, q, r \in S$. If τ is continuous with respect to the topology of Sibley's metric (1), the strong neighbourhood system $\mathcal{N}$ determines a Hausdorff topology, called the **strong topology** for S. Actually, the requirement that τ be continuous may be weakened: τ need only be continuous on the boundary, viz. $\tau(F, G_n) \to F$ whenever $G_n \to \varepsilon_0$.

If τ is continuous, the **strong** x**-vicinity** $(x > 0)$ is the subset of $S \times S$ defined via

$$U(x) := \{(p, q) \in S \times S: d_S(F_{p,q}, \varepsilon_0) < x\}$$

and the **strong vicinity system** for S is the union $\mathcal{U} := \bigcup_{x>0} U(x)$. It can be proved (see [12, Section 12.1]) that, if $(S, \mathcal{F}, \tau)$ is a PM space with a continuous triangle function τ, the strong vicinity system $\mathcal{U}$ is the base for a ***Hausdorff uniformity*** for S [Ke], called the **strong uniformity** and that this, or, equivalently, the strong topology is metrizable.

A sequence $\{p_n\}$ of elements in S is said to **converge strongly** to $p \in S$, if, for every $x > 0$ there is a natural number $n_0 = n_0(x)$ such that p_n belongs to $N_p(x)$ for all $n \geqslant n_0$. A sequence $\{p_n\}$ converges strongly to $p \in S$ if, and only if, $d_S(F_{p_n,p}, \varepsilon_0) \to 0$ as n tends to $+\infty$, or, equivalently (see [12, Section 4.3]), if, and only if, for every $x > 0$, and for every $n \geqslant n_0(x)$, $F_{p_n,p}(x) > 1 - x$.

It is often useful to consider a different system of neighbourhoods of the point $p \in S$; these are defined by

$$N'_p(\varepsilon, \delta) := \{q \in S\colon F_{p,q}(\varepsilon) > 1 - \delta\} \tag{5}$$

for $\varepsilon > 0$ and $\delta > 0$. Because of the properties of the metric d_S (see [12, Section 4.3]), one has $N'_p(x, x) = N_p(x)$ for $x > 0$ and $N_p(\min\{\varepsilon, \delta\}) \subset N'_p(\varepsilon, \delta)$ for all $\varepsilon, \delta > 0$; thus, the two systems (4) and (5) define the same topology. Similarly, the strong uniformity is equivalent to that generated by the ε, δ-vicinity system defined by

$$U_{\varepsilon,\delta} := \{(p, q)\colon F_{p,q}(\varepsilon) > 1 - \delta\}.$$

Every PM space $(S, \mathcal{F}, \tau)$ with a continuous triangle function τ has a ***completion***, in the sense that it is isometric to a dense subset of a PM space $(S_0, \mathcal{F}_0, \tau)$ which is complete in the strong topology. The completion is unique up to isometries. If $(S, \mathcal{F}, \tau)$ is complete, and if $\mathcal{U}$ is the strong uniformity, then $(S, \mathcal{U})$ is a complete uniform space; moreover, if d is a metric that metrizes $(S, \mathcal{F}, \tau)$, then (S, d) is a complete metric space. But more is true: if $(S, \mathcal{F}, \tau)$ is a PM space, if $(S_0, \mathcal{F}_0, \tau)$ is its completion, if $\mathcal{U}$ and $\mathcal{U}_0$ are their respective strong uniformities, then $(S_0, \mathcal{U}_0)$ is the completion of $(S, \mathcal{U})$. And, if d is a metric that metrizes $(S, \mathcal{F}, \tau)$ and if (S', d') is the completion of (S, d), then a metric d_0 on S_0 exists that metrizes $(S_0, \mathcal{F}_0, \tau)$ and which is such that (S_0, d_0) and (S', d') are isometric.

It is possible to define products, both finite and countably infinite, of PM spaces.

Let $(S_1, \mathcal{F}_1, \tau)$ and $(S_2, \mathcal{F}_2, \tau)$ be PM spaces under the same triangle function τ and let the map $\mathcal{F}_1 \tau \mathcal{F}_2 : (S_1 \times S_2)^2 \to \Delta_+$ be defined by

$$(\mathcal{F}_1 \tau \mathcal{F}_2)(p, q) := \tau(\mathcal{F}_1(p_1, q_1), \mathcal{F}_2(p_2, q_2))$$

for all $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $S_1 \times S_2$. The **τ-product** of the two PM spaces $(S_1, \mathcal{F}_1, \tau)$ and $(S_2, \mathcal{F}_2, \tau)$ is then defined as the pair $(S_1 \times S_2, \mathcal{F}_1 \tau \mathcal{F}_2)$. This is a PM space under τ. The strong topology on the product space is equivalent to the product topology of the factor spaces. The associativity of every triangle function τ allows one to define the τ-product of any finite number of PM spaces under the same triangle function.

There are several ways of constructing the product of a countable family of PM spaces. They are essentially due to Radu, Alsina and Schweizer. But of these, only one [1] has the following properties: (i) the product of a sequence of PM spaces under a given triangle function τ, viz. $\tau_n = \tau$ for all $n \geqslant 2$, is a PM space under the same triangle function τ; (ii) the strong topology on the product space coincides with the product topology. In order to define the product of a countable product the concepts of "serial iterates" of a triangle function τ and of a "φ-transform" are needed.

Given a triangle function τ, its **serial iterates** $\{\tau^n\colon n \geqslant 2\}$ are defined as follows: for $n = 2$, $\tau^2 := \tau$, while, for $n \geqslant 3$, $\tau^n : \Delta_+^n \to \Delta_+$ is defined recursively by

$$\tau^n(F_1, F_2, \dots, F_n) := \tau(\tau^{n-1}(F_1, F_2, \dots, F_{n-1}), F_n),$$

for $F_1, F_2, \dots, F_n$ in Δ_+. Given a sequence $\{F_n\colon n \in \mathbf{N}\}$ in Δ_+, the limit with respect to Sibley's metric of the sequence $\{\tau^n(F_1, F_2, \dots, F_n)\colon n \in \mathbf{N}\}$ always exists (although it may be trivial, i.e., equal to ε_∞) and is denoted by $\tau^{(\infty)}\{F_n\}$.

For $\beta \in]0, +\infty]$, let $C_i(\beta)$ be the set of continuous strictly increasing functions φ from $[0, \beta]$ to $\overline{\mathbb{R}}_+ = [0, +\infty]$; any function φ in $C_i(\beta)$ has an inverse φ^{-1}. If F is a distribution function in Δ_+ and φ is in $C_i(\beta)$, let $F \circ \varphi$ be defined, for $x \geqslant 0$, by

$$(F \circ \varphi)(x) := \begin{cases} F(\varphi(x)), & x \in [0, \beta], \\ \lim_{x \to \beta, x < \beta} F(\varphi(x)), & x = \beta, \\ 1, & x > \beta, \end{cases}$$

if $\beta < +\infty$, and by $F(\varphi(x))$ for all $x \geqslant 0$ if $\beta = +\infty$.

Given a triangle function τ, a function φ in $C_i(\beta)$ is said to be **τ-superadditive** if, for all $F, G \in \Delta_+$, $\tau(F, G) \circ \varphi \geqslant \tau(F \circ \varphi, G \circ \varphi)$. It turns out that if $\tau = \tau_T$ for some continuous t-norm T, or if $\tau = *$, then φ is τ-superadditive if, and only if, it is superadditive, i.e., $\varphi(x + y) \geqslant \varphi(x) + \varphi(y)$ for all x and y in $[0, \beta]$.

If $(S, \mathcal{F}, \tau)$ is a PM space and $\varphi \in C_i(\beta)$ is τ-superadditive the **φ-transform** of $(S, \mathcal{F}, \tau)$ is the PM space $(S, \mathcal{F} \circ \varphi, \tau)$; if τ is continuous, then the strong topologies of $(S, \mathcal{F}, \tau)$ and $(S, \mathcal{F} \circ \varphi, \tau)$ are equivalent.

Let $\{(S_n, \mathcal{F}_n, \tau)\colon n \in \mathbf{N}\}$ be a sequence of PM spaces under the same continuous triangle function τ, which is also assumed to satisfy the condition $\tau(\varepsilon_a, \varepsilon_b) \geqslant \varepsilon_{a+b}$, for all $a, b \in \overline{\mathbb{R}}_+$.

Let $\{\beta_n\colon n \in \mathbf{N}\}$ be a sequence of strictly positive numbers such that the series $\sum_{n \in \mathbf{N}} \beta_n$ converges, say to β. For every $n \in \mathbf{N}$, choose a τ-superadditive function φ_n in $C_i(\beta_n)$ and let $(S_n, \mathcal{G}_n, \tau)$ be the φ_n-transform of $(S_n, \mathcal{F}_n, \tau)$. For every pair p and q of sequences in $S := \prod_{n \in \mathbf{N}} S_n$, define a map $\mathcal{G} : S \times S \to \Delta_+$ via

$$G_{p,q} := \tau^{(\infty)}\{G_{p_n,q_n}\}.$$

Then the product of the sequence $\{(S_n, \mathcal{F}_n, \tau)\colon n \in \mathbf{N}\}$ is the pair $(S, \mathcal{G})$, which turns out to be a PM space under the same τ. Moreover, basically because $G_{p,q}$ is greater than or equal to ε_β for all p and q in S, the strong topology of $(S, \mathcal{G}, \tau)$ and the product topology coincide.

In a PM space $(S, \mathcal{F}, \tau)$ with a continuous τ, the **probabilistic diameter** D_A of a nonempty subset A of S is the function defined on $\overline{\mathbb{R}}$ by $D_A(+\infty) = 1$ and, for every $x > 0$

$$D_A(x) := \lim_{y \to x, y < x} \inf\{F_{p,q}(y): p, q \in A\}. \tag{6}$$

The introduction of the probabilistic diameter allows one to classify nonempty subsets of S with respect to boundedness, see [7]; this classification differs from the classical one valid in metric spaces: a subset A of the PM space $(S, \mathcal{F}, \tau)$ is

(B1) **certainly bounded** if, and only if, $D_A(x_0) = 1$ for some $x_0 \in]0, +\infty[$;
(B2) **perhaps bounded** if, and only if, $D_A(x) < 1$ for every $x \in]0, +\infty[$ and $\lim_{x \to +\infty} D_A(x) = 1$;
(B3) **perhaps unbounded** if, and only if, the limit $\lim_{x \to +\infty} D_A(x)$ belongs to the open interval $]0, 1[$;
(B4) **certainly unbounded** if, and only if, $D_A = \varepsilon_0$.

A set $A \subset S$ is said to be **bounded in distribution** (briefly **D-bounded**) if it is either certainly or perhaps bounded.

With the help of the probabilistic diameter, Kuratowski's measure of noncompactness (see [Kur]) was generalized to PM spaces by Bocşan and Costantin.

In a PM space $(S, \mathcal{F}, \tau)$ the **Kuratowski measure of noncompactness** c maps the non-empty subsets A of S into Δ_+ and is defined by $\alpha_A(x) := \sup\{\varepsilon \geqslant 0: \exists n \in \mathbf{N}, \exists A_1, A_2, \ldots, A_n \; A \subset \bigcup_{j=1}^n A_j, D_{A_j}(x) \geqslant \varepsilon\}$.

One has $\alpha_A \geqslant D_A$; moreover in a Menger space $(S, \mathcal{F}, \tau_T)$ with a continuous t-norm T, a set A and its closure in the strong topology have the same Kuratowski measure of noncompactness, $\alpha_{\bar{A}} = \alpha_A$. Let K_A denote the set of distribution functions F for which there exists a finite cover $A_1, A_2, \ldots, A_n$ of A, with $D_{A_j} \geqslant F$ $(j = 1, 2, \ldots, n)$. Then, for every $x \geqslant 0$, $\alpha_A(x) = \sup\{F(x): F \in K_A\}$. The Kuratowski function α may be used in order to characterize precompactness in PM spaces (see [4, Section 1.2.1]).

One sees from (5) that the definition of the strong topology contains the tacit assumption that statements about arbitrarily small distances can be made with arbitrarily high probabilities. This rather strong assumption was relaxed by Tardiff who introduced a profile function c which belongs to Δ_+ and whose value for any x is interpreted as the maximum probability with which one can make statements about distances less than x. Replacing ε_0 by c in (4) leads to a generalized topology, specifically a ***closure space*** in the sense of Čech.

Two concepts closely related to that of a PM space are those of a "Probabilistic Normed Space" (briefly PN space) and of a "Probabilistic Inner Product Space", (briefly PIP space).

PN spaces were introduced by Šerstnev by means of a definition that was closely modeled on the theory of (classical) normed spaces, and used to study the problem of best approximation in statistics. The definition below is more general; it has been proposed in [3].

DEFINITION 4. A **Probabilistic normed space** is a quadruple (V, ν, τ, τ^*), where V is a real vector space, τ and τ^* are continuous triangle functions with $\tau \leqslant \tau^*$ and ν is a map (the **probabilistic norm**) from V into Δ_+, such that for every choice of p and q in V the following hold:

(N1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);
(N2) $\nu_{-p} = \nu_p$;
(N3) $\nu_{p+q} \geqslant \tau(\nu_p, \nu_q)$;
(N4) $\nu_p \leqslant \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0, 1]$.

A PN space is called a **Šerstnev space** if it satisfies (N1), (N3) and the following condition

$$\forall \alpha \in \mathbb{R} \setminus \{0\} \forall x > 0, \quad \nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right),$$

which clearly implies (N2) and also (N4) in the strengthened form

$$\forall \lambda \in [0, 1], \quad \nu_p = \tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p}).$$

A PN space in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for a suitable continuous t-norm T and its conorm T^* is called a **Menger PN space**.

Given a probabilistic norm ν on S a probabilistic distance $\mathcal{F}$ on S is defined through $F_{p,q} = \nu_{p-q}$. Thus all the metric and topological concepts introduced for PM spaces can be recovered for PN spaces. Every PN space has a completion [5]. The same concepts of boundedness apply as in the case of PM spaces, although, in the case of PN spaces, they are based on the consideration of the probabilistic radius rather than that of the probabilistic diameter (6); the **probabilistic radius** R_A of a set $A \subset S$ is defined by $R_A(+\infty) = 1$ and, for $x > 0$, by

$$R_A(x) := \lim_{y \to x, y < x} \inf\{\nu_p(y): p \in A\}.$$

In a PN space there is an easy characterization of a D-bounded set A: A is D-bounded if, and only if, there exists a proper distance distribution function G, i.e., one for which $\lim_{x \to +\infty} G(x) = 1$, such that $\nu_p \geqslant G$ for every $p \in A$.

Many of the examples of PM spaces have a corresponding PN space: thus, for instance, there are simple PN spaces and EN-spaces – the analogue of E-spaces [6] – in which the target space is now a normed space $(V, \|\cdot\|)$. EN-spaces have an interesting application to functional analysis. Let L^0 be the linear space of equivalence classes of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ and define a map $\nu: L^0 \to \Delta_+$ by

$$\nu_f(x) := P\{\omega \in \Omega: |f(\omega)| < x\} \quad x > 0. \tag{7}$$

Then (L^0, ν) is a Šerstnev space under τ_W and the norms of L^p ($p \in [1, +\infty[$) and Orlicz spaces all derive from the same probabilistic norm (7) through

$$\forall f \in L^p, \quad \|f\|_p = \left(\int_{\mathbb{R}_+} x^p \, \mathrm{d}\nu_f(x) \right)^{1/p},$$

$$\forall f \in L^\varphi,$$

$$\|f\|_\varphi = \inf\left\{ k > 0\colon \int_{\mathbb{R}_+} \varphi\left(\frac{x}{k}\right) \mathrm{d}\nu_f(x) \leqslant 1 \right\}.$$

PIP spaces according to the definition below, were introduced in [2].

DEFINITION 5. A **Probabilistic inner product space** is a quadruple $(V, \mathcal{G}, \tau, \tau^*)$, where V is a real linear space, τ and τ^* are multiplications on Δ such that $\tau \leqslant \tau^*$ and $\mathcal{G} : V \times V \to \Delta$ is a map such that, if $G_{p,q}$ denotes the value of $\mathcal{G}$ at the pair (p, q) and, if the function $\nu : V \to \Delta_+$ is defined via

$$\nu_p(x) := \begin{cases} G_{p,p}(x^2), & x > 0, \\ 0, & x \leqslant 0, \end{cases}$$

the following condition hold for all $p, q, r \in V$:

(P1a) $G_{p,p} \in \Delta_+$ and $G_{\theta,\theta} = \varepsilon_0$, where θ is the null vector in V;
(P1b) $G_{p,p} \neq \varepsilon_0$, if $p \neq \theta$;
(P2) $G_{\theta,p} = \varepsilon_0$;
(P3) $G_{p,q} = G_{q,p}$;
(P4) $\forall x \in \mathbb{R}$, $G_{-p,q}(x) = 1 - \lim_{y \to x, y > x} G_{p,q}(-y)$;
(P5) $\nu_{p+q} \geqslant \tau(\nu_p, \nu_q)$;
(P6) $\forall \alpha \in [0, 1]$ $\nu_p \leqslant \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$;
(P7) $\tau(G_{p,r}, G_{q,r}) \leqslant G_{p+q,r} \leqslant \tau^*(G_{p,r}, G_{q,r})$.

It is immediate that (V, ν, τ, τ^*) is a PN space and that the probabilistic norm ν derives from the probabilistic inner product $\mathcal{G}$.

An E-space in which the target space, the metric space (M, d), has been replaced by a normed space $(M, \|\cdot\|)$ or by an inner product space $(M, \langle\cdot,\cdot\rangle)$, provides an example of a PN space, or, respectively, of a PIP space.

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e-16 Approach Spaces

1. Basic concepts

An approach space is a set equipped with a notion of distance between points and sets.

Given a set X, a function $\delta : X \times 2^X \to [0, \infty]$ is called a **distance** (on X) if it satisfies the following properties:

(D1) $\forall x \in X$: $\delta(x, \{x\}) = 0$,
(D2) $\forall x \in X$: $\delta(x, \emptyset) = \infty$,
(D3) $\forall x \in X, \forall A, B \in 2^X$: $\delta(x, A \cup B) = \min(\delta(x, A), \delta(x, B))$,
(D4) $\forall x \in X, \forall A \in 2^X, \forall \varepsilon \in [0, \infty]$: $\delta(x, A) \leqslant \delta(x, A^{(\varepsilon)}) + \varepsilon$, where $A^{(\varepsilon)} := \{x \in X: \delta(x, A) \leqslant \varepsilon\}$.

The pair (X, δ) is called an **approach space** [7].

A closely related concept is that of a κ-metric as introduced by Ščepin [14, 15]. A ***Tychonoff space*** X is called a κ**-metrizable space** if there exists a function $\rho : X \times RC(X) \to \mathbb{R}$, where $RC(X)$ stands for the collection of ***regularly closed sets*** in X, fulfilling the properties:

(1) $\forall x \in X, \forall F \in RC(X)$: $\rho(x, F) = 0$ if and only if $x \in F$,
(2) $\forall x \in X$, $\forall F, F' \in RC(X)$: $F \subset F' \Rightarrow \rho(x, F') \leqslant \rho(x, F)$,
(3) $\forall F \in RC(X)$: $\rho(\cdot, F)$ is continuous,
(4) $\forall (F_j)_j$ increasing totally ordered:

$$\rho\left(x, \mathrm{cl}\left(\mathrm{int}\left(\bigcup_{j \in J} F_j\right)\right)\right) = \inf_{j \in J} \rho(x, F_j).$$

Such a function is called a κ**-metric** for X. The class of κ-metrizable topological spaces contains all ***metrizable*** topological spaces. Analogous ideas also appeared in the work of Nagata [11], and Naimpally and Pareek [12] where a similar concept under the name of annihilator was used. An **annihilator** basically is more general than a κ-metric, only fulfills the first and third properties above, and can be defined on $X \times C(X)$ where $C(X)$ is an arbitrary collection of closed sets.

It seems to have been known to J.-I. Nagata already in 1956 that a T_1-space X is metrizable if and only if it has a continuous annihilator ρ on $X \times 2^X$ such that both $\inf\{\rho(x, F): F \in \mathcal{F}\}$ and $\sup\{\rho(x, F): F \in \mathcal{F}\}$ are continuous in x for an arbitrary collection $\mathcal{F}$ of closed sets [11].

Naimpally and Pareek characterized ***(semi)stratifiable***, ***quasi-metric***, ***developable***, ***first-countable*** and ***Nagata spaces*** via supplementary continuity conditions on annihilators [13].

Much work in the field of κ-metrics (and annihilators) is aimed at finding supplementary conditions on a κ-metric or an annihilator to insure metrizability of a given space. Thus a typical result, obtained by T. Isiwata [3], is that a κ-metrizable space which is ***stratifiable*** is metrizable. Somewhat later J. Suzuki, K. Tamano and Y. Tanaka showed that every κ-metrizable ***CW-complex*** also is metrizable [16].

Another type of result obtained in this context concerns stability of the notion of κ-metrizability. Thus in the original papers of Ščepin [14, 15] it is shown that any product of κ-metrizable spaces is again κ-metrizable and that the ***open perfect*** image of a κ-metrizable space is again κ-metrizable.

Approach spaces form the objects of a category, the morphisms of which are defined as follows. If (X, δ_X) and (Y, δ_Y) are approach spaces and $f : X \to Y$ is a function, then f is called a **contraction** if for any $x \in X$ and $A \subset X$ we have $\delta_Y(f(x), f(A)) \leqslant \delta_X(x, A)$. The category of approach spaces and contractions is denoted **AP**.

As topological spaces, approach spaces can be characterized in several different ways. The two most important other characterizations are gauges and limit operators.

A **gauge** $\mathcal{G}$ is a collection of extended pseudo-quasi-metrics (meaning the value ∞ is permitted), which is saturated in the sense of the following condition:

(G) if d is an extended pseudo-quasi-metric such that $\forall x \in X, \forall \varepsilon > 0$ and $\forall \omega < \infty\ \exists e \in \mathcal{G}$ such that $d(x, \cdot) \wedge \omega \leqslant e(x, \cdot) + \varepsilon$ then $d \in \mathcal{G}$.

Given a gauge the associated distance is derived via the formula:

$$\delta(x, A) := \sup_{d \in \mathcal{G}} \inf_{a \in A} d(x, a).$$

A **limit operator** λ is a function which to each ***filter*** $\mathcal{F}$ on X associates a function $\lambda\mathcal{F} : X \to [0, \infty]$ fulfilling the conditions:

(L1) $\forall x \in X$: $\lambda(\mathrm{stack}\, x)(x) = 0$,
(L2) for any family $(\mathcal{F}_j)_{j \in J}$ of filters on X: $\lambda(\bigcap_{j \in J} \mathcal{F}_j) = \sup_{j \in J} \lambda\mathcal{F}_j$,
(L3) for any filter $\mathcal{F}$ on X and any collection of filters $(\mathcal{S}(x))_{x \in X}$ on X:

$$\lambda\big(\mathcal{D}(\mathcal{S}, \mathcal{F})\big) \leqslant \lambda\mathcal{F} + \sup_{x \in X} \lambda\big(\mathcal{S}(x)\big)(x),$$

where stack x stands for the filter of all sets containing x and where $\mathcal{D}(\mathcal{S}, \mathcal{F}) := \bigvee_{F \in \mathcal{F}} \bigcap_{x \in X} \mathcal{S}(x)$ is the **Kowalsky diagonal filter**. Given a limit operator, the associated distance is derived via the formula:

$$\delta(x, A) := \inf\big\{\lambda\mathcal{F}(x): \mathcal{F} \text{ an ultrafilter, } A \in \mathcal{F}\big\}.$$

Contractions too can be characterized by means of gauges and limit operators. If (X, δ) and (X', δ') are approach spaces with corresponding gauges and limit operators $\mathcal{G}, \mathcal{G}'$, λ and λ' and $f : X \to X'$, then f is a contraction if and only if for any filter $\mathcal{F}$ on X: $\lambda'(f(\mathcal{F})) \circ f \leqslant \lambda\mathcal{F}$, if and only if for any $d' \in \mathcal{G}'$: $d' \circ (f \times f) \in \mathcal{G}$.

2. Some fundamental results

Approach spaces, as topological spaces, form a so-called ***topological category***. Especially, this implies that arbitrary initial and final structures exist. Given approach spaces (determined by their gauges) $(X_j, \mathcal{G}_j)_{j\in J}$ and a **source**

$$\left(f_j : X \to (X_j, \mathcal{G}_j)\right)_{j\in J}$$

in **AP**, then the **initial approach structure** on X is determined by the smallest gauge containing the collection

$$\mathcal{B} := \left\{ \sup_{j\in K} d_j \circ (f_j \times f_j) \colon K \in 2^{(J)}, \; \forall j \in K \colon d_j \in \mathcal{G}_j \right\},$$

where $2^{(J)}$ stands for the set of all finite subsets of J.

Given a topological space $(X, \mathcal{T})$, a natural approach space is associated with it by defining the distance $\delta_{\mathcal{T}} : X \times 2^X \to [0,\infty]$ by

$$\delta_{\mathcal{T}}(x, A) := \begin{cases} 0, & x \in \mathrm{cl}_{\mathcal{T}}(A), \\ \infty, & x \notin \mathrm{cl}_{\mathcal{T}}(A). \end{cases}$$

A function between topological spaces then is continuous if and only if it is a contraction between the associated approach spaces. The functor

$$\begin{array}{rcl} \mathbf{TOP} & \to & \mathbf{AP}, \\ (X, \mathcal{T}) & \mapsto & (X, \delta_{\mathcal{T}}), \\ f & \mapsto & f, \end{array}$$

is a full embedding of **TOP** into **AP**. **TOP** is actually embedded as a bireflective and bicoreflective subcategory of **AP**. For any space $(X, \delta) \in |\mathbf{AP}|$, its **TOP**-bicoreflection is given by $\mathrm{id}_X : (X, \delta^{tc}) \to (X, \delta)$, where δ^{tc} is the distance associated with the topological closure operator given by $\mathrm{cl}_\delta(A) := \{x \in X \mid \delta(x, A) = 0\}$.

Given an extended pseudometric space (X, d), a natural approach space is associated with it via the usual distance

$$\delta_d : X \times 2^X \to [0,\infty] \colon (x, A) \to \inf_{a\in A} d(x, a).$$

A function between extended pseudometric spaces then is nonexpansive if and only if it is a contraction between the associated approach spaces. The functor

$$\begin{array}{rcl} p\mathbf{MET}^\infty & \to & \mathbf{AP}, \\ (X, d) & \mapsto & (X, \delta_d), \\ f & \mapsto & f, \end{array}$$

is a full embedding of $p\mathbf{MET}^\infty$ into **AP**. The category $p\mathbf{MET}^\infty$ of extended pseudometric spaces and nonexpansive functions is embedded as a bicoreflective subcategory of **AP**. For any space $(X, \delta) \in \mathbf{AP}$, its $p\mathbf{MET}^\infty$-bicoreflection is given by id_X: $(X, \delta^{mc}) \to (X, \delta)$, where δ^{mc} is the distance determined by the extended pseudometric

$$d_\delta : X \times X \to [0,\infty] \colon (x, y) \to \delta\big(x, \{y\}\big) \vee \delta\big(y, \{x\}\big).$$

$p\mathbf{MET}^\infty$ is not embedded bireflectively, especially it is not closed under the formation of infinite products. An infinite product of extended pseudometric spaces is a "genuine" approach space, i.e., in general neither topological nor extended pseudometric. The relationship between an extended pseudometric and the underlying topology is recaptured in **AP** via a canonical functor, namely the **TOP**-bicoreflector restricted to $p\mathbf{MET}^\infty$. In the case of an extended pseudometric space the **TOP**-bicoreflection is the underlying topological space. In the same way, given an arbitrary approach space (X, δ), (X, δ^{tc}) is the underlying topological approach space and the topology generating (X, δ^{tc}) is the topology underlying δ. The following diagram clarifies the situation, where **UAP**, the subcategory of so-called **uniform approach spaces** stands for the ***epireflective*** hull of $p\mathbf{MET}^\infty$ in **AP**. The horizontal arrows are embeddings and the vertical arrows are ***forgetful functors***.

$$\begin{array}{ccccccc} p\mathbf{MET}^\infty & \longrightarrow & \mathbf{UAP} & \longrightarrow & \mathbf{AP} & \longleftarrow & \mathbf{TOP} \\ \downarrow & & \downarrow & & \downarrow & & \\ pm\mathbf{UNIF} & \longrightarrow & \mathbf{UNIF} & \longrightarrow & q\mathbf{UNIF} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ pm\mathbf{TOP} & \longrightarrow & \mathbf{CREG} & \longrightarrow & \mathbf{TOP} & & \end{array}$$

A typical concept defined in the setting of approach spaces is the measure of compactness, which puts the so-called Hausdorff measure of noncompactness in its proper setting [6]. If (X, d) is a pseudometric space and $A \subset X$ then

$$m_H(A) := \inf\left\{ \varepsilon \in \mathbb{R}^+ \colon \exists x_1, \ldots, x_n \in X \colon A \subset \bigcup_{i=1}^n B(x_i, \varepsilon) \right\}$$

is called the **Hausdorff measure of noncompactness**. Given an arbitrary approach space (X, δ), the **measure of compactness** of X is defined as

$$\mu_c(X) := \sup_{\mathcal{F}\text{ ultrafilter}} \inf_{x\in X} \lambda\mathcal{F}(x).$$

The idea behind this definition is that compactness in **TOP** means that every ultrafilter must converge and the information given by μ_c is based on the verification for all ultrafilters of what their "best" convergence points are. In the case of a pseudometric approach space, μ_c coincides with m_H. If $(X, \delta_{\mathcal{T}})$ is a topological approach space, then it is compact if and only if $\mu_c(X) = 0$. If (X, δ_d) is a pseudometric approach space, then it is totally bounded if and only if $\mu_c(X) = 0$ and it is bounded if and only if $\mu_c(X) < \infty$. If $(X_j, \delta_j)_{j \in J}$ is a family of approach spaces, then

$$\mu_c\left(\prod_{j \in J} X_j\right) = \sup_{j \in J} \mu_c(X_j),$$

which is essentially the ***Tychonoff Product Theorem***.

Another fundamental concept is completeness. A filter $\mathcal{F}$ in an approach space (X, δ) is called a **Cauchy filter** if $\inf_{x \in X} \lambda\mathcal{F}(x) = 0$ and $\mathcal{F}$ is said to be a **convergent filter** (with limit x) if $\lambda\mathcal{F}(x) = 0$, i.e., if it converges to x in the topological bicoreflection of (X, δ). An approach space (X, δ) is said to be a **complete approach space** if every Cauchy filter converges. Any topological approach space is complete. If (X, δ_d) is a pseudometric approach space, then it is complete if and only if (X, d) is complete.

An approach space is said to have a particular topological property, e.g., Hausdorff, if its underlying topology has that property.

The fact that **AP** contains both **TOP** and p**MET**$^\infty$ as full and isomorphism-closed subcategories has as consequence that there exist natural constructions of **completion** and of **compactification**, at least for uniform approach spaces. It is beyond the scope of this text to define these constructions in full [7].

The notion of completion in **AP** coincides with the usual completion in the case of pseudometric approach spaces. It is an ***epireflection*** from the subcategory of Hausdorff uniform approach spaces to the subcategory of complete Hausdorff uniform approach spaces.

The compactification in **AP** is the counterpart of the ***Čech–Stone compactification*** in **TOP**. For topological spaces this compactification coincides with the Čech–Stone compactification, and in general the **TOP**-coreflection of this compactification is the ***Smirnov compactification*** of an associated ***proximity space***. In the special case of a (metric) ***Atsuji space***, the topological coreflection of the compactification in **AP** coincides with the Čech–Stone compactification of the topological coreflection. This implies that, e.g., $\beta\mathbb{N}$ can be endowed with a distance which extends the usual metric on $\mathbb{N}$, and which has the Čech–Stone compactification as topological coreflection, in other words, which "distancizes" the Čech–Stone topology of $\beta\mathbb{N}$. The compactification is an epireflection from the subcategory of Hausdorff uniform approach spaces to the subcategory of compact Hausdorff uniform approach spaces.

3. Applications in other fields of mathematics

Approach spaces find applications in those fields of mathematics where initial structures of metrizable topological spaces, in particular products, occur, especially if the metrizable topological spaces are endowed with canonical metrics. In those instances instead of having to work with the underlying topological structure, since in general initial metrics do not exist, one can remain working at the numerical level, by performing the initial construction in **AP**. This results in a canonical approach structure which is strongly linked to the given metrics on the original spaces and which has as underlying topology the initial topology. Examples of this situation are the following.

If X is a normed space then both X and its dual X^* can be equipped with canonical approach structures with underlying topologies the ***weak topology*** and the ***weak* topology***. Also most other topologies considered on X or X^*, such as, e.g., the Mackey topology or any topology describing uniform convergence on a particular class of subsets, can be derived from natural approach structures. In all cases the metric coreflection is given by the norm (or the dual norm) [10].

If X is a metric space then the hyperspace of closed sets can be endowed with various natural approach structures having as underlying topology, e.g., the ***Wijsman topology***, the ***Attouch–Wets topology*** and the ***proximal topology***. In all these cases the metric coreflection is given by the ***Hausdorff metric*** [9].

If X is a separable metric space then the so-called ***weak topology*** on the space of measures on X can be derived from various natural approach structures constructed with specific collections of continuous real-valued functions. The metric coreflection always is the so-called L^1-metric [7].

Probabilistic Metric Spaces have underlying approach structures in exactly the same way that ordinary metric spaces have underlying topologies. The topologies underlying these approach structures are precisely the topologies which are used in the context of probabilistic metric spaces [1].

Spaces of random variables with values in a metric space have natural approach structures with topological coreflection the topology of convergence in probability and with metric coreflection the so-called metric of equality almost everywhere [7].

Finally, another direction of research in this field is of a categorical nature. Several categorically better behaved categorical hulls, such as the ***Cartesian closed*** topological hull, the ***extensional*** topological hull and the quasitopos hull of **AP** have recently been described [4, 5]. Monoidal closed structures are being studied and it has recently been shown by M.M. Clementino and D. Hofmann that approach spaces appear as the lax algebras of the ultrafilter monad over numerical relations [2].

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F: Special properties

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f-1 Continuum Theory

1. Introduction

Continuum theory contains the study of all continua (i.e., all ***compact*** and ***connected*** spaces) and maps between them. The vitality of an area is often determined by the existence of major open problems which are either natural or connected to other areas of mathematics. We will write our survey from this perspective, focusing on problems and briefly describing what is known. Because of limited space we will restrict ourselves mainly to metric continua and we will refrain from describing many other related results which were inspired by these problems.

2. One-dimensional Continuum Theory

All continua naturally belong to one of the following two types. A **decomposable continuum** is one that is the union of two proper subcontinua. It is an **indecomposable continuum** otherwise. All ***locally connected*** continua are decomposable while all indecomposable continua are nowhere locally connected. Hence, decomposable continua include the class of continua with nice local properties, while indecomposable continua includes the class with complicated local structure. It is well known that all locally connected continua are ***arcwise connected*** and hence, between every pair of points there is a very simple irreducible subcontinuum (i.e., an ***arc***) containing them. It is know that indecomposable, and even hereditarily indecomposable continua, of ***dimension*** bigger than one exist, but such continua are more difficult to construct (for recent simpler constructions see [24, p. 210]). Since we are dealing with one-dimensional continua in this section both classes are common and both will be well represented.

It is known that each one-dimensional continuum can be represented as an ***inverse limit*** on graphs. We will call such continua **graph-like**. Special classes of one-dimensional continua are the **tree-like**, **arc-like** and **circle-like** continua (inverse limits on trees, arcs and circles, respectively).

The simplest non-trivial continuum is an arc. On the other extreme the **pseudo-arc** $\mathcal{P}$ is an example of a hereditarily indecomposable arc-like continuum. It was shown by Moise [27] that it is **hereditarily equivalent** (i.e., homeomorphic to all of its subcontinua). Bing [8] showed that it is homogeneous and topologically unique. Homogeneous continua have been extensively studied but the following Conjecture remains open:

CONJECTURE 1. *Suppose that X is a homogeneous tree-like continuum, then X is the pseudo-arc.*

A positive solution to Conjecture 1 would complete the classification of all homogeneous plane continua. It is known that such a continuum must be hereditarily indecomposable. Moreover, any planar example must have span zero (defined below). It is also unknown if the arc and the pseudo-arc are the only examples of hereditarily equivalent continua. It is known that every hereditarily equivalent continuum is tree-like and, if not an arc, hereditarily indecomposable. It is a difficult open problem to find a useful characterization of an arc-like continuum. The most promising is the notion of a continuum of span zero and is due to Lelek. Let X be a continuum, then the **span** of X, denoted by $\mathrm{span}(X)$, is the least upper bound of all numbers $\varepsilon \geqslant 0$ for which there exists a continuum C and maps $f, g : C \to X$ such that $f(C) \subset g(C)$ and for each $c \in C$, $d(f(c), g(c)) \geqslant \varepsilon$ (more precisely this is the definition of semi-span but we will not distinguish between the various forms of span). A positive solution to the problem below would imply a positive solution to Conjecture 1 for planar continua:

CONJECTURE 2. *Suppose that X is a continuum such that* $\mathrm{span}(X) = 0$*, then X is an arc-like continuum.*

It is not difficult to show that all arc-like continua have zero span. It is also known [29] that all continua of span zero are atriodic, tree-like and the continuous image of the pseudo-arc.

A continuum X has the **fixed-point property** provided each continuous self map of X fixes a point. A classical result by Brouwer states that each n-cube has the fixed point property. Hamilton [14] showed that all arc-like continua have the fixed-point property. Surprisingly, Bellamy [6] has constructed a tree-like continuum without the fixed-point property. It is known that each hereditarily decomposable tree-like continuum has the fixed-point property. The following is another long standing conjecture:

CONJECTURE 3. *Each non-separating plane continuum has the fixed-point property.*

Bell [4] has essentially shown that if the answer is no, there exists an example with an indecomposable boundary. Cartwright and Littlewood [11] have shown that each orientation preserving homeomorphism of the plane must have a fixed point in each invariant, non-separating subcontinuum. This was generalized by Bell [5] to all planar homeomorphisms. This result was recently extended to the class of all positively oriented compositions of monotone and open maps of the plane. This class includes all holomorphic maps (see also [1]).

Invariance of certain properties of continua have been extensively studied for traditional classes of maps (i.e., ***open maps***, ***monotone maps***, ***light maps***, etc.). A useful additional class of maps are **confluent maps**: a map $f : X \to Y$ between continua is confluent provided for each continuum $K \subset Y$ each component of $f^{-1}(K)$ maps onto K. It is not difficult to see that all ***open maps*** between continua are confluent and each monotone map is trivially confluent. Confluent map share many properties with these classes. For example they preserve end-points, atriodicity and tree-likeness. In some respects they are nicer. For example, while the inverse limit of open maps is not necessarily open, the inverse limit of confluent maps is always confluent. Lelek and Read [22] have shown that each confluent map onto a locally connected continuum is the composition of a monotone map and a light, open map. A positive solution to the following problem would complete the classification of all homogeneous plane continua:

PROBLEM 4. *Is the confluent image of an arc-like continuum always arc-like*?

It is known [23] that the confluent image of an arc-like continuum is atriodic, weakly-chainable and tree-like. For a given class $\mathcal{C}$ of continua, a continuum X is called **confluently $\mathcal{C}$-like** if it can be obtained as an inverse limit $\varprojlim(C_n, f_m^n)$, where each $C_n \in \mathcal{C}$ and each f_m^n is a confluent and onto map. Oversteegen and Prajs have recently shown that each continuum which admits for each $\varepsilon > 0$ a confluent ε-map to a class $\mathcal{C}$ of graphs is confluently $\mathcal{C}$-like. It is also shown that each locally connected curve is confluently graph-like.

During the last 15 years maps between continua have been studied from the point of view of a dynamical system. For example Kennedy and Yorke [19] have shown that there exists an open subset of the function space of a manifold such that each member has a hereditarily indecomposable attractor. Laminations are other well studied examples of attractors. A continuum X is called a **lamination** if it is locally the product of a Cantor set and an arc. Hagopian [13] has shown that each homogeneous lamination is a solenoid. It is well known that non-homogeneous laminations exist (see for example recent work by Barge and Diamond [3] who have obtained a classification of such laminations using tiling spaces).

Laminations often admit an **expansive homeomorphism** (i.e., a homeomorphism for which there exists $c > 0$ such that for each $x \neq y$ there exists $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > c$). Plykin [30] has shown that there exists a planar lamination which admits an expansive homeomorphism while Fokkink and Oversteegen have shown [12] that any non-trivial planar lamination must have at least 4 complementary domains. It is known that no infinite-dimensional compactum admits an expansive homeomorphism [25]. It is an important result, due independently to Hiraide and Lewowicz [15, 21], that the 2-sphere also does not admit an expansive homeomorphism. Answering a question of Williams, Kato [17] has shown that arc-like continua do not admit expansive homeomorphisms. This was recently significantly extended by Mouron to all tree-like continua. The following problem remains open:

PROBLEM 5. *Does the Menger Universal Curve admit an expansive homeomorphism*?

Continuum theory has also been used to study Julia sets of polynomial maps. In many cases these Julia sets are locally connected continua. It follows in this case that they are hereditarily locally connected. This can be used to elucidate certain dynamical properties. Many interesting problems remain for non-locally connected Julia sets. For example, all known examples of connected Julia sets of polynomials are hereditarily decomposable continua.

PROBLEM 6. *Suppose that J is a connected Julia set of a complex polynomial. Is J (hereditarily) decomposable*?

It is known [18] that there exist non-arcwise connected Cremer Julia sets for complex polynomials.

Let us assume for simplicity that P is a complex polynomial of degree 2 and that P has a fixed point x such that $P'(x) = \exp(2\pi\alpha i)$ with α irrational. Then either P is linearizable at x (i.e., there exists a maximal open disc Δ such that $P \restriction \Delta$ is conformally conjugate to an irrational rotation of the unit disc; the *Siegel* case and Δ is called a **Siegel Disk**), or P is not linearizable at x (i.e., $x \in J$; the *Cremer* Case and x is called a **Cremer point**). The following is an important open problem in complex dynamics:

PROBLEM 7. *Suppose that $\partial\Delta$ is the boundary of a Siegel disc. Is $\partial\Delta$ always a simple closed curve*?

Rogers [32] has shown that the boundary of a Siegel disc is either an indecomposable continuum or admits a finest monotone map to a circle.

An extensive study has been made of **λ-dendroids** (hereditarily decomposable tree-like continua), **dendroids** (i.e., arcwise connected λ-dendroids) and **dendrites** (locally connected dendroids). All λ-dendroids have the fixed-point property. Several subclasses of dendroids (for example smooth) have been introduced and extensively studied. However, the following problem remains open:

PROBLEM 8. *Suppose that D is a dendroid. Does there exist for each $\varepsilon > 0$ an ε-retraction $r : D \to T$, where $T \subset D$ is a tree*?

Dendrites appear naturally as the Julia sets of certain complex polynomials. A map $f : X \to X$ is called **transitive** if there exists a point $x \in X$ such that its orbit (i.e., the set $\{x, f(x), f^2(x), \ldots\}$) is dense. A map $f : X \to X$ is **strongly transitive** if the set $\bigcup_n f^{-n}(y)$ is dense for each $y \in X$. It is well known that if a dendrite D is the Julia set of a complex polynomial P, then $P \restriction D$ is an open and strongly transitive map of D onto itself (in fact it is a "branched covering" map) with a dense set of periodic points. We will call such maps **polynomial-like**. The following problem is open:

PROBLEM 9. *Suppose that $f : D \to D$ is a polynomial-like map of a dendrite D onto itself. Does every branch point of D either have a finite orbit or is eventually mapped to a critical point*?

It follows from Thurston's No-wandering Triangle Theorem that Problem 9 has a positive solution if there exists a point $v \in D$ such that $f^{-1}(v)$ is a single point while for all other points $y \in D \setminus \{v\}$, $|f^{-1}(y)| = 2$ (i.e., f is a degree 2 branched covering map with a unique branch point). This argument can be extended to all polynomial like branched covering maps with a single branchpoint. Problem 9 is closely related to the well-known open problem on the existence of wandering triangles for higher degree covering maps of the circle

3. Higher-dimensional continuum theory

At first glance, continuum theory seems to be largely concerned with one-dimensional continua. That this is not an artificial restriction is exemplified by the Hahn–Mazurkewicz Theorem: *Every locally connected continuum is the continuous image of the arc*. Moreover, simple topological conditions often imply that the topological space in question is one-dimensional, or at least finite-dimensional. An important static example is Rogers' theorem: *Homeogeneous, hereditarily indecomposable continua are tree-like* (i.e., one-dimensional and of trivial shape) [31]. An important dynamic example is Mañé's theorem: *If a compact metric space admits an expansive homeomorphism, then it is finite-dimensional*.

Nevertheless, there are several interesting questions that belong to the theory of continua of dimension > 1. Most of these questions belong equally well to geometric topology and/or algebraic topology as well as continuum theory. Rather than try to be exhaustive, we will focus on four questions that have proven very resistant to solution, and seem to have connections to other areas of topology.

PROBLEM 10 (Rogers). *Is there a homogeneous indecomposable continuum of dimension > 1?*

CONJECTURE 11 (Bing–Borsuk). *Every homogeneous ENR is a topological manifold.*

CONJECTURE 12 (Hilbert–Smith). *A Cantor group cannot act effectively on a manifold.*

CONJECTURE 13 (Tymchatyn). *If X is a connected, locally arcwise connected* (*separable*) *metric space, then X admits a convex metric.*

Problem 10 is open even in dimension 2. Conjecture 11 is true in dimensions 1 and 2. Conjecture 12 is true in dimensions 1 and 2. Conjecture 13 is open even in dimension 1, but does hold under the added assumption that X is uniquely arcwise connected and without the assumption of separability.

Homogeneous indecomposable continua
With regard to Problem 10, Rogers' theorem, quoted above, shows that a continuum X providing an affirmative answer could not be hereditarily indecomposable. Even (or maybe, particularly), the case where the continuum in question is cell-like is open. At this point, there does not seem to be any promising direction for finding an answer.

There are non-metric indecomposable homogeneous continua of arbitrarily large dimension, but the approach does not offer any insight into the metric case.

Homogeneous ENRs
An **ENR** is a Euclidean neighbourhood retract. Partial results abound in the case of the **Bing–Borsuk Conjecture**. In dimensions 1 and 2 the proof is by Bing and Borsuk [9]. One important result in higher dimensions is that of Jakobsche: *In dimension* 3, *the Bing–Borsuk Conjecture is stronger than the Poincaré Conjecture* [16]. Jakobsche shows that the "fake" 3-sphere S resulting from the failure of the Poincaré conjecture can be used to construct, by an inverse limit argument involving connected sums of ever-more copies of S, a homogeneous compact ENR that is not a manifold.

John Bryant has long investigated the weaker conjecture: *A homogeneous ENR must be a homology manifold.* Bryant (anticipated by Bredon) proved. *If X is a homogeneous, finite-dimensional ENR, and X has finitely generated local homology, then X is a homology manifold* [10]. One might expect that in dimensions > 5, greater progress might have been made (because the proof, by Smale, in dimension > 5 that the Poincaré Conjecture is true is more manageable), but such is not the case.

Hilbert–Smith Conjecture
The **Hilbert–Smith Conjecture** grows out of Hilbert's Fifth Problem on continuous transformation groups, one part of which asks if a locally Euclidean group acts effectively on a manifold, must it be a Lie group in some coordinates? While Hilbert's Fifth Problem was solved by Montgomery and Zippin and by Gleason [28], the variation of Smith has proved resistant.

The weaker conjecture, replacing "effectively" by "freely", is also unproven at present. The truth of the conjecture is easy to demonstrate in dimension 1, and there are several quite different proofs in dimension 2, each leading to a distinct line of investigation of the higher-dimensional cases. The conjecture is known to be true with additional hypotheses on the actions, including Lipschitz actions and quasiconformal actions.

A proof of the Hilbert–Smith conjecture, using classically continuum-theoretic covering techniques, has been claimed by L.F. McAuley, but the proof has not held up to close inspection. Beverly Brechner and her students have long worked on developing a stronger 3-dimensional prime end theory with the aim of generalizing to one higher dimension the prime end proof that reduces the dimension 2 case to dimension 1 [20].

Robert Edwards has taken a classifying space approach that has produced some intriguing partial results, and a generalization of the conjecture. Edwards' **Free-Set Z-Set Conjecture** (FSZS) states *Given any action by a Cantor group on an ENR, the free set of the action is a homology Z-set in the ENR*. A **homology Z-set** is one whose removal does not change the homology of any open set in the ENR. The FSZS conjecture reduces to the Hilbert–Smith conjecture when the ENR is a manifold.

The question is made more interesting by the fact that a Cantor group can act effectively on the Hilbert cube, and on **Menger manifolds** [2] (manifolds modelled on ***Menger Universal Spaces***).

Convex metrics

A metric d on a space X is a **convex metric** provided that for all $x, z \in X$, there is an arc A from x to z such that for all $y \in A$, $d(x,z) = d(x,y) + d(y,z)$. It is a theorem of Bing that every Peano continuum admits a convex metric. Mayer and Oversteegen showed that every uniquely and locally arcwise connected space admits a convex metric, dropping compactness, but, essentially, requiring one-dimensionality and trivial shape [26]. Both proofs use partitioning techniques in which the elements of the partition reiterate the connectedness of the space.

There are deep connections to the partitioning techniques used in Bestvina's characterization of Menger manifolds, to Hilbert cube manifolds, and to dimension-raising map questions. Related is the question of Tymchatyn and Alex Chigogidze: *If X is a continuum that is LC^{n-1} and has DD^nP does X admit small mesh partitions into C^{n-1} and LC^{n-1} elements with boundaries of elements being Z_{n-1} sets in those elements*?

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f-2 Continuum Theory (General)

Because continuum theory is extremely diffuse I have confined myself to sampling a few topics mostly involving maps and with which I have some familiarity. In that way I hope to give the reader some sense of the flavour of the subject – of some of the recurrent ideas and their interconnections. I have not tried to state best possible results or to properly assign credit for results.

A ***compactum*** is a compact Hausdorff space. A **continuum** is a connected compactum. We can classify continua by their local and global connectivity properties. A continuum is a **locally connected continuum** if it has a base of connected open sets. In metric continua local connectedness is equivalent to local arc connectedness. Among the continua which are everywhere far from being locally connected are the indecomposable continua. An **indecomposable continuum** is not the union of two of its proper subcontinua. A compactum is **hereditarily indecomposable** if each of its subcontinua is indecomposable. The nice compacta are those with very strong local connectivity properties, e.g., ***polyhedra***, ***manifolds*** and ANRs. At the opposite end of the spectrum are the hereditarily indecomposable compacta. Continuum theorists are fascinated by the fact that these opposite ends of the spectrum are never very far apart. Hereditarily indecomposable compacta are initially very counterintuitive but because they contain few internal connections they are often easy to work with.

Another classification of continua is obtained by looking at finite open covers. If $\mathcal{P}$ is a class of compact, connected polyhedra, a continuum X is said to be $\mathcal{P}$**-like** if for each open cover $\mathcal{U}$ of X there is a continuous surjection $f : X \to P \in \mathcal{P}$ with fibers refining $\mathcal{U}$. A metric continuum X is $\mathcal{P}$-like iff it is homeomorphic to the inverse limit of an inverse sequence (X_i, f_{ij}) of members of $\mathcal{P}$ with continuous surjections for bonding maps. In the non-metric case the situation is more complex (Mardešić, 1963). A continuum X is $\mathcal{P}$-like iff it is homeomorphic to the inverse limit of an inverse system $(X_\alpha, f_{\alpha\beta}, A)$ where the X_α are metric $\mathcal{P}$-like continua and the $f_{\alpha\beta}$ are continuous surjections. A **graph** is a finite, connected 1-dimensional polyhedron. A **tree** is a simply connected graph. The graphlike continua are exactly the 1-dimensional continua. Treelike continua are the 1-dimensional continua of trivial shape.

Continuous maps preserve a few covering properties of spaces like compactness and connectedness. ***Open map***s (i.e., images of open sets are open) and ***closed map***s preserve local connectedness. Open maps also respect certain 1-dimensional algebraic invariants of compacta. However, open maps are not sufficiently general for our purposes. If X is a continuum let 2^X (respectively $C(X)$) denote the space of closed subsets (respectively subcontinua) of X with the Vietoris topology. A continuous surjection $f : X \to Y$ of continua is a **monotone map** if $f^{-1}(y)$ is connected for each y in Y. It is a **confluent map** (**weakly confluent map**) if for each $C \in C(Y)$ each (some) component of $f^{-1}(C)$ is mapped onto C. Hence, f is weakly confluent when the induced map $C(f) : C(X) \to C(Y)$ is a surjection and f is confluent if each continuum in the range is covered under f by each component of its preimage.

Confluent maps generalize both open maps and monotone maps of compacta. Light confluent maps onto a locally connected continuum are open. Hence, by the monotone-light factorization theorem confluent maps onto locally connected continua are just compositions of a monotone map with an open map. Both confluent maps and weakly confluent maps are preserved by composition. However, neither confluent nor weakly confluent maps are preserved by products (Lelek, 1976). Mackowiak (1975) constructed (weakly) confluent maps of compacta whose product with the identity map on the unit interval is not (weakly) confluent. It is a question of Lelek if the product of two confluent maps is again confluent if both ranges are treelike.

If X and Y are compacta and Y is locally connected the sets of monotone surjections and of confluent maps of X to Y are closed in the space of all maps of X to Y (with the compact open topology). If X and Y are metric compacta then the set of weakly confluent maps of X to Y is closed (Mackowiak, 1976).

Confluent maps of compacta preserve 1-dimensional algebraic properties. If $f : X \to Y$ is a confluent map of compacta and $g : Y \to G$ is a map to a graph G then $g \circ f$ inessential (i.e., homotopic to a constant map) implies g is inessential [3]. In particular, confluent images of treelike continua are treelike. Hence, there is no confluent map of an arc to a circle. It is an important open question whether confluent images of arclike continua are arclike.

A map $f : X \to M$ where M is a manifold is said to be **AH-essential** if f homotopic to $g : X \to M$ relative to the boundary of M implies g is a surjection. AH-essential maps detect covering dimension. In particular, a Tychonov space X has covering dimension at least n if and only if there exists an AH-essential map of X onto the n-cell. Each AH-essential map of a compactum onto a connected manifold is weakly confluent (see also Marsh et al., 1989). We get Mazurkiewicz's theorem that each compactum of dimension at least 2 contains an indecomposable continuum since each continuum that maps onto a nondegenerate indecomposable continuum must itself contain a nondegenerate indecomposable continuum.

It is known that monotone open maps and even cell-like maps can raise dimension of compacta. For example the 1-dimensional Menger curve can be mapped by a monotone and open map onto any locally connected metric continuum (Wilson, 1972). However, weakly confluent maps cannot raise the dimension of continua embeddable in 2-manifolds (because no 2-manifold contains a subcontinuum which maps onto a non-planar ***solenoid***, i.e., inverse limit of an inverse sequence of circles with open bonding maps of degree at least 2).

In general, weakly confluent maps are much weaker than confluent maps. However, if the domain and range are nice and have enough general position then weakly confluent maps carry the same information as open maps. In particular, if $f:M \to Y$ is a map of a compact, connected, PL, n-manifold, with $n \geqslant 3$ onto an ANR Y with the **disjoint arcs property** (i.e., each pair of maps of an arc into Y can be approximated by maps with disjoint images) then the following three conditions are equivalent: (i) f is homotopic to a weakly confluent map, (ii) f is homotopic to an open map, and (iii) the image under f of the fundamental group of M has finite index in the fundamental group of Y (Grispolakis et al., 1986).

A continuum Y is in Class(C) (respectively Class(W)) if each continuous surjection $f:X \to Y$ of each continuum X is confluent (respectively weakly confluent). Class(C) is the class of hereditarily indecomposable continua. Characterizations of Class(W) are a little more complicated. The metric continuum Y is in Class(W) iff whenever Y is embedded in a metric continuum X and $\{Y_i\}$ converges to Y in $C(X)$ then $\{C(Y_i)\}$ converges to $C(Y)$. The Class(W) is large and includes many of the continuum theorists' favorite examples: arclike continua, the Case-Chamberlin curve and nonplanar solenoids [3].

A subcontinuum K is a **terminal subcontinuum** in a continuum Y if whenever $L \in C(Y)$ meets K and $L \not\subset K$ then $K \subset L$. If $K \in C(Y)$ is terminal in a continuum Y and $f:X \to Y$ is a continuous surjection where X is also a continuum then each component of $f^{-1}(K)$ is mapped onto K. So f is confluent at K. The continuum Y is in Class(C) if and only if each of its subcontinua is terminal. There is an analogous characterization of subcontinua which are "W-embedded" in a continuum Y (Grispolakis et al., 1986).

An ***Absolute Retract*** AR (**Approximative Absolute Retract** AAR) is a metric continuum X such that if $X \subset Y$ where Y is a metric space (and $\varepsilon > 0$) then there is a map $f:Y \to X$ such that (f moves no point of X more than ε) $f(x) = x$ for each $x \in X$. We say X is an **Absolute Terminal Retract** ATR if X is a retract of each continuum Y in which X is embedded as a terminal subcontinuum. Charatonik, Charatonik and Prajs have recently shown that AANRs are exactly the ATRs. ARs for the class of tree-like continua turn out to be AAR so they have the fixed-point property. Continua which are inverse limits of finite trees with confluent bonding maps (e.g., cone over the Cantor set and Knaster's simplest indecomposable continuum) are ARs for treelike continua. Mackowiak (1984) showed that the pseudo-arc (i.e., hereditarily indecomposable, metric, arclike continuum) is AR for the class of hereditarily indecomposable continua.

In 1951 Bing showed that most metric continua are hereditarily indecomposable. Pseudoarcs form a dense G_δ-set in the hyperspace of subcontinua of R^n, $n \geqslant 2$. Krasinkiewicz (1996) called a continuum X a **free continuum** if each map of a metric continuum into X can be approximated by maps with hereditarily indecomposable fibers. Levin [6] showed that the interval is free, in order to prove that $C(X)$ is infinite-dimensional for each metric continuum X of dimension at least 2. This answered a question that had stood more than fifty years. Song et al. (2000) have shown that each compact polyhedron is free. It is an open question whether each ANR is free.

The **pseudo-arc** is the topologically unique, hereditarily indecomposable, metric, arclike continuum. It is also homogeneous and homeomorphic to each of its non-degenerate subcontinua. M. Smith (1985) constructed a non-metric, hereditarily indecomposable, arclike continuum which is homogeneous and homeomorphic to each of its non-degenerate subcontinua.

Each metric continuum of dimension at least two contains a non-degenerate hereditarily indecomposable continuum. Smith (1988) has shown that there are continua X such that no product of copies of X contains a non-degenerate hereditarily indecomposable continuum.

The **composant** of a point p in a continuum X is the union of all proper subcontinua of X which contain p. Metric, non-degenerate, indecomposable continua contain $\mathfrak{c}$ composants. Bellamy (1973) has constructed non-metric, indecomposable continua with one composant. Smith (1978) has constructed indecomposable continua with arbitrarily large numbers of composants. On the other hand, Bellamy (1973) showed that each continuum connected by generalized arcs is decomposable. Smith has shown that there is an indecomposable non-metric continuum such that each of its proper subcontinua is metrizable.

In [4] Hart, van Mill and Pol use Krasinkiewicz and Minc's notion of crooked maps to the interval to give unified constructions of a variety of hereditarily indecomposable continua. They use order theoretic methods to get results for hereditarily indecomposable continua of uncountable weight from corresponding results for metric hereditarily indecomposable continua.

Several topologists have recently been studying the properties of maps between hyperspaces induced by maps between metric compacta. One result of W.J. Charatonik is the following: If $f:X \to Y$ is a confluent map of continua and Y has the arc approximation property (e.g., Y is locally connected) then $C(f):C(X) \to C(Y)$ is confluent.

Inverse limits of even very simple objects are not at all well understood. De Man (1995) showed that any two composants (in this situation maximal one to one images of the real line) of any two nonplanar solenoids are homeomorphic. There has been considerable recent work done on inverse limits of inverse sequences of arcs with one bonding map which is a PL map with one interior extreme point. In Ingram [5] it is shown when such an inverse limit contains an indecomposable continuum. Raines and Kailhofer have independently obtained a classification of such inverse limits in case critical points of the bonding map are periodic.

If X and Y and are nondegenerate metric continua and if Y is locally connected there is a map of X onto Y by Urysohn's Lemma and the Hahn–Mazurkiewicz Theorem. If Y is not locally connected then it may be problematic to determine if there is a map of X onto Y. If Y admits an essential map to a graph G (e.g., Y is the compactification of the halfline $[0, \infty)$ spiralling counterclockwise onto a circle and the map is the projection of Y to the limit circle) then unbounded spiralling of Y measured in the universal cover of G is an obstruction to map X onto Y. Cook [1] has used this technique to construct a rigid continuum, i.e., a continuum whose only selfmaps are the identity map and the constant maps. If Y is treelike then one may resort to the patterns that sequences of open covers (with small mesh) of Y follow in a given cover of Y. Oversteegen [10] has used this technique to recognize arclike continua among the treelike continua. Bellamy has shown [7] that there is an uncountable family $\mathcal{C}$ of treelike continua no one of which maps onto any other. His continua are spirals of the halfline $[0, \infty)$ onto the wedge of three arcs. In fact no metric continuum maps onto uncountably many members of $\mathcal{C}$. In the nonmetric setting the situation is very different. Dow and Hart [2] showed that the Čech–Stone remainder $[0, \infty)^*$ of the halfline $[0, \infty)$ maps onto every continuum of weight at most ω_1. Obersnel (2000) extended their result by proving that $[0, \infty)^*$ has this property hereditarily.

A compactum X is said to be **ordered** if it admits a linear order such that the ***order topology*** coincides with the given topology. If it is, in addition, connected then we speak of an **ordered continuum** or an **arc** (tacitly assuming the space in nondegenerate). Each ordered compactum embeds in an arc. Arcs are locally connected and are characterized as nondegenerate continua having exactly two non-**cut point**s, i.e., points that do not disconnect it. A space is said to be **arcwise connected** if for any two points one can find an arc that contains them both.

Each metric arc is homeomorphic to the closed unit interval. The Cantor set is an ordered metric compactum and its continuous images are exactly all of the metric compacta. The classical Hahn–Mazurkiewicz Theorem characterized the continuous images of the closed unit interval as the locally connected metric continua. It was not until 1960 that it was realized that the classes IOA (respectively IOC) of continuous images of arcs (respectively ordered compacta) are extremely restricted outside the metric setting. Mardešić (1960) showed that arc-connectedness is preserved by continuous maps, hence, not all locally connected continua are in IOA. There even exists a locally connected, nonmetric continuum (e.g., Gruenhage, 1985) that contains no proper, nondegenerate, locally connected subcontinua.

Treybig (1965) showed that for spaces in IOC weight equals density. Each locally connected continuum in IOC is in IOA [8]. A continuum is a **cyclic continuum** if it contains no cut point. A **cyclic element** of a continuum X is a maximal cyclic subcontinuum of X. Cornette (1974) showed that a locally connected continuum is in IOA iff each of its cyclic elements is in IOA.

A ***dendron*** is a continuum obtained from a tree by replacing some edges in the tree by nonmetric arcs. A continuum X is said to be **approximated by dendrons** if it contains a family $\mathcal{D}$ of dendrons directed by inclusion whose union is dense in X and such that each member of $\mathcal{D}$ has only "short" branches.

A **T-set** in a continuum X is a closed subset A such that each component of $X \setminus A$ has an exactly two point boundary. Nikiel [8] showed that each compact metric subset of a cyclic continuum X in IOA is contained in a metric T-set. In particular, if X is a non-metric IOA then some two point subset disconnects X. A cyclic continuum X can be **approximated by T-sets** if there is a special increasing sequence $\{A_n\}$ of T-sets whose union is dense in X.

In [8] Nikiel building on work of Treybig and Ward proved the Fundamental Theorem for the class IOA. Namely, for a locally connected continuum X the following are equivalent: (i) X is in IOA, (ii) X can be approximated by dendrons, (iii) each cyclic element of X can be approximated by T-sets.

Nikiel's theorem enabled the study of IOAs by inverse sequences and was the key ingredient in the solution of most of the outstanding problems about the class IOA. Let P be a topological property. A space X is said to be ***rim-P*** if X has a base of open sets whose boundaries have P.

Grispolakis et al. (1993) showed that each separator of an IOA X between two disjoint closed sets A and B contains a metric separator of X between A and B. In particular, IOAs are rim-metrizable. Nikiel et al. (1991) showed that for each X in IOC the three usual dimension functions coincide, i.e., $\dim(X) = \operatorname{ind}(X) = \operatorname{Ind}(X)$. Moreover, $\dim(X) > 1$ implies $\dim(X) = \dim(Z)$ for some compact, metric subset Z of X. It is a result of Simone that a continuum in IOA that contains no non-degenerate metric subcontinuum is rim-finite. Bula et al. (1992) showed that each X in IOC is ***supercompact***. Very little is known about homogeneous, non-metric continua. However, if X is a homogeneous, non-metric continuum in IOA then X is a simple closed curve, i.e., each two point set disconnects X (Nikiel et al., 1993). In [9] it is shown that if $\mathcal{S} = (X_n, f_n)$ is an inverse sequence where each X_n is in IOA and each f_n is a monotone surjection then the inverse limit of $\mathcal{S}$ is again in IOA. Loncar has given extensions of this result to more general inverse systems.

Arcs are of course ***rim-finite***. L.E. Ward (1976) and Pearson (1975) showed that rim-finite continua are in IOA. In [9] it is proved that each one-dimensional IOA X is homeomorphic to the inverse limit of an inverse sequence (X_n, f_n) where each X_n is rim-finite and each f_n is a monotone surjection. It is known that each metric continuum is the monotone image of a one-dimensional metric continuum. It is an open question whether each continuum in IOA is the monotone image of a one-dimensional continuum in IOA. Rim-finite continua are hereditarily locally connected, i.e., all connected subsets are locally connected. Nikiel (1989) showed that each hereditarily locally connected continuum is in IOA and also rim-countable. A set Y is ***scattered*** if each closed non-empty subset has an isolated point. Each compact countable set is scattered. It is known that each rim-scattered IOA is rim-countable. Drozdovskii and Filippov (1994) have shown there is a locally connected, rim-metrizable and rim-scattered continuum which is not rim-countable. Drozdovskii (1999) constructed a locally connected rim-countable continuum which is not arc-connected.

Nikiel et al. (1995) have shown that arbitrary continuous functions do not preserve rim-metrizability of locally connected continua. However, the locally connected weakly confluent image of a rim-metrizable continuum is rim-metrizable (Tuncalı, 1991). Mardešić (1962) showed that light maps of locally connected continua preserve weight. Tuncalı (1991) obtained the same conclusion but replaced local connectedness in the domain by rim-metrizability. Treybig showed that if an IOC X contains the product $A \times B$ of two infinite compacta then A and B are metrizable. Tuncalı (1990) proved that the continuous image of a locally connected, rim-scattered continuum does not contain the product of a non-metric compactum and a perfect set. Also, if Z is the weakly confluent image of a rim-metrizable continuum then Z does not contain the product of a non-metric compactum with a non-degenerate continuum. It seems that rim-properties deserve further study. In particular what properties of IOAs carry over to continuous images of locally connected rim-metrizable continua?

Tymchatyn and Yang (2003) consider non-compact spaces with very strong separation properties. Let $(X, \mathcal{T})$ be a connected Hausdorff space with the property $\mathcal{P}$: Each separator of X between $a, b \in X$ contains a finite separator between a and b. Then there is a coarser topology $\mathcal{F}$ on X such that $(X, \mathcal{F})$ embeds in a continuum Y which has $\mathcal{P}$. (Note that X as a subspace of Y is rim finite while in the original topology X need not be locally connected or even finite dimensional.)

The next three problems go back to the 1930s and are unsolved even in the case of locally connected metric continua. They are all combinatorial and involve local cut points. Whyburn (1942) studied the distribution of local cut points in metric continua as the 0-dimensional case of a general "cyclic" element theory.

A surjective map $f : X \to Y$ is said to be ***irreducible*** if no proper closed subset of X maps onto Y. Ward (1977) showed that each locally connected metric continuum is the irreducible image of some metric dendron. Ward's proof and the method of T-set approximation can be used to show that each continuum in IOA is the irreducible image of some dendron. However, it is still an open question as to what locally connected metric continua are irreducible images of an arc. Euler showed that each connected graph with at most two odd vertices is the irreducible image of an arc. Harrold 1940 showed that each locally connected metric continuum X whose non-local cut points are dense in X is an irreducible image of an arc. Bula et al. (1994) have studied this problem via T-sets.

Menger's second n-arc theorem asserts that if in the complete, locally connected, metric space X no set with fewer than n points separates X between two points a and b then X contains n independent arcs from a to b, i.e., each pair of these arcs meet precisely in the set $\{a, b\}$. Whyburn has shown that if X is a complete, connected, locally connected, metric space with no local cut-points then each pair of points of X is contained in a Cantor set of independent arcs. However, if X contains local cut points then it an open question under what conditions X contains infinitely many independent arcs between two points (see F.B. Jones, 1980).

If X is a locally connected metric continuum which contains no local cut points then it is a theorem of Whyburn (1931) that each Cantor set in X is contained in an arc. Give necessary and sufficient conditions that a metric locally connected continuum have each Cantor set contained in an arc.

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f-3 Dimension Theory (General Theory)

The paper is devoted to Dimension theory of non-metrizable spaces. The case of metrizable ones is considered in the separate article "Dimension theory (metrizable spaces)" written by E. Pol.

Dimension theory is a very specific part of General Topology connected with both Geometry (by means of comparisons of the topological dimension with geometric one) and Algebraic Topology (by means of the cohomological dimension).

Dimension theory was being created during about the last 90 years. The definition of the topological dimension was not a simple matter. The first steps to the "correct" definition were made by H. Poincaré (1903, 1912), L.E.J. Brouwer (1913) and H. Lebesgue (1911). The first important progress in Dimension theory was obtained by Brouwer (1911) and Lebesgue (1911, 1921) who proved that n-cube and m-cube are not homeomorphic if $n \neq m$.

The first "correct" definition of the topological dimension was presented by P. Urysohn (1922) and K. Menger (1923) in the form of the small inductive dimension ind (see below). Their definition and results laid the foundations of Dimension theory.

The development of Dimension theory was very complicated and very many topologists participated in this work. A rather detailed information about the development of Dimension theory for the cases of ***normal*** and ***Tychonoff*** spaces may be found in the historical and bibliographic notes of [E] and [2]. Concerning the case of general spaces see [7, 11].

Below 'space', 'map' and 'compactum' are used instead of '***topological space***', '***continuous map***' and '***compact Hausdorff*** space', respectively. All ***regular***, normal and ***paracompact*** spaces considered are assumed to be Hausdorff.

Dimension theory assigns to a space either an integer $\geqslant -1$ (in this case the space is called **finite-dimensional**) or a symbol ∞ (in this case the space is called infinite-dimensional) which is named the dimension of the space. There are three basic approaches to the definition of the dimension of a space which give three dimensions of a space – the covering dimension dim (Lebesgue, Urysohn, Čech), the large inductive dimension Ind (Poincaré, Brouwer, Urysohn, Menger, Čech) and the small inductive dimension ind (Urysohn, Menger). In the class of ***separable metrizable*** spaces (i.e., in the case of the classical dimension theory) all three these approaches are equivalent. In the case of general (non-metrizable) spaces they are essentially different. The main goals of Dimension theory of general spaces were and are: (1) the extension of the basic definitions and theorems of the classical dimension theory as far as possible (in particular, by means of a modification of correspondent definitions and by means of the introduction of new dimension-like invariants) with constructing proper counterexamples to clarify the limits of such extension; (2) the study of new dimensional effects arising only in general spaces (for example, factorization theorems (see below) or relations between dimensional properties in various classes of spaces). To this moment these goals are reached on the whole but many interesting (and sometimes simply formulated but difficult) problems remain open.

We begin with the definition of the main (the most interesting outside the class of metrizable spaces) dimension – the covering dimension.

A family λ of subsets of a set has **order** n, $n = -1, 0, 1, \dots,$ if n is the largest integer such that λ contains $n + 1$ elements with non-empty intersection.

Recall that a subset O of a space X is called **functionally open** (or a **cozero set**) if $X \setminus O$ is **functionally closed** ($=$ a **zero set**), i.e., there exists a map $f : X \to \mathbb{R}$ such that $X \setminus O = f^{-1}(0)$. Below "FO" will be used instead of "functionally open".

The **covering dimension** – also called **Čech–Lebesgue dimension** – $\dim X$ of a space X is equal either to the least integer n such that every finite FO (i.e., consisting of FO sets) ***cover*** of X has a finite FO ***refinement*** of order $\leqslant n$ or to ∞, if there is no such n. (Evidently, $\dim X = -1$ iff $X = \emptyset$.)

For ***Tychonoff space***s this definition was suggested by M. Katětov (1950) and, without changing, was extended to arbitrary spaces by K. Morita (1975). Katětov's definition is a slight modification of Čech's definition of dim for normal spaces. Čech's definition may be obtained from the one given above by replacing FO covers with open covers. In the class of normal spaces both definitions are equivalent. Moreover, in this class, $\dim X \leqslant n$ iff every ***locally finite*** open cover of X has an open refinement of order $\leqslant n$. (Hence, for a paracompact space X, $\dim X \leqslant n$ iff every open cover of X has an open refinement of order $\leqslant n$.) For arbitrary spaces this result may be reformulated in the following way: $\dim X \leqslant n$ iff any locally finite FO cover of X has a locally finite FO refinement of order $\leqslant n$.

We need the following notions to formulate the basic characterizations of dim: (a) for a cover ε of a space X, a map $f : X \to Y$ is called an ε**-map** if for any $y \in Y$ there exists its neighbourhood Oy such that $f^{-1}[Oy]$ is contained in an element of ε; (b) for two disjoint subsets A and B of a space X, a closed subset C of X is called a **partition** between A and B (in X) if $X \setminus C$ is the union of two disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

For any space X the following conditions are equivalent: (1) $\dim X \leqslant n$; (2) (the theorem on ε-maps) for any finite FO cover ε of X there exists an ε-map to (even onto) a compact ***polyhedron*** of (geometric) dimension $\leqslant n$; (3) (the theorem on essential maps) for any map f of X to an $(n+1)$-cube I^{n+1} there exists a map $g : X \to S^n$, where S^n is the boundary of I^{n+1}, such that $gx = fx$ for all $x \in f^{-1}[S^n]$; (4) (the theorem on partitons) for any disjoint pairs of functionally closed in X sets A_i and B_i, $i = 1, \ldots, n+1$, there exist functionally closed partitions C_i between A_i and B_i with $\bigcap\{C_i : i = 1, \ldots, n+1\} = \emptyset$; (5) for any finite (even locally finite) FO cover ε of X, there exists a FO refinement $\delta = \bigcup\{\delta_i : i = 1, \ldots, n+1\}$, where all δ_i are ***discrete***. For normal X: in (2), ε may be supposed open (instead of FO), (3) may be reformulated in the following way: (3_{nor}) (the theorem on an extension of maps to n-sphere) for any closed in X set A and its map f to S^n there exists a continuous extension of f to X and, in (4), A_i, B_i and C_i may be taken closed. Points (2) and (3) express P.S. Alexandroff's geometrization of the notion of dimension (he obtained his correspondent results for metric compacta in 1926, 1930); point (3_{nor}) is a "bridge" between Cohomological Dimension Theory and Algebraic Topology.

The **large inductive dimension – Brouwer–Čech dimension** – $\operatorname{Ind} X$ (respectively, the **small inductive dimension – Menger–Urysohn dimension** – $\operatorname{ind} X$) for a normal (respectively, for a regular) space X:

(a) equals to -1 iff X is empty;
(b) is not greater than n for an integer $n \geqslant 0$ if for any closed set in X (respectively, for any point of X) and any its neighbourhood U there exists its neighbourhood V such that $V \subset U$ and $\operatorname{Ind} \operatorname{bd} V \leqslant n$ (respectively, $\operatorname{ind} \operatorname{bd} V \leqslant n$);
(c) $\operatorname{Ind} X = n$ (respectively, $\operatorname{ind} X = n$) if $\operatorname{Ind} X \leqslant n$ (respectively, $\operatorname{ind} X \leqslant n$) and $\operatorname{Ind} X > n-1$ (respectively, $\operatorname{ind} X > n-1$), i.e., $\operatorname{Ind} X \leqslant n-1$ (respectively, $\operatorname{ind} X \leqslant n-1$) does not hold;
(d) $\operatorname{Ind} X = \infty$ (respectively, $\operatorname{ind} X = \infty$) if $\operatorname{Ind} X > n$ (respectively, $\operatorname{ind} X > n$) for any integer n.

For Ind point (b) may be reformulated in the following way: (b′) for any disjoint pair of closed in X sets A and B there exists a partition C between A and B such that $\operatorname{Ind} C \leqslant n$. The definition of Ind for normal spaces was formulated by Čech in 1931. Note that $\dim X$ and $\operatorname{Ind} X$ express global properties of a space and ind reflects the behavior of it at its points. Beyond the class of separable metrizable spaces, ind has bad properties and so, mainly, its connections with dim and Ind are interesting here.

Always $\dim X \leqslant \operatorname{Ind} X$, $\operatorname{ind} X \leqslant \operatorname{Ind} X$ and $\dim X = 0$ implies $\operatorname{Ind} X = 0$ for any normal space X; $\dim X = 0$ implies $\operatorname{ind} X = 0$ for any regular space X. (See also RELATIONS BETWEEN dim, ind AND Ind below).

1. The sum theorem

A family λ of subsets of a space X is called **uniformly locally finite** (respectively, **uniformly locally countable**) if there exists a ***locally finite*** FO cover μ such that every element of μ intersects a finite (respectively, countable) number of elements of λ only. A family λ in X is called **σ-uniformly locally finite** (respectively, **σ-uniformly locally countable**) if λ is the countable union of uniformly locally finite (respectively, uniformly locally countable) subfamilies. Each ***σ-locally finite*** (respectively, ***σ-locally countable***) family of subsets of any paracompact space is σ-uniformly locally finite (respectively, countable).

(Σ) Let X be a space, λ a cover of X and $\dim L \leqslant n$ for all $L \in \lambda$. Then $\dim X \leqslant n$ in each of the following cases:

(1) λ is σ-locally finite and all $L \in \lambda$ are FO;
(2) λ is σ-uniformly locally countable (in particular, λ is countable) and all $L \in \lambda$ are ***C^*-embedded*** in X;
(3) X is paracompact, λ is σ-locally countable and closed (i.e., all $L \in \lambda$ are closed in X);
(4) X is normal, λ is σ-locally finite (in particular, countable) and closed;
(5) X is ***weakly paracompact*** normal, λ is locally countable and closed;
(6) X is normal, λ consists of two elements one of which is closed in X.

(Point 6. follows from the more general result: if X is normal, A is closed in X, $\dim A \leqslant n$ and $\dim B \leqslant n$ for any closed in X set $B \subset X \setminus A$, then $\dim X \leqslant n$.) It is not known whether $\dim X \leqslant n$ for Tychonoff X and for locally finite λ such that all its elements are C^*-embedded in X.

There exists a Tychonoff space X with $\dim X > 0$ such that it is the union of two zero sets F_i with $\dim F_i = 0$, $i = 1, 2$. The sum theorem for Ind and ind is not true even in the following strong form: there exists a compactum L such that $\operatorname{Ind} L = \operatorname{ind} L = 2$, $L = L_1 \cup L_2$, L_i is closed in L and $\operatorname{Ind} L_i = 1$, $i = 1, 2$. There exists a metrizable space R with $\operatorname{ind} R = 1$ which is the union of two closed subsets R_i with $\operatorname{ind} R_i = 0$, $i = 1, 2$.

The sum theorem for Ind is true in special ***hereditarily normal*** spaces. A Hausdorff space X is called **strongly hereditarily normal** if for any pair of its separated subsets A and B (i.e., $A \cap (\operatorname{cl} B) = \emptyset = (\operatorname{cl} A) \cap B$) there exist their disjoint neighbourhoods U and V such that both of them are the unions of ***point-finite*** families of open ***F_σ-sets***. All strongly hereditarily normal spaces are hereditarily normal. ***Perfectly normal*** and hereditarily ***paracompact*** (and even hereditarily ***weakly paracompact*** hereditarily normal) spaces are strongly hereditarily normal. If X is strongly hereditarily normal, then points 4 and 5 in (Σ) are true for Ind instead of dim.

For an arbitrary hereditarily normal space X we have $\dim X \leqslant n$ if there exists a closed in X set A such that $\dim A \leqslant n$ and $\dim(X \setminus A) \leqslant n$. The same assertion is true if we substitute Ind for dim.

2. The addition theorem

Let we have a space $X = A \cup B$, then

(1) $\dim X \leqslant m + n + 1$ if X and A are normal, $\dim A \leqslant m$ and $\dim C \leqslant n$ for any closed in X set $C \subset X \setminus A$;

(2) $\operatorname{Ind} X \leqslant \operatorname{Ind} A + \operatorname{Ind} B$ if X is normal and A, B are closed in X;
(3) $\operatorname{ind} X \leqslant \max\{\operatorname{ind} A, \operatorname{ind} B\} + 1$ if X is regular and A, B are closed in X;
(4) for $d = \dim$, Ind and ind we have $d(X) \leqslant m + n + 1$ if X is hereditarily normal and $d(A) \leqslant m$ and $d(B) \leqslant n$ (in particular, $d(X) \leqslant n$ if $X = \bigcup\{X_i\colon i = 0, 1, \ldots, n\}$ and $d(X_i) \leqslant 0$ for all i.)

3. The subset theorem (the monotonicity of the dimension)

The unique advantage of ind over dim and Ind consists in the following: $\operatorname{ind} A \leqslant \operatorname{ind} X$ for any regular space X and any its subset A. The similar inequality for Ind and any normal X is true only for any closed set A in X. There exists a hereditarily normal space P with subsets P_n such that $\dim P = \operatorname{Ind} P = 0$ but $\dim P_n = \operatorname{Ind} P_n = n$ for $n = 1, 2, \ldots, \infty$. Under the Continuum Hypothesis (CH), there exists a hereditarily normal compactum F with $\dim F = 0$ containing subsets U_{mn} with $\dim U_{mn} = m$, $\operatorname{Ind} U_{mn} = n$, $1 \leqslant m \leqslant n$. The inequalities $\dim A \leqslant \dim X$ and $\operatorname{Ind} A \leqslant \operatorname{Ind} X$ hold for any strongly hereditarily normal space X and its subset A.

There are some interesting cases of the monotonicity of dim in general spaces. For example, $(*)$ $\dim A \leqslant \dim X$ for any space X and its subset A if A is the union of σ-uniformly locally countable in A (in particular, countable) family of C^*-embedded in X sets. Hence $(*)$ holds for any normal X and an F_σ-set A in it.

Rather general conditions sufficient for $(*)$ are provided by the placement of an A in X. A subset A of a space X is called: (1) **z-embedded** if for any FO in A set V there exists an FO in X set U such that $V = A \cap U$; (2) **d-right** (respectively, **d-posed**) if, for any FO in A set V, there exists a σ-locally finite in A cover ν of V such that, for each $O \in \nu$, a FO set $U(O)$ in X may be found in such a way that O is closed–open in $A \cap U(O)$ (respectively, $V = A \cap U(O)$). Evidently: C^*-embeddedness $\Longrightarrow$ z-embeddedness $\Longrightarrow$ d-posedness $\Longrightarrow$ d-rightness; if a subset A of a space X is the union of a σ-locally finite in A family of FO in X sets, then A is d-posed. A space X is called **completely paracompact** if it is Tychonoff and for every open cover μ of X there exist ***star-finite*** open covers ν_i, $i \in \mathbb{N}$, of X such that $\bigcup\{\nu_i\colon i \in \mathbb{N}\}$ contains a refinement of μ. Every ***strongly paracompact*** space is completely paracompact. It is possible to show that every completely paracompact subset of a Tychonoff space is d-right. It may be proved that: (1) $(*)$ holds if A is d-right (in X); (2) for a subset A of a space X with $\dim X = 0$ we have $\dim A \leqslant 0$ iff A is d-right.

4. The product theorem

Let Π be the ***topological product*** of spaces X and Y, not both empty.

For FO sets $U \subset X$ and $V \subset Y$ the subset $U \times V$ of Π is called an **FO rectangle** in Π. A closed-open subset of an FO rectangle in Π is called an **FO rectangular piece** in Π. The product Π is called **piecewise rectangular** (respectively, **rectangular**) if each its finite FO cover has a σ-locally finite refinement consisting of FO rectangular pieces (respectively, FO rectangles). For Tychonoff X and Y the product Π is piecewise rectangular (respectively, rectangular) iff the subset Π of $\beta X \times \beta Y$ is d-right (respectively, d-posed), where βZ denotes the ***Čech–Stone compactification*** of a Tychonoff space Z. The product Π is piecewise rectangular, if, for example, X and Y are Tychonoff and $X \times Y$ is completely paracompact; Π is rectangular, in particular, in the following cases: X and Y are ***paracompact Σ-spaces*** (in particular, paracompact M- (or p-) spaces (for example, compact spaces)); X is metrizable and $X \times Y$ is normal (moreover, if in this situation a space Z has a ***perfect map*** onto X, then the product $Z \times Y$ is also rectangular); X is paracompact, Y is ***Lašnev*** and $X \times Y$ is normal; the projection of the product Π onto the factor X is closed (for example, Y is compact or Y is ***countably compact*** and X is a ***sequential*** space); Y is ***locally compact*** paracompact. The following assertions explain our consideration of (piecewise) rectangular products: (1) $(**)$ $\dim \Pi \leqslant \dim X + \dim Y$ if Π is piecewise rectangular (in particular, rectangular); (2) if $\dim X = \dim Y = 0$, then $\dim \Pi = 0$ iff Π is piecewise rectangular. Note that $(**)$ holds if Y is paracompact and is a countable union of locally compact closed subsets.

Let us add the following "exact" product theorems: the equality $\dim(X \times Y) = \dim X + \dim Y$ holds if (1) X is the n-cube, $n = 1, 2, \ldots$, (even an arbitrary ***CW-complex***) or (2) X is paracompact and is a countable union of locally compact closed subsets and $\dim Y = 1$. (See also RELATIONS BETWEEN dim, Ind AND ind.)

QUESTION. Is it true that $\dim(X \times Y) \leqslant \dim X + \dim Y$ for paracompact $X \times Y$ (even for $X = Y$)?

There are counterexamples for the product theorem for dim. Among them we have the following ones: (1) for every $n = 1, 2, \ldots, \infty$ there exists a separable and ***first-countable Lindelöf*** space X_n such that $\dim X_n = 0$, X_n^2 is normal but $\dim X_n^2 = n$; (2) there exist a separable metrizable space X and a separable first-countable Lindelöf space Y such that $\dim X = 0 = \dim Y$ but $\dim(X \times Y) = 1$. For Ind and ind such a kind of examples may be presented even in the class of compacta: there exist a compactum X with $\operatorname{Ind} X = \operatorname{ind} X = \dim X = 1$ (even a ***linearly ordered*** compactum X) and a compactum Y with $\operatorname{Ind} Y = \operatorname{ind} Y = \dim Y = 2$ such that $\operatorname{ind}(X \times Y) = 4 \leqslant \operatorname{Ind}(X \times Y)$. Note that the inequalities (1) $\operatorname{ind} X < \infty$, $\operatorname{ind} Y < \infty$ and (2) $\operatorname{Ind} Z < \infty$, $\operatorname{Ind} T < \infty$ for (1) regular spaces X and Y and for (2) paracompact M- (or p-) space Z and normal space $Z \times T$ imply the inequalities (1) $\operatorname{ind} X \times Y < \infty$ and (2) $\operatorname{Ind} Z \times T < \infty$ respectively.

We say that the **finite sum theorem** (FST) for Ind (respectively, for ind) holds in a space X if $\operatorname{Ind}(A \cup B) = \max\{\operatorname{Ind} A, \operatorname{Ind} B\}$ (respectively, $\operatorname{ind}(A \cup B) = \max\{\operatorname{ind} X, \operatorname{ind} Y\}$) for any closed in X sets A and B. For example, FST for Ind holds in X if X is strongly hereditarily normal or if X is normal and $\operatorname{Ind} X = 1$. We have $\operatorname{Ind}(X \times Y) \leqslant$

$\operatorname{Ind} X + \operatorname{Ind} Y$ if the product $X \times Y$ is piecewise rectangular, the space $X \times Y$ is normal and FST for Ind holds in X and Y (all this is true if, for example, (1) X is metrizable and Y is perfectly normal or (2) X and Y are paracompact Σ-spaces and either strongly hereditarily normal or $\operatorname{Ind} X = \operatorname{Ind} Y = 1$). If FST for ind holds in regular spaces X and Y, then $\operatorname{ind}(X \times Y) \leqslant \operatorname{ind} X + \operatorname{ind} Y$.

The inequality $\operatorname{ind}(X \times Y) \leqslant \operatorname{ind} X + \operatorname{ind} Y$ is true for any Tychonoff space X and a separable metrizable space Y but there exists a regular space M such that $\operatorname{ind} M = 2$ but $\operatorname{ind}(X \times I) = 4$, where $I = [0, 1]$.

Question. Let X be a compactum. Is it true that $\operatorname{Ind}(X \times I) \leqslant \operatorname{Ind} X + 1$?

5. Compactification theorems, universal compacta, factorization theorems for maps to compacta

The equality $\dim \beta X = \dim X$ holds for any Tychonoff space X and $\operatorname{Ind} \beta X = \operatorname{Ind} X$ is true for any normal space X. If FST for Ind in a normal space X holds, then $\operatorname{ind} \beta X = \operatorname{Ind} X$. There exists even metrizable space R such that $\operatorname{ind} R = 0$ but $\operatorname{ind} \beta R = 1$.

For any Tychonoff (respectively, normal) space X there exists a Hausdorff ***compactification*** cX of X such that $\dim cX = \dim X$ (respectively, $\operatorname{Ind} cX \leqslant \operatorname{Ind} X$) and $w(cX) = w(X)$, where $w(Z)$ denotes the ***weight*** of a space Z. The similar result for ind is true if $\operatorname{ind} X = 0$. There exists a perfectly normal space M such that $\operatorname{ind} M = 1$ but $\operatorname{ind} cM = \infty$ for every Hausdorff compactification cM of M. Add yet two results: for any normal space X and any family ν of closed sets in X with $|\nu| \leqslant w(X)$, (1) there exists a Hausdorff compactification cX such that $w(cX) = w(X)$ and $\dim(\operatorname{cl} F) \leqslant \dim F$ for any $F \in \nu$, (2) if, additionally, ν is locally finite, then we may suppose that $w(\operatorname{cl} F) = w(F)$ for any $F \in \nu$.

The previous results about dim and Ind are connected with the existence of "universal" compacta: for each integer $n \geqslant 0$ and infinite cardinal number τ there exists a compactum D^n_τ (respectively, I^n_τ) such that every Tychonoff (respectively, normal) space X with $\dim X \leqslant n$ (respectively, $\operatorname{Ind} X \leqslant n$) and $w(X) \leqslant \tau$ has a ***topological embedding*** in D^n_τ (respectively, I^n_τ). (As D^0_τ may be taken the ***Cantor cube*** of weight τ.) The proofs of these two results are based on the use of the following two factorization theorems: for any space (respectively, normal space) X and its map f to a compactum Z, there exists a compactum Y and maps $g : X \to Y$, $h : Y \to Z$ such that $f = h \circ g$, $\dim Y = \dim X$ (respectively, $\operatorname{Ind} Y \leqslant \operatorname{Ind} X$) and $w(Y) = w(Z)$. The same assertion is true for ind if X is a compactum.

6. The covering dimension and inverse systems (in particular, inverse sequences)

If a space X is ***Dieudonné complete*** and $\dim X \leqslant n$, then (#) X is the ***limit*** of an ***inverse system*** $\{X_\alpha, \pi^\beta_\alpha; A\}$ consisting of metrizable spaces X_α with $\dim X_\alpha \leqslant n$.

Now let a space X be the limit of an inverse system $S = \{X_\alpha, \pi^\beta_\alpha; A\}$. This system is called **piecewise rectangular** (**rectangular**) if for each FO finite cover μ of X there exists its σ-locally finite refinement ν such that, for any $V \in \nu$, an FO set $U(V)$ in some X_α, $\alpha = \alpha(V)$, may be found such that V is closed-open in $\pi_\alpha^{-1} U(V)$ (respectively, $V = \pi_\alpha^{-1} U(V)$). If all X_α are Tychonoff and X is completely paracompact, then S is piecewise rectangular; if all X_α are paracompact and all π^β_α are perfect then S is rectangular. It is proved that: (1) if S is piecewise rectangular and $\dim X_\alpha \leqslant n$ for all α, then $\dim X \leqslant n$; (2) if $\dim X_\alpha \leqslant 0$ for all α, then $\dim X \leqslant 0$ iff S is piecewise rectangular. Hence for any completely paracompact (in particular, compact) space X condition (#) is sufficient for the inequality $\dim X \leqslant n$. An inverse sequence $S = \{X_i, \pi^{i+1}_i; \mathbb{N}\}$ with the limit X is rectangular in the following cases: (1) all X_i are perfectly normal; (2) X is countably compact, all X_i are normal and all π^{i+1}_i are onto maps; (3) X is ***countably paracompact***, all X_i are normal and either all π^{i+1}_i are open or all of them are closed.

We conclude this point with two "negative" results: (1) there exists a compactum X with $\dim X = \operatorname{ind} X = \operatorname{Ind} X = 1$ which is not the limit of any inverse system consisting of 1-dimensional polyhedra; (2) for any $n = 1, 2, \ldots, \infty$ there exists an inverse sequence $S_n = \{X_i(n), \pi^{i+1}_i(n); \mathbb{N}\}$ with limit X_n such that X_n is normal, all $X_i(n)$ are ***Lindelöf***, first countable, separable with $\dim X_i(n) = 0$ but $\dim X_n = n$.

7. Cantor manifolds, intermediate dimensions

A compactum X with $\dim X = n \geqslant 1$ is called an ***n*-dimensional Cantor-manifold** if $\dim P \geqslant n - 1$ for any closed in X set P separating X (i.e., if $X \setminus P$ is not ***connected***). Every connected compactum X with $\dim X = 1$ is a 1-dimensional Cantor-manifold. Every compactum X with $\dim X = n \geqslant 1$ contains an n-dimensional Cantor-manifold. It follows from this that every compactum X with $\dim X = n$ has at least one ***component*** C with $\dim C = n$.

For any $n > 1$, there exists a first-countable compactum X_n with $\dim X_n = n$ such that for every its closed subset A either $\dim X = 0$ or $\dim A = n$ and so X_n has no closed subsets with intermediate dimensions between 0 and n. (Under the ***Continuum Hypothesis*** (CH), there are infinite compacta in which every closed subset is either finite or has infinite covering dimension.) This result reveals an essentially non-inductive nature of the covering dimension because, as it follows directly from the definitions: if $\operatorname{Ind} X = n$ (respectively, $\operatorname{ind} X = n$), $n = 1, 2, \ldots,$ then for any k between 0 and n there exists a closed set A_k in X with $\operatorname{Ind} A_k = k$ (respectively, $\operatorname{ind} A_k = k$).

8. Dimension and maps

For a ***closed map*** f of a normal space X onto a space Y we have (that Y is normal and): (a) $\dim X \leqslant \dim Y + \dim f$

(where the **dimension** $\dim f$ **of a map** $f: X \to Y$ is equal to $\sup\{\dim f^{-1}(y)\colon y \in Y\}$), if Y is weakly paracompact; (b) $\dim Y \leqslant \dim X + k$ if $|f^{-1}(y)| \leqslant k+1$ for all $y \in Y$ and $k = 1, 2, \ldots$. For any $n = 1, 2, \ldots$, there exists a perfect map f with $\dim f = 0$ of a normal, locally compact and countably compact space X_n with $\dim X_n = n$ onto the space ω_1 (with $\dim \omega_1 = 0$). There exists a method to reduce the situation of a closed maps between normal spaces to the situation of maps between metrizable compacta to obtain the assertions from points (a) and (b) (and their strengthens) (see [5]).

For every Tychonoff space $Y \neq \emptyset$, there exists a perfect map of a Tychonoff space $X = X(Y)$ with $\operatorname{ind} X = 0$, $w(X) \leqslant w(X)$ onto Y and, additionally, it may be supposed that $\dim X = 0$ if Y is completely paracompact.

Question. Is every paracompact space Y a perfect image of a paracompact space $X = X(Y)$ with $\dim X \leqslant 0$ and $w(X) \leqslant w(Y)$?

For every paracompact space Y with $\dim Y \geqslant 1$ there exists a perfect and ***open map*** onto Y of a paracompact space $Z = Z(X)$ with $\dim Z = 1$. If, additionally, Y is completely paracompact (in particular, compact), then we may suppose that $w(X) \leqslant w(Y)$.

9. Reducing the case of general spaces to the cases of simpler ones

The Tychonoff functor
For any space X there exist the Tychonoff space τX and the onto map $\tau_X : X \to \tau X$ (they are unique in the natural sense) such that for any map f of X to a Tychonoff space Y there exists a map $g : \tau X \to Y$ such that $f = g \circ \tau_X$. (Evidently, for any map $f : X \to Y$ the unique map $\tau f : \tau X \to \tau Y$ is defined such that $\tau f \circ \tau_X = \tau_Y \circ f$. The correspondence $X \to \tau X$ for any space X and $f \to \tau f$ for any map f is called the **Tychonoff functor**.) Often this allows to consider the Tychonoff space τX instead of a space X in the theory of the dimension dim because $\dim X = \dim \tau X$ and for any FO set O in X we have $\tau_X^{-1} \tau_X(O) = O$ and $\tau_X(O)$ is FO in τX.

The factorization theorem for maps to metrizable spaces
The following factorization theorem allows to consider metrizable spaces instead of arbitrary ones: for any space X and its map f to a metrizable space Z there exist a metrizable space Y and maps $g : X \to Y$ and $h : Y \to Z$ such that $f = h \circ g$, $\dim Y = \dim X$ and $w(Y) \leqslant w(Z)$.

10. Relations between dim, Ind and ind

Note that some correspondent information was given after the definition of Ind and ind.

For every completely paracompact (in particular, strongly paracompact, or Lindelöf, or compact) space X we have: $\dim X \leqslant \operatorname{ind} X$; the conditions $\operatorname{ind} X = i$ and $\operatorname{Ind} X_i = i$, $i = 0, 1$, are equivalent; if, additionally, X is strongly hereditarily normal, then $\operatorname{ind} X = \operatorname{Ind} X$.

For any integers l, m and n with $n \geqslant l \geqslant 0$ and $n \geqslant m > 0$ there exist normal spaces $X = X(l, m, n)$ with $\operatorname{ind} X = l$, $\dim X = m$, and $\operatorname{Ind} X = n$. For any m and n with $1 \leqslant m \leqslant n$ there exist first-countable (even perfectly normal under (CH)) compacta $X = X(m, n)$ with $\dim X = m$ and $\operatorname{ind} X = n$. There exist compacta F_n with $\dim F_n = 1$, $\operatorname{ind} F_n = n$ and $\operatorname{Ind} F_n = 2n - 1$, $n = 2, 3, \ldots$. Besides, there exists a compactum X with $\dim X = 1$, $\operatorname{ind} X = 2$ and $\operatorname{Ind} X = 4$.

QUESTION. Does there exists a compactum X_n with $\operatorname{ind} X_n = 2$, $\operatorname{Ind} X_n = n$ for any $n > 4$?

For any integer $m > 4$, there exists (under $(\diamondsuit)$) a normal, countably compact, separable and perfectly normal ***manifold*** $M = M(m)$ with $\operatorname{ind} M = 4$, $\dim M = m$, $\operatorname{Ind} M = m + 2$ ([4] contains a survey of results concerning dimensional properties of topological manifolds).

The equalities $\dim X = \operatorname{ind} X = \operatorname{Ind} X$ hold for any CW-complex X.

The equivalence of the following properties (all or some of them) was stated for some spaces X close (in some sense) to metrizable ones: (α) $\dim X \leqslant n$; (β) $\operatorname{Ind} X \leqslant n$; ($\gamma$) $\Delta X \leqslant n$ (i.e., there exists a closed map f of a paracompact space X_0 of dimension $\dim X_0 = 0$ onto X with $|f^{-1}(x)| \leqslant n + 1$ for all $x \in X$); (δ) X is the union of subspaces X_i with $\dim X_i \leqslant 0$, $i = 0, 1, \ldots, n+1$. For example, (α)–(δ) are equivalent for μ-spaces. (Note that a space is σ-metrizable if it is the countable union of closed metrizable subspaces and μ-spaces are subspaces of the countable products of paracompact σ-metrizable spaces. All μ-spaces are paracompact and perfectly normal and all Lašnev spaces are μ-spaces.)

If a normal space X has a closed map f to a metrizable space (even to a μ-space) with $\dim f \leqslant 0$, then (∗∗∗) $\dim X = \operatorname{Ind} X$. This result is used in obtaining the following ones concerning ***topological group***s (we consider only separated topological groups, i.e., their spaces are Tychonoff). A topological group G is called **almost metrizable** if there exists a compactum $C \neq \emptyset$ in it and neighbourhoods O_i, $i \in \mathbb{N}$, of C such that for every its neighbourhood O there exists i with $O_i \subset O$. All ***Čech-complete*** (and so all locally compact and all compact) groups are almost metrizable. If a group G is almost metrizable and H is its ***closed*** subgroup, then for the ***quotient space*** $X = G/H$ (in particular, for G) we have (∗∗∗). Hence if G is locally compact (and so X is strongly paracompact), then $\dim X = \operatorname{ind} X = \operatorname{Ind} X$. The equality $\operatorname{ind} G = \operatorname{Ind} G$ is true for any Lindelöf Σ-group G (i.e., G is a continuous image of a Tychonoff space having a perfect map onto a separable metrizable space) and there exists a Lindelöf Σ-group G with $\dim G < \operatorname{ind} G = \infty$.

QUESTION. Does there exist an **algebraically homogeneous compactum** X (i.e., $X = G/H$ for some topological group G and its closed subgroup H) with different some of dimensions dim, ind and Ind?

There exist ***topologically homogeneous*** first-countable compacta C_n with $\dim C_n = 1$, $\operatorname{ind} C_n = n$, $n = 2, 3, \ldots$.

Recall that a compactum X is **Dugundji** if, for any paracompact space Y with $\dim Y = 0$, its closed subset F and a map $\varphi : F \to X$, there exists a continuous extension $\Phi : Y \to X$ of φ. All Tychonoff products of metrizable compacta and all compact G_δ-subsets of topological groups (in particular, all compact groups) are Dugundji compacta and the dimensions dim and ind coincide for them. But there exist Dugundji compacta D_n with $\dim D_n = 1$, $\operatorname{ind} D_n = n$, $n = 1, 2, \ldots$.

If X_i is a compactum with $\operatorname{Ind} X_i = 1$, $i = 1, \ldots, n$, then for the product Π of all X_i we have $\dim \Pi = \operatorname{ind} \Pi = \operatorname{Ind} \Pi = n$, $n = 2, 3, \ldots$. There exist a first-countable compactum X_n with $\dim X_n = \operatorname{ind} X_n = n$ and a metrizable compactum Y_n with $\dim Y_n = n$ such that $\dim X_n \times Y_n = \dim X_n^2 = 2n - 1 < 2n = \operatorname{ind} X_n \times Y_n \leqslant \operatorname{ind}\ X_n^2$, $n = 2, 3, \ldots$. For $n = 1, 2, \ldots$, there exists a subgroup G_n of $\mathbb{R}^{n+1}$ with $\dim G_n = \dim(G_n)^\omega = \dim(G_n)^\tau = n$ and $\operatorname{ind}(G_n)^\tau = \infty$ for any $\tau > \omega$. The groups $(G_n)^\tau$ are not normal. If we take the ***Σ-product*** $\Sigma(G_n)^\tau$ of τ copies of G_n for $\tau > \omega$, then the group $\Sigma(G_n)^\tau$ is a ***collectionwise normal Fréchet space*** with the ***Souslin property*** and with $\dim \Sigma(G_n)^\tau = n$, $\operatorname{ind} \Sigma(G_n)^\tau = \infty$.

11. The local dimension loc dim

For a space X, we shall write: $\operatorname{loc\,dim} X = -1$ iff $X = \emptyset$ and $\operatorname{loc\,dim} X = n$, for $n = 0, 1, 2, \ldots$, if every point $x \in X$ has a FO neighbourhood Ox with $\dim Ox \leqslant n$ and there exists at least one point $x \in X$ with $\dim Ox \geqslant n$ for every FO neighbourhood Ox of x. We write $\operatorname{loc\,dim} X = \infty$ if $\operatorname{loc\,dim} X \neq n$ for any $n = -1, 0, 1, \ldots$. For any normal X, we have $\operatorname{loc\,dim} X \leqslant n$ iff every point $x \in X$ has a neighbourhood Ox with $\dim(\operatorname{cl} Ox) \leqslant n$.

Evidently, $\operatorname{loc\,dim} X \leqslant \dim X$ for every space X and there exist normal, locally compact and countably compact spaces X_n with $\operatorname{loc\,dim} X_n = 0$, $\dim X_n = n$, $n = 1, 2, \ldots$. Under CH there exist hereditarily separable, perfectly normal, locally compact and locally countable (hence $\operatorname{loc\,dim} X_n = 0$) space X_n with $\dim X = n$, $n = 1, 2, \ldots$. If X is a normal and weakly paracompact space or is a space of a topological (not necessary separated) group, then $\operatorname{loc\,dim} X = \dim X$.

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f-4 Dimension of Metrizable Spaces

The Euclidean spaces R^n and R^m with $n \neq m$ are not homeomorphic, therefore their linear dimension is a topological invariant. This fundamental fact was proved by L.E.J. Brouwer in 1911. Two years later Brouwer gave a proof of the tiling principle of H. Lebesgue (stated by Lebesgue in 1911 with an incomplete proof): every finite family of closed sets of diameter less than 1 covering the n-cube $\mathbb{I}^n$ contains $n+1$ elements with non-empty intersection, while there exists a ***covering*** of $\mathbb{I}^n$ by closed sets of arbitrarily small diameter such that the intersection of every $n+2$ of them is empty.

Several years earlier, in 1903, H. Poincaré presented a viewpoint that most basic properties of dimension can be expressed in terms of cuts by sets of smaller dimension (B. Bolzano's papers from 1844 contain similar ideas, but were published only hundred years later). The inductive character of the dimension was a starting point for P.S. Urysohn and K. Menger, who independently laid down in the 1920s firm foundations for the modern dimension theory. The first efforts were concentrated on ***compact metrizable*** spaces, and subsequently, W. Hurewicz and L.A. Tumarkin extended the basic results to ***separable*** metrizable spaces. The ***paracompactness*** theorem of A.H. Stone made it possible in the 1950s to develop the dimension theory for non-separable metrizable spaces.

All spaces considered in this article are metrizable. The article by B.A. Pasynkov in this volume presents the dimension theory for general topological spaces. The reader is referred to [4] for the history of the emergence of dimension theory. The basic monographs on the dimension theory are [2, 7, 8, 14, 15] and [KI] (§25–27, 45); [7] provides also a comprehensive bibliography and rich historical information on the topic.

Menger and Urysohn introduced what is now called the **small inductive dimension** ind, defined as follows:

(a) $\operatorname{ind} X = -1$ if and only if $X = \emptyset$,
(b) $\operatorname{ind} X \leqslant n$, where $n = 0, 1, 2, \ldots$, if for every point $x \in X$ and each neighbourhood V of the point x in X there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\operatorname{ind} \operatorname{Bd} U \leqslant n-1$,
(c) $\operatorname{ind} X = n$ if $\operatorname{ind} X \leqslant n$ and it is false that $\operatorname{ind} X \leqslant n-1$,
(d) $\operatorname{ind} X = \infty$ if $\operatorname{ind} X \leqslant n$ does not hold for any $n \in \mathbb{N}$.

If M is a subspace of $\mathbb{R}^n$, then $\operatorname{ind} M = n$ if and only if the interior of M in $\mathbb{R}^n$ is non-empty.

There are two other dimension functions, which coincide with ind for metrizable separable spaces: the covering dimension dim, related to the Lebesgue tiling principle, and the large inductive dimension Ind, based on a notion of separation stronger than the one used in the definition of ind. The coincidence of these three dimensions is one of the cornerstones of the dimension theory of separable metrizable spaces.

To define dim, we first agree that the **order** of a family $\mathcal{U}$ of subsets of a space X, denoted by $\operatorname{ord}\mathcal{U}$, is the largest integer $n \in \{-1, 0, 1, 2, \ldots\}$ such that the family $\mathcal{U}$ contains $n+1$ sets with a non-empty intersection; if no such integer exists, then $\operatorname{ord}\mathcal{U} = \infty$.

The **covering dimension** dim (the Čech–Lebesgue dimension) is defined in the following way:

(a) $\dim X \leqslant n$, where $n = -1, 0, 1, 2, \ldots$, if every finite open cover of X has a finite open *refinement* of order $\leqslant n$,
(b) $\dim X = n$, if $\dim \leqslant n$ and it is false that $\dim X \leqslant n-1$,
(c) $\dim X = \infty$, if $\dim X \leqslant n$ does not hold for any $n \in \mathbb{N}$.

In the case of metrizable spaces, one can equivalently consider in (a) arbitrary covers.

The **large inductive dimension** Ind (the Brouwer–Čech dimension) is defined as follows:

(a) $\operatorname{Ind} X = -1$ if and only if $X = \emptyset$,
(b) $\operatorname{Ind} X \leqslant n$, where $n = 0, 1, 2, \ldots$, if for every closed set $A \subset X$ and each open set $V \subset X$ containing A there exists an open set $U \subset X$ such that $A \subset U \subset V$ and $\operatorname{Ind} \operatorname{Bd} U \leqslant n-1$,
(c) $\operatorname{Ind} X = n$ if $\operatorname{Ind} X \leqslant n$ and it is false that $\operatorname{Ind} X \leqslant n-1$,
(d) $\operatorname{Ind} X = \infty$ if $\operatorname{Ind} X \leqslant n$ does not hold for any $n \in \mathbb{N}$.

A **partition** L between two disjoint subsets A and B of X is a subset of X such that $X \setminus L$ is the union of two disjoint sets U and V open in X with $A \subset U$ and $B \subset V$. Condition (b) in the definition of Ind can be equivalently stated as follows:

(b') $\operatorname{Ind} X \leqslant n$, where $n = 0, 1, 2, \ldots$, if for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\dim L \leqslant n-1$.

For any metrizable X, $\operatorname{Ind} X = \dim X$ (the **Katětov–Morita Theorem**) and $\operatorname{ind} X \leqslant \operatorname{Ind} X$. There exist, however, non-separable ***completely metrizable*** spaces with $\operatorname{ind} X = 0$ and $\operatorname{Ind} X = 1$ (**Roy's Example** Δ). We refer the reader to [12, 13, 10] for more information about the gap between ind and Ind.

In the sequel, speaking about dimension we shall always mean the covering dimension dim.

In $\mathbb{R}^{2n+1}$ there is a compact n-dimensional space M_n^{2n+1}, the ***Menger Universal Space***, such that every separable metrizable n-dimensional space X embeds in M_n^{2n+1} (S. Lefschetz, H.G. Bothe, Menger for $n = 1$). In particular, any separable n-dimensional X embeds in $\mathbb{R}^{2n+1}$ (S. Lefschetz, G. Nöbeling, L. Pontrjagin and G. Tolstova) and X

has a metrizable ***compactification*** of the same dimension (Hurewicz).

Any n-dimensional compact space X contains an n-dimensional **Cantor manifold** Y, i.e., a subspace Y such that for every partition L in Y between any two points, $\dim L \geqslant n-1$ (Hurewicz and Menger, Tumarkin). In particular, the **dimensional kernel** $K(X)$ of an n-dimensional compact space X, i.e., the complement of the set of points in X which have arbitrarily small neighbourhoods with at most $(n-2)$-dimensional boundaries, is n-dimensional for any compact X (Menger). For any separable metrizable space X, $\dim K(X) \geqslant \dim X - 1$ (Menger). There exist, however, n-dimensional subsets X of $\mathbb{R}^{n+1}$ with $\dim K(X) = n-1$, i.e., **weakly n-dimensional space**s (S. Mazurkiewicz, W. Sierpiński for $n=1$); for a thorough discussion of this topic see [11].

Any compact set that is an irreducible partition between two points in $\mathbb{R}^n$ is an $(n-1)$-dimensional Cantor manifold (P.S. Alexandroff, P.S. Urysohn for $n=3$). If one removes from $\mathbb{R}^n$ an $(n-2)$-dimensional set M then any two points in $\mathbb{R}^n \setminus M$ can be joined by a ***continuum*** missing M (Mazurkiewicz).

An n-dimensional space may not contain any non-trivial continuum. Among striking examples of such 1-dimensional spaces are: the graphs of derivatives discontinuous on every non-trivial interval (B. Knaster and K. Kuratowski), the set of points in ***Hilbert space*** l^2 with all coordinates rational (**Erdös space**) and the group of the autohomeomorphisms of the ***Sierpiński Carpet*** (B. Brechner, L.G. Oversteegen and E.D. Tymchatyn). The celebrated ***Knaster–Kuratowski Fan*** also falls in this category. For each n there exist separable completely metrizable n-dimensional spaces X such that any two points of X can be separated by the empty partition (Mazurkiewicz, Sierpiński for $n=1$).

For a compact $X \subset \mathbb{R}^m$ one has $\dim X \leqslant n$ if and only if for every $\varepsilon > 0$ there exists a map $f : X \to \mathbb{R}^m$ onto a ***polyhedron*** of dimension $\leqslant n$ such that f is an **ε-translation** (i.e., the Euclidean distance between x and $f(x)$ is always less than ε), or equivalently, if X can be removed from any $(m-n-1)$-dimensional polyhedron in $\mathbb{R}^m$ by arbitrarily small ε-translations (Alexandroff). There exists, however, a 2-dimensional subset of $\mathbb{R}^3$ that can be mapped by ε-translations onto 1-dimensional polyhedra, for arbitrary $\varepsilon > 0$ (K. Sitnikov). This phenomenon is related to the metric dimension $\mu\dim$ we shall discuss later on. There is also a 2-dimensional compact subset of $\mathbb{R}^4$ that can be removed from any 2-dimensional affine subspace by arbitrarily small ε-translations (A.N. Dranishnikov).

Another important approximation of n-dimensional compact spaces by polyhedra is provided by the fact that any such space is the inverse limit of an inverse sequence of n-dimensional polyhedra (H. Freudenthal).

Basic general properties of dimension dim in the class of metrizable spaces (of arbitrary ***weight***) are:

(*the subspace theorem*) if $Y \subset X$ then $\dim Y \leqslant \dim X$,

(*the countable sum theorem*) if X is the union of a sequence of closed subspaces $F_1, F_2, \ldots$ with $\dim F_i \leqslant n$ for $i = 1, 2, \ldots$, then $\dim X \leqslant n$,

(*the locally finite sum theorem*) if X is the union of a ***locally finite*** family $\{F_s : s \in S\}$ of closed subspaces such that $\dim F_s \leqslant n$, then $\dim X \leqslant n$,

(*the addition theorem*) if X is the union of two arbitrary subspaces X_1 and X_2, then $\dim X \leqslant \dim X_1 + \dim X_2 + 1$,

(*the decomposition theorem*) if $\dim X \leqslant n$ then $X = \bigcup_{i=1}^{n+1} X_i$, where $\dim X_i \leqslant 0$ for $i = 1, 2, \ldots, n+1$,

(*the enlargement theorem*) for every subspace Y of X with $\dim Y \leqslant n$ there exists a G_δ-set Y^* in X such that $Y \subset Y^*$ and $\dim Y^* \leqslant n$ (which yields *the completion theorem*: for every X there exists a completely metrizable space $\widetilde{X}$ containing a dense subspace homeomorphic to X such that $\dim \widetilde{X} = \dim X$ and $\mathrm{weight}\widetilde{X} = \mathrm{weight}X$),

(*the Cartesian product theorem*) if X and Y are nonempty, then $\dim(X \times Y) \leqslant \dim X + \dim Y$,

(*the theorem on the dimension of inverse limits*) if X is the inverse limit of an inverse sequence of spaces X_i such that $\dim X_i \leqslant n$ for $i = 1, 2, \ldots$, then $\dim X \leqslant n$.

Even the weaker form of the second property, the finite sum theorem, is not valid for ind in general. The failure of the completion theorem for ind is discussed in [12, 13, 10].

Concerning the Cartesian products, there exist 2-dimensional compact spaces (*the Pontrjagin surfaces*) X, Y with $\dim(X \times Y) = 3$ and 4-dimensional compact metric ***Absolute Retract***s Z and T such that $\dim(Z \times T) = 7$ (Dranishnikov). Also, for every $n \in \mathbb{N}$ there exists an n-dimensional subspace X of $\mathbb{R}^{n+1}$ such that $\dim X^{\aleph_0} = n$ (R.D. Anderson, J.E. Keisler). Such X can be in addition a connected subgroup of $\mathbb{R}^{n+1}$ (J. Keesling), or completely metrizable (J. Kulesza). However, $\dim(X \times Y) = \dim X + 1$ if Y is the interval (K. Morita) or X is compact and Y is 1-dimensional (Morita, Hurewicz for separable spaces). On the other hand, there exists a 2-dimensional subset X of $\mathbb{R}^3$ and a 1-dimensional continuum Y such that $\dim(X \times Y) = 2$ (A.N. Dranishnikov, D. Repovš and E.V. Ščepin). We refer the reader to [5], Section 1, for an outline of deep relations between the dimension of the product of compact spaces X and Y and properties of the spaces of maps of these spaces into $\mathbb{R}^m$ with $m = \dim X + \dim Y$.

An important topic in the dimension theory are **universal space**s, i.e., members of a given class of spaces containing topologically every space from this class. Menger's universal space M_n^{2n+1} has been already mentioned. For every $n < m$ there are analogues M_n^m of these spaces in $\mathbb{R}^m$ (M_1^2 is the celebrated Sierpiński Carpet), which are universal in the class of all n-dimensional compact subspaces of $\mathbb{R}^m$ (M.A. Štan'ko). In particular, the Sierpiński Carpet is universal for the ***curves*** (i.e., one-dimensional continua) in the plane (Sierpiński). The topological properties of the spaces M_n^m are of significant importance, cf. [3].

The **Nöbeling universal space** N_n^{2n+1}, consisting of points in $\mathbb{R}^{2n+1}$ with at most n rational coordinates, is universal in the class of all n-dimensional separable metrizable spaces (Nöbeling). An analogue of N_n^{2n+1} for the class of n-dimensional metrizable spaces of weight τ is the **Nagata universal space**, the subspace of the countable product $J(\tau)^{\aleph_0}$ of the ***hedgehog*** $J(\tau)$ with τ spines, consisting of points with at most n rational coordinates different from 0. The space $B(\tau) \times N_n^{2n+1}$, where $B(\tau)$ is the countable product of discrete spaces of cardinality τ, is universal for the class of n-dimensional metrizable spaces with ***σ-star-finite*** open base (J. Nagata). For each space X in this class, $\operatorname{ind} X = \dim X$ (A.V. Zarelua).

Universal spaces can be obtained also by means of *the Pasynkov factorization theorem*: for every continuous map $f: X \to Y$ of a metrizable space X into a metrizable space Y there exist a metrizable space Z of weight not exceeding the weight of Y with $\dim Z \leqslant \dim X$, and continuous maps $g: X \to Z$ and $h: Z \to Y$ such that $f = hg$.

An important link with algebraic topology is provided by the characterization of dimension in terms of maps to the Euclidean spheres $\mathbb{S}^n$: $\dim X \leqslant n$ if and only if for every closed subspace A of X and every continuous map $f: A \to \mathbb{S}^n$ there exists a continuous *extension* $F: X \to \mathbb{S}^n$ of f over X (P.S. Alexandroff, Hurewicz for separable spaces). Replacing the spheres by the ***Eilenberg–MacLane complexes*** $K(G, n)$, one arrives at the definition of the ***cohomological dimensions***, introduced by Alexandroff (in terms of homology groups). The consideration of even more general ***CW-complex***es leads to the ***extensional dimension*** theory (cf. [6] and the article of A.N. Dranishnikov in this volume).

An **essential map** $f: X \to \mathbb{I}^n$ onto the n-cube is a continuous map such that any continuous $g: X \to \mathbb{I}^n$ coinciding with f on the preimage $f^{-1}(\partial\mathbb{I}^n)$ of the boundary of the cube maps X onto $\mathbb{I}^n$. The existence of an essential map from X onto $\mathbb{I}^n$ is equivalent to $\dim X \geqslant n$ (Alexandroff). This is closely related to the above theorem on extending maps to spheres and to the characterization of dimension by means of partitions: $\dim X \leqslant n$ if and only if for every sequence $\{(A_i, B_i)\}_{i=1}^{n+1}$ of pairs of disjoint closed subsets of X there exists a sequence $\{L_i\}_{i=1}^{n+1}$ of closed subsets of X such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$ (E. Hemingsen, S. Eilenberg and E. Otto for separable spaces).

The Alexandroff theorem on ε-translations has the following general counterpart: $\dim X \leqslant n$ if and only if for every open cover $\mathcal{E}$ of X there exists an $\mathcal{E}$-map of X onto a polyhedron of dimension $\leqslant n$, where $f: X \to Y$ is an **$\mathcal{E}$-map** if there exists an open cover $\mathcal{U}$ of Y such that $f^{-1}(\mathcal{U})$ is a *refinement* of $\mathcal{E}$ (Kuratowski).

Combinatorial aspects of the dimension are captured by the following theorem: for $X \neq \emptyset$, $\dim X \leqslant n$ if and only if for every locally finite open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of X there exists an open cover $\mathcal{V}$ of X which is the union of $n+1$ discrete families $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{n+1}$, where $\mathcal{V}_i = \{V_{i,s}\}_{s\in S}$ and $V_{i,s} \subset U_s$ for $s \in S$ and $i = 1, 2, \ldots, n+1$ (P.A. Ostrand).

Metrizability of a ***regular*** space X is equivalent to the existence of a σ-locally finite base $\mathcal{B}$. If, in addition, $\operatorname{ord}\{\operatorname{Bd} U\colon U \in \mathcal{B}\} \leqslant n-1$, or $\dim \operatorname{Bd} U \leqslant n-1$ for every $U \in \mathcal{B}$, then $\dim X \leqslant n$, and in any metrizable n-dimensional space such bases always exist (K. Morita).

Other characterizations of the dimension of metrizable spaces are expressed in terms of sequences of certain covers. To state the results, let us agree that, given a cover $\mathcal{U}$ of X and $A \subset X$ we define $\operatorname{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon A \cap U \neq \emptyset\}$, and for any fixed metric on X, $\operatorname{mesh}\mathcal{U}$ is the supremum of the diameters of elements of $\mathcal{U}$ with respect to the metric. The inequality $\dim X \leqslant n$ is equivalent to each of the following properties:

(i) there exists a metric generating the topology of X and a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X such that $\operatorname{mesh}\mathcal{U}_i \leqslant 1/i$, $\operatorname{ord}\mathcal{U}_i \leqslant n$ and $\mathcal{U}_{i+1} \prec \mathcal{U}_i$ (P. Vopěnka),

(ii) there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X such that $\operatorname{ord}\mathcal{U}_i \leqslant n$ and, for every $U \in \mathcal{U}_{i+1}$, there exists $V \in \mathcal{U}_i$ such that $\operatorname{St}(U, \mathcal{U}_{i+1}) \subset V$ and $\{\operatorname{St}(x, \mathcal{U}_i)\colon i \in \mathbb{N}\}$ is a neighbourhood basis of each point x in X (J. Nagata),

(iii) there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X such that $\operatorname{ord}\mathcal{U}_i \leqslant n$, $\mathcal{U}_{i+1} \prec \mathcal{U}_i$ and, for every $x \in X$ and its neighbourhood U there exists a neighbourhood V in X such that $\operatorname{St}(V, \mathcal{U}_i) \subset U$ (K. Nagami and J.H. Roberts).

The **metric dimension** $\mu\dim(X, \rho)$ of the metric space (X, ρ) is the smallest integer n such that for every $\varepsilon > 0$ the space X admits an open covering $\mathcal{U}_\varepsilon$ consisting of sets of diameter $< \varepsilon$ with $\operatorname{ord}\mathcal{U}_\varepsilon \leqslant n$; $\mu\dim(X, \rho) = \infty$ if there is no such integer. For compact spaces, $\mu\dim(X, \rho) = \dim X$, and always $\mu\dim(X, \rho) \leqslant \dim X \leqslant 2\mu\dim(X, \rho)$ (M. Katětov). The Sitnikov example shows that the upper bound can be attained for subsets of $\mathbb{R}^3$ (and hence the assumption $\mathcal{U}_{i+1} \prec \mathcal{U}_i$ in characterization (i) of the dimension is indispensable). The reader is referred to [14, Chapter 7] for a comprehensive discussion of metric-dependent dimension functions, corresponding to various basic properties of the dimension.

The following extensions of classical Hurewicz theorems (proved by him for separable spaces) are among most fundamental results concerning maps in dimension theory:

(*theorem on dimension-raising maps*) if $f: X \to Y$ is a closed map of a metrizable space X onto a metrizable space Y such that all fibers $f^{-1}(y)$ have cardinality at most k for some integer $k \geqslant 1$, then $\dim Y \leqslant \dim X + (k-1)$ (Morita),

(*theorem on dimension-lowering maps*) if $f: X \to Y$ is a closed map of a metrizable space X into a metrizable space Y and there exists an integer $k \geqslant 0$ such that $\dim f^{-1}(y) \leqslant k$ for every $y \in Y$, then $\dim X \leqslant \dim Y + k$ (Morita, Nagami).

Open maps can raise the dimension. The Pontrjagin surface is an image of a one-dimensional compactum under an open map with 0-dimensional fibers (A.N. Kolmogoroff). There exist open maps with 0-dimensional fibers from one-dimensional compacta onto the square (L. Keldyš, I. M. Kozlovskiĭ). The Menger curve M_1^3 can be mapped onto the Hilbert cube by an open map all of whose fibers are ***Cantor set***s (D.C. Wilson) or by an open map with connected fibers (Anderson). However, open maps with countable fibers defined on compact spaces preserve the dimension (Alexandroff).

Refinable maps are also known to preserve covering dimension. A map f between compact metric spaces X and Y is refinable if for every $\varepsilon > 0$ there is an ***ε-map*** $g: X \to Y$ such that $d(f(x), g(x)) < \varepsilon$ for all x. This concept can be extended to more general spaces and refinable maps are useful not only in dimension theory but also in the theories of ***continua***, ***Absolute Neighbourhood Retracts*** and ***shape***, see [9].

Each metrizable space X with $\dim X \leqslant n$ has a parametrization on a zero-dimensional space by a closed map with at most $(n+1)$-point fibers, i.e., there exists a metrizable space Z with $\dim Z \leqslant 0$ and $wZ \leqslant wX$ and a closed map with fibers of cardinality $\leqslant n+1$ of Z onto X (Morita, Hurewicz for separable spaces).

For metric spaces (X,ρ), $\dim X \leqslant n$ is equivalent to the existence of a **uniformly** 0**-dimensional** map $f: X \to \mathbb{I}^n$ into the Euclidean cube, i.e., the map f such that for any $y \in \mathbb{I}^n$ and $\varepsilon > 0$ there exists a neighbourhood U of y in Y whose preimage $f^{-1}(U)$ is the union of a family of its open pairwise disjoint subsets of diameter less than ε (Katětov; see [15, III.10]). For compact spaces, uniformly 0-dimensional maps can be replaced in this characterization by maps with 0-dimensional fibers (Hurewicz).

There is an important relation between the topological dimension and the metric Hausdorff dimension (see [8] or [15]): a separable metrizable space X has dimension $\leqslant n$ if and only if X is homeomorphic to a subset of I^{2n+1} whose Hausdorff $(n+1)$-dimensional measure is equal to 0 (E. Szpilrajn (Marczewski)). It follows that for any metrizable separable space X with $\dim X \leqslant n$ there is a metric on X such that for all $r > 0$, except for a set of Lebesgue measure null, the boundary of the r-ball about any point in X has dimension $\leqslant n-1$.

The dimension can also be characterized by the existence of special metrics generating the topology of a metrizable space. The property $\dim X = 0$ is equivalent to the existence of a **non-Archimedean metric** on X, i.e., a metric ρ such that $\rho(x,y) \leqslant \max(\rho(x,z), \rho(z,y))$ for every $x,y,z \in X$ (J. de Groot). For $n \geqslant 1$, the property $\dim X \leqslant n$ is equivalent to the existence of a metric ρ on X satisfying any one of the following conditions, where $B(x,r)$ is the open r-ball about x:

(i) for each point $x \in X$ and each $r > 0$, $\dim \mathrm{Bd}\, B(x,r) \leqslant n-1$; moreover, $\overline{\bigcup_{x\in X_0} B(x,r)} = \bigcup_{x \in X_0} \overline{B(x,r)}$ for every $X_0 \subset X$ (Nagata),

(ii) for every point $x \in X$, each $r > 0$ and every sequence $y_1, y_2, \ldots, y_{n+2}$ of $n+2$ points of X satisfying the inequality $\rho(y_i, B(x, r/2)) < r$ for $i = 1, 2, \ldots, n+2$, there exist different indices $i, j \leqslant n+2$ such that $\rho(y_i, y_j) < r$ (Nagata),

(iii) for every point $x \in X$ and every sequence $y_1, y_2, \ldots, y_{n+2}$ of $n+2$ points of X there exist different indices $i, j \leqslant n+2$ such that $\rho(y_i, y_j) \leqslant \rho(x, y_i)$ (Nagata, Ostrand).

Several sets of axioms characterizing the dimension in some subclasses of metrizable spaces have been formulated. The reader is referred to [1] for a discussion of this topic.

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f-5 Dimension Theory: Infinite Dimension

A space is **infinite-dimensional** if it fails to have finite ***covering dimension***. Many important spaces fit into this class and dimension theory provides a certain stratification of it. With the exception of CW-complexes, for simplicity *we assume all spaces below are metrizable*, even if weaker conditions suffice for the validity of the results discussed. (Unfortunately, we thus miss a substantial body of research dealing with more general spaces.) By a complete space we mean one which is completely metrizable. We abbreviate infinite-dimensional to i.d., finite-dimensional to f.d. and countable-dimensional (see below) to c.d.

Closest to being finite-dimensional are spaces that are **locally finite-dimensional**, that is, admit an open cover by f.d. sets. Next come **strongly countable-dimensional** spaces, i.e., countable unions of closed f.d. subspaces. Dropping the requirement that the subspaces be closed leads to a wider class of **countable-dimensional** spaces. Useful examples of strongly c.d. spaces are ***cell complex***es (with CW-topology), realizations of ***simplicial complex***es (CW or metric topology) and various spaces of PL-maps of one compact polyhedron to another (with the ***compact-open topology***; see R. Geoghegan [7]). Moreover, if the star of each vertex of a simplicial complex is finite-dimensional (a union of finitely many simplices) then the arising realization is locally f.d. (locally f.d. and locally compact, respectively). Thus canonical maps of a space into ***nerves*** of locally finite coverings take values in strongly c.d. spaces. These maps lie behind the proofs of the following general results, which display further the importance of the above classes of spaces: (a) C.H. Dowker [5] proved that any open covering of a space has an open ***refinement*** with a locally f.d. nerve, (b) K. Kuratowski proved that for any separable spaces X and Y and a map $f : A \to Y$ of a closed subset A of X, there exists another metric space $Z \supset Y$ such that $Z \setminus Y$ is a locally finite polyhedron and f extends to a map $X \to Z$, and (c) A. Lelek proved that any separable complete space X admits a compactification cX such that $cX \setminus X$ is a countable union of f.d. polyhedra [6, §5.3].

Each c.d. space has **Haver's property C** (is a C**-space**), i.e., for any sequence $(\mathcal{U}_n)_{n=1}^{\infty}$ of its open covers there exists an open cover of the form $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$, where each $\mathcal{V}_n$ consists of pairwise disjoint sets contained in members of $\mathcal{U}_n$. (W.E. Haver, D.F. Addis, J.H. Gresham.) Each C-space in turn is **weakly infinite-dimensional** (or: A-weakly i.d., for Alexandroff), i.e., for any sequence (A_n, B_n) of pairs of its disjoint closed subsets there exist continuous functions f_n into the segment $[-1, 1]$, such that $f_n(A_n) \subset \{-1\}$, $f_n(B_n) \subset \{1\}$ and $\bigcap_n f_n^{-1}(0) = \emptyset$. (We note that f.d. spaces *are* weakly i.d.) In our category of metric spaces, each of the above properties turns out to be local; this follows from the characterizations discussed later.

A basic example of a space which is **strongly infinite-dimensional** (i.e., not weakly i.d.) is the Hilbert cube, the infinite countable product of closed segments. It is not known if every weakly i.d. compactum is a C-space. A compact C-space, however, may fail to be c.d., as shown by an example of R. Pol (see [6, §6.1.]). A countable-dimensional space in turn may fail to be strongly c.d.; this is witnessed by the ***adjunction space*** obtained from the Cantor set C by attaching to the members of a dense null collection of compacta in C cubes of increasing dimensions, using appropriate surjections (Yu. Smirnov, see [6, §5.1]).

On (some) i.d. spaces various dimension functions are defined. Two of them are the **small** and the **large transfinite dimension**, denoted below by ind and Ind. The dimension ind is defined inductively with respect to the ordinal number α by demanding that $\operatorname{ind} X = -1$ iff $X = \emptyset$ and $\operatorname{ind} X \leqslant \alpha$ iff each point $x \in X$ has a base of neighbourhoods U such that $\operatorname{ind} \operatorname{Bd}(U) < \alpha$; in the definition of Ind points are replaced by closed sets. For a compact space X, if $\operatorname{ind} X$ exists, then so does $\operatorname{Ind} X$ and $\operatorname{Ind} \leqslant \omega_0 \cdot \operatorname{ind} X$. (The reverse implication and the inequality $\operatorname{ind} X \leqslant \operatorname{Ind} X$ hold without assuming X to be compact.) Classical results of W. Hurewicz (for ind) and Smirnov (for Ind) assert the following: (a) for separable spaces X, if $\operatorname{ind} X$ exists, then it is a countable ordinal and X is c.d., and for Ind the same holds true without assuming X to be separable; (b) conversely, $\operatorname{ind} X$ exists provided X is c.d. and complete, and $\operatorname{Ind} X$ exists provided X additionally is compact. Also, a space with large inductive dimension contains a compact subset such that all closed sets missing it are f.d. (See [6, §§7.1 and 7.2].)

The linear span l_2^f of the vectors $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$, $\ldots$ of the Hilbert space l_2 is strongly c.d. and has no ind, while the disjoint union of the cubes $[0, 1]^n$, $n \in \mathbb{N}$, is locally f.d. space, has no Ind and is a subspace of its one-point compactification having Ind equal to ω_0. For each countable ordinal α, **Smirnov's compactum** S_α is defined by induction on α: S_0 is a point, $S_\alpha = S_{\alpha-1} \times [0, 1]$ if α is a successor, and otherwise S_α is the one-point compactification of the disjoint union of the S_β's, $\beta < \alpha$. One has $\operatorname{Ind} S_\alpha = \alpha$ for every $\alpha < \omega_1$ (Smirnov) and $\operatorname{ind} S_\alpha \neq \operatorname{Ind} S_\alpha$ when $\alpha = \omega_0 + 3$ (L.A. Luxemburg).

Each component of S_α being a cube, essential maps onto S_α can naturally be defined. For a weakly i.d. compactum X, the set of ordinals α such that a closed subset of X admits an essential map onto S_α is bounded in ω_1 (R. Pol [14]); if X is c.d., this bound is not greater than $\operatorname{Ind} X$ (D.W. Henderson [9]).

The bound above is the Henderson–Borst index of a space. There also exist other interesting functions that stratify i.d. spaces. Such are the closely related Luzin–Sierpiński index introduced by R. Pol [14], the transfinite kernel dimension and transfinite covering dimension, described in [6, §7.3], or the order dimension, recently introduced by F.G. Arenas. Also, the ***cohomological dimension*** with respect to various groups of coefficients can serve this purpose, for as known from A. Dranishnikov's results it may attain finite values on an i.d. space.

For each of the classes of spaces, determined by the stratifications considered, natural questions arise about the behaviour of the class with respect to taking unions or Cartesian products of its elements, as well as about taking subspaces or their images under maps. (One considers also products or unions of spaces belonging to different classes.) Other important problems are: (a) if the *compactification theorem* for the class is valid, that is, if each member of the class is a dense subset of a compact member, (b) if the *completion theorem* is valid for this class (same as above, but with "complete" replacing "compact") and (c) if in the class there exist spaces universal with respect to embeddings. Below we review some of the results regarding these and similar problems.

1. Compactifications and completions

By results of Hurewicz, Lelek and A.W. Schurle (see [6, §§5.3 and 6.1]), the compactification theorem holds true for the classes of complete separable weakly i.d. spaces, of complete separable c.d. spaces, and of complete separable strongly c.d. spaces. (The completeness assumption is essential: any completion of the space l_2^f contains topologically the Hilbert cube.) A separable complete space has a weakly i.d. compactification iff all its compact subsets are weakly i.d., which can happen also for strongly i.d. spaces. (See Section 4, below.)

A subspace of a given transfinite dimension Ind is contained in a G_δ-set of the same dimension, and for ind this is true if the subspace is separable (Luxemburg). Thus, an appropriate completion theorem is valid for ind and for Ind. It is valid also for the class of locally f.d. spaces, while for separable locally f.d. spaces there is a local compactification theorem (B.R. Wenner).

Compactification theorem holds true for separable spaces of a given (transfinite) dimension Ind, but not for those of a given dimension ind (Luxemburg). See [6, §7.2]. A separable complete space containing all Smirnov compacta contains the Hilbert cube (R. Pol), whence there is no completion theorem for a nontrivial class containing all these compacta.

W. Olszewski invented a very useful technique of extending countably many maps, with values in countable CW-complexes, while keeping sufficient control on the extensions. This allowed him to prove the existence of an universal space and a G_δ-enlargement theorem for the ***cohomological dimension*** or, more generally, for the ***extensional dimension*** (the separable case). See [13] and the articles on the above subjects in this volume.

2. Universal spaces

Within spaces of a given weight, there exist universal spaces for (strongly) c.d. spaces (J. Nagata) and for locally f.d. spaces (Luxemburg). Explicit descriptions of these spaces for various weights are also known (Smirnov, Nagata, Wenner, L. Ju. Bobkov, Olszewski). In particular, the space l_2^f is universal for strongly c.d. separable spaces. See [6, §§5.3 and 5.5].

Results of R. Pol assert that for each countable ordinal α, there exists: (a) a universal space for the class of separable spaces with transfinite dimension ind equal to α, and (b) a countable dimensional compactum X_α containing all compacta of dimension ind (dimension Ind, respectively) equal to α. If, however, α is a limit ordinal then such compacta X_α must have the respective dimension larger than α and one cannot replace ind by Ind in a) (Olszewski). Some other results on the non-existence of universal spaces related to transfinite dimension were given by Luxemburg, see [6, §7.2]. Under the Continuum Hypothesis, there is also no universal weakly i.d. separable space (R. Pol).

3. Maps

If each fibre of a closed surjective map $f: X \to Y$ has an isolated point or is of cardinality smaller than continuum and X is c.d. (weakly i.d., respectively), then so is Y. (E.G. Skljarenko, A.V. Arkhangel'skiĭ, K. Nagami, Olszewski). Similarly, an open map with finite fibres preserves the property of being c.d. and of being strongly c.d. (Arkhangel'skiĭ). In the other direction, if $f: X \to Y$ is a closed map and Y has property C, then X is a C-space (or is weakly i.d.) provided all fibres of f have the respective property (Y. Hattori and K. Yamada). Similarly, if X is compact and all the fibres of a map $f: X \to Y$ are countable, then X is weakly i.d. provided Y is (R. Pol). See [6, §§5.4 and 6.3].

In addition to these results on preservation of classes of spaces by maps, an important role is played by constructions of "dimension increasing maps" with good properties. The initial constructions of R.D. Anderson (of a monotone open map of the Menger curve onto the Hilbert cube) and of L. Keldyš (of a ***light open*** map of a one-dimensional continuum onto the square) have been put into a wider context by E.V. Ščepin. A map $f: X \to Y$ is said to be $\mathcal{K}$**-soft**, where $\mathcal{K}$ is a class of spaces, provided that for any map $u: A \to X$ of a closed subset A of a space $B \in \mathcal{K}$, and for any map $v: B \to Y$ which extends fu, there exists a map $B \to X$ extending u and such that its composition with f is equal to v. If the preceding condition is satisfied with $A = \emptyset$, the map is said to be $\mathcal{K}$**-invertible**. (For $\mathcal{K}$ consisting of a convergent sequence softness is equivalent to openness; monotone open maps can be characterized similarly.) An insightful short note of I.M. Kozlovskii [10] and his specific map of a torus onto a disc have been taken by A.N. Dranishnikov as a starting point for a series of results, culminating in his construction of an $\mathcal{K}_{n-1}$-soft and $\mathcal{K}_n$-invertible map, f, of the Menger n-dimensional curve onto the Hilbert cube, where $\mathcal{K}_m$ stands for the class of all m-dimensional compacta. Moreover, f is a **universal map** for the class $\mathcal{G}_n$ of all maps of n-dimensional compacta, for each $g \in \mathcal{G}_n$ can be viewed as a restriction of f. See [2] for details and for other results on soft maps.

Related is M. Zarichnyj's result [16] stating that, for the class $\mathcal{K}$ of all strongly c.d. spaces, there exists among all maps of σ-compact members of $\mathcal{K}$ a universal one, which is $\mathcal{K}$-soft and has the Hilbert cube as its image. A modification of this result played a very essential role in R. Cauty's investigations of the topological properties of compact convex subsets of linear metric spaces. (Cf. J.E. West's article on Absolute Retracts for this volume.)

The important problem of the existence of a ***cell-like map*** of a f.d. compactum onto one which is i.d. has been solved by Dranishnikov, in a construction culminating the work initiated by R.D. Edwards and J. Walsh. The domain of a dimension increasing cell-like map may be 3-dimensional (Dranishnikov) or even 2-dimensional (J. Dydak, Walsh), and the image may be strongly i.d. (Dranishnikov). However, the domain cannot be a manifold of dimension $\leqslant 3$ (G. Kozlowski and Walsh), nor can the image be a C-space (F. Ancel). By refinements of results going back to S. Smale and to C. Lacher, the image of a ***CE-map*** contains no f.d. subspaces of dimension higher than that of its domain. For more information see articles on extensional dimension and on cell-like maps in this volume.

For maps of i.d. spaces attention is focused on G. Kozlowski's ***hereditary shape equivalences***, which (by his results) may be defined as restrictions, to full inverse sets, of CE-maps between Absolute Retracts. These can change both transfinite dimensions ind and Ind of c.d. compacta (J.J. Dijkstra), as well as the property of being strongly c.d.; however, they do preserve property C (Ancel, Dijkstra, J. van Mill, J. Mogilski). See [3] and references quoted there.

The projection to the orbit space of a compact group action on a 2-dimensional compactum can have a strongly i.d. range (Dranishnikov and West). This phenomenon is related to the torsion of the group acting; see [4].

4. Hereditarily i.d. spaces

Hurewicz proved that any c.d. complete space either is f.d. or contains closed subspaces of arbitrarily high finite dimension; it remains unknown if completeness assumption is necessary here. He showed also that Continuum Hypothesis is equivalent to the existence of an i.d. space containing no uncountable subspaces of finite dimension. Within ZFC and in the class of complete spaces the following striking results are known: (a) there exist complete, separable, strongly i.d. spaces which are ***totally disconnected*** and thus contain no compacta of positive finite dimension (L. Rubin, R. Schori, J. Walsh, R. Pol), and (b) there exist **Walsh compacta**: strongly i.d. compact spaces, containing no weakly i.d. subspaces other than 0-dimensional ones (Walsh); moreover, any strongly i.d. separable space contains a closed subspace which is hereditarily strongly i.d. in this sense (Rubin). The beautiful constructions of spaces lacking weakly i.d. subsets depended on a diagonal-type reasoning that can be traced back to S. Mazurkiewicz; cf. [6, §6.2]. There exist by now various approaches to such results; see [6, §5.2] for R. Pol's version, [11] for a description in terms of essential maps onto products of manifolds, and [12] for a short construction of **Henderson's compacta**: ones that contain no weakly i.d. subcompacta other than 0-dimensional ones. Lelek's compactification of the space described in a) yields **Pol's example** mentioned earlier, and from its construction and properties it follows that neither weak infinite-dimensionality nor property C passes to subspaces. (Henderson's construction answered a question of Tumarkin and Mazurkiewicz, and Pol's one of Alexandroff, each open for over 40 years.)

5. Finite unions and Cartesian products

A product of two separable spaces, one of which is weakly i.d. and the other is 0-dimensional, may be strongly i.d. Under the Continuum Hypothesis one may take the space of irrational numbers for one of the above two spaces and a C-space for the other (E. Pol). Still open is however Alexandroff's problem whether the product of two compact weakly i.d. spaces may lack this property.

The functions $d = \text{ind}$ and $d = \text{Ind}$ have the property that if $C = A \cup B$ and $d(A)$ and $d(B)$ do exist, then $d(C)$ does also and it may be bounded from above by a function of $d(A)$ and $d(B)$. This yields related bounds of $d(X \times Y)$ (Toulmin and later improvements by other authors; see [6, §7.2] and [1]). Smirnov's compactum S_{ω_0+1} has Ind equal to $\omega_0 + 1$ and is the union of two closed subsets of Ind equal to ω_0 (B.T. Levšenko), which violates the possibility of extending to the transfinite case the max-estimate of $\text{Ind}(A \cup B)$.

6. Retractions and selections

The possibilities of constructing extensions of maps or, more generally, (continuous) selections or near-selections of set-valued functions, get drastically reduced if the domain is i.d. An important exception offers E. Michael's selection theorem, stating that a ***lower semi-continuous*** (l.s.c.; see [E, p. 63]) function with values in convex closed subsets of a Banach space admits a continuous selection (the domain may even be paracompact, rather than metrizable). In the absence of convexity of the values there are other well-known results of Michael, in which however dimension restrictions on the domain need to be imposed. Haver indicated that his property C may in certain cases replace these restrictions. So, a map of a closed subset A of a space B can continuously be extended over B provided the target space is locally contractible and the boundary of A has property C; this and Kuratowski's theorem stated earlier implies that locally contractible C-spaces are Absolute Neighborhood Retracts (Haver, Gresham). If F is a l.s.c. function of a C-space X, taking values in CE-compacta in an ANR-space Y, then there is a map $X \to Y$ whose graph is contained in a given neighbourhood of the graph of F. (Cf. Haver [8].) Recently, V.V. Uspenskiĭ [15] showed that each C-space, X, has the following property: if a function F defined on X takes values in aspherical subsets of a space Y, and the set $\{x \in X\colon F(x) \supset K\}$ is open for any compact set $K \subset Y$, then F has a continuous selection.

Of great importance are methods of detecting, or disproving, the ANR-property of spaces lacking property C. See the article on Absolute Retracts in this volume. Enlightening examples of spaces failing to be ANRs (or of CE-maps failing to be hereditary shape equivalences) were given by Daverman and Walsh.

7. Characterizations

A space is c.d. iff it is the union of an increasing sequence of 0-dimensional sets, and also iff it is an image of a 0-dimensional space under a closed map with finite fibres (Nagata). Strongly c.d. spaces, locally f.d. spaces and spaces having Ind can similarly be characterized as images of 0-dimensional spaces under maps with some additional properties. (Engelking, Hattori). The proofs depend on the fact that these spaces can be detected by the existence of a σ-discrete base satisfying certain finiteness intersection property of the boundaries of its members (Nagata, Skljarenko, Engelking). See [6, §§5.1, 5.4, 5.5 and 7.1].

Strong infinite-dimensionality of a compactum X can obviously be characterized by the existence of an ***essential map*** onto the Hilbert cube. (See [11].) There exists however no reasonable characterization of the properties $\operatorname{ind} X = \omega_0$ and $\operatorname{Ind} X = \omega_0$ in terms of the existence of essential maps onto compacta (Dijkstra [3]). Uspenskiĭ showed that C-spaces are characterized by the selection property of his theorem stated in Section 6.

An old problem of Alexandroff may take the following form: can the infinite-dimensionality of a compact space X be detected by some (co)homological invariants? Alexandroff posed it with the invariant being the homological dimension over the integers, in which case a negative answer was given by Dranishnikov. S. Nowak's question, if the cohomological dimension over the stable homotopy group may serve as the invariant, is open. It is known that it may if X is either strongly i.d. or has property C (Dranishnikov).

8. Estimating dimension

This, of course, is of central importance for several of the topics discussed. The existence of a space of dimension $\geqslant n$ implies the fixed-point property of the n-cube, ordinarily proved using tools of algebraic topology. In many important cases these tools are applied directly: Dranishnikov's proof of the existence of an i.d. compactum having finite cohomological dimension rests on K-theory; the proof of its refinement by Dydak and Walsh rests on the fact that spheres can non-trivially (homotopically) be mapped to $\Omega^3 S^3$, but certain Eilenberg–MacLane spaces cannot.

An interesting recent set-theoretic method of estimating dimension depends on using Bing's theorem stating that any two disjoint closed subsets of a compact space can be ***separated*** by a compactum, each component of which is ***hereditarily indecomposable***. The underlying idea is that the lattice of subcontinua of such separators is very simple, which enables additional control of dimension and of functions. In this way it was shown, amongst other things, that the hyperspace of any 2-dimensional compactum is i.d. (M. Levin and Y. Sternfeld). See [12] for another application.

9. Further remarks

(a) As noticed by L. Rubin, the literature on i.d. spaces can hardly be said to determine a theory (which exists in the f.d. case). A more systematic approach is offered by infinite-dimensional topology, the domain of which is however restricted to especially regular i.d. spaces: to manifolds modeled on the Hilbert cube or on some linear metric spaces, and to certain of their subspaces. The present article should be read in conjunction with that on i.d. topology in this volume. Of special interest, from the point of view of dimension theory, are the ***absorbing sets***. After an initial development in a different setting by Anderson, Bessaga, Pełczyński, Toruńczyk and West, they became a standard tool of producing examples of homogeneous universal spaces for various classes of spaces, including those discussed here.

(b) Dimension theory of i.d. spaces is the subject of a large portion of R. Engelking's book [6], which is a basic recent reference on this subject. This book describes the full development in the area, without restricting attention to the metric case.

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f-6 Zero-Dimensional Spaces

Definitions and basic facts

A ***topological space*** is said to be **zero-dimensional** (or 0-**dimensional**) if it is a non-empty T_1-***space*** with a base consisting of ***clopen*** sets, i.e., sets that are simultaneously ***closed*** and ***open***. Zero-dimensional spaces were defined by Sierpiński in [14]; they form a widely studied class of topological spaces. They occur in areas as diverse as non-Archimedean analysis and the study of Boolean algebras. Also, many counterexamples in topology are zero-dimensional, though often more as a matter of convenience than intentionally.

A subset of a topological space is clopen if and only if its boundary is empty. Hence zero-dimensional spaces are exactly those having their ***small inductive dimension*** equal to zero, $\operatorname{ind} X = 0$. Obviously, zero-dimensional spaces are ***completely regular***. Let $\operatorname{Clop}(X)$ denote the field of all clopen subsets of a space X. Any point x of a zero-dimensional space X corresponds to an ***ultrafilter*** $\mathcal{F}_x$ on $\operatorname{Clop}(X)$ defined by $\mathcal{F}_x = \{A \in \operatorname{Clop}(X)\colon x \in A\}$, or, equivalently, with a $\{0,1\}$-valued ***measure*** μ_x defined by $\mu_x(A) = 1$ iff $A \in \mathcal{F}_x$. This correspondence is one-to-one and determines an embedding of the space X into the ***Cantor cube*** $D_\kappa = \{0,1\}^\kappa$, where κ is an infinite cardinal number. If, moreover, X is compact then for every ultrafilter $\mathcal{F}$ on $\operatorname{Clop} X$ there is a point $x \in X$ such that $\mathcal{F} = \mathcal{F}_x$. Zero-dimensional spaces are exactly all nonempty subspaces of the Cantor cubes D_κ. For a fixed κ the space D_κ is a ***universal space*** for all zero-dimensional spaces of ***weight*** at most κ.

We say that a space X is **strongly zero-dimensional** if it is a non-empty completely regular space such that every finite cover by ***cozero sets*** has a finite refinement, which is a partition of X by clopen sets. Hence a space X is strongly zero-dimensional if and only if its ***covering dimension*** is equal to zero, $\dim X = 0$. A topological space X is strongly zero-dimensional if and only if it is a nonempty completely regular space such that any two functionally separated subsets are separated by a clopen set. Recall that two sets A and B are **functionally separated** if there is a continuous real-valued map f on X such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$. Strongly zero-dimensional spaces form a subclass of the zero-dimensional ones. Conversely, if X is ***Lindelöf*** and zero-dimensional then it is strongly zero-dimensional. While zero-dimensionality is a hereditary property and is preserved by products, strong zero-dimensionality is not hereditary even for zero sets and it is not productive even in the class of hereditarily normal spaces (see [13], [E, 7.4.6]).

In the sequel, by a space we mean a non-empty topological space.

Examples

(i) A subspace of the real line is zero-dimensional (in fact, strongly zero-dimensional) if and only if it does not contain any non-degenerate interval, e.g., the rational numbers $\mathbb{Q}$ and the irrational numbers $\mathbb{P}$ form a decomposition of the real line into two zero-dimensional spaces.

(ii) Every completely regular space of size less than 2^ω is strongly zero-dimensional.

(iii) A zero-dimensional ***GO-space*** is strongly zero-dimensional, e.g., the ***Sorgenfrey line*** is such a space.

(iv) Every ***scattered compact Hausdorff*** space is strongly zero-dimensional.

(v) ***Stone space***s of ***Boolean algebra***s are exactly all compact zero-dimensional spaces.

Related notions

Let us note that a space X is ***connected*** if the field $\operatorname{Clop}(X)$ is as small as possible, i.e., if the only clopen subsets are X and $\emptyset$. Zero-dimensionality is among a class of properties that strongly negate connectedness. Weaker negations are hereditary disconnectedness introduced by F. Hausdorff in 1914 [7] and total disconnectedness introduced by W. Sierpiński in 1921 [14]. A space is said to be **hereditarily disconnected** if any subspace of size at least two is disconnected, or, equivalently, if the only connected components are singletons. A space X is called **totally disconnected** if the field $\operatorname{Clop}(X)$ separates its points.

A space X is **extremally disconnected** if the ***closure*** of every open subset is clopen in X, i.e., the field $\operatorname{Clop}(X)$ coincides with the algebra of all ***regular open*** sets. This notion was introduced in 1937 by M. H. Stone [15].

Let X be a topological space. The following implications characterize the relationship among the notions defined above: X is extremally disconnected and $T_3 \Longrightarrow X$ is strongly zero-dimensional $\Longrightarrow X$ is zero-dimensional $\Longrightarrow$ X is totally disconnected $\Longrightarrow X$ is hereditarily disconected. None of the implications can be reversed and counterexamples exist even in the class of metric spaces, see [E]. However, an example of a metrizable zero-dimensional space which is not strongly zero-dimensional is very difficult to find. Only a few spaces of this kind are known up to now, they were constructed by P. Roy, J. Kulesza and S. Mrówka.

Important fact

In the realm of non-empty ***locally compact paracompact*** (especially, ***compact***) spaces hereditary disconnectedness, total disconnectedness, zero-dimensionality and strong zero-dimensionality are equivalent, see [E]. Thus compact zero-dimensional spaces and compact totally disconnected spaces are synonymous.

Compactifications

The ***Čech–Stone compactification*** of a space X is zero-dimensional if and only if X itself is strongly zero-dimensional. Similarly, the Čech–Stone compactification of a completely regular space is extremally disconnected if and only if the space itself is extremally disconnected. There is a reflection of the category of zero-dimensional spaces into a subcategory of compact zero-dimensional spaces, the corresponding compactification is sometimes called the **Banaschewski compactification** [2]. In fact, the Banaschewski compactification of a zero-dimensional space X is just the Stone space of the field $\mathrm{Clop}(X)$, i.e., the space of all ultrafilters on $\mathrm{Clop}(X)$.

Metrizable case

Every metrizable space has a dense subset which as a subspace is itself a strongly zero-dimensional space: it has a ***σ-discrete dense*** subset and every σ-discrete metrizable space is strongly zero-dimensional. A metric space is strongly zero-dimensional if and only if it has a base $\mathcal{B}$ which, together with the ordering by inverse inclusion, is a tree. Generally, spaces having such bases are called ***non-Archimedean***. This notion was introduced by A.F. Monna [11].

For an infinite cardinal number κ, the **Baire space of weight** κ, $B(\kappa)$, is the product κ^ω of ω copies of the discrete space κ with the metric given by $\varrho(x, y) = 1/(\min\{i \in \omega\colon x_i \neq y_i\} + 1)$ for $x \neq y$. The space $B(\kappa)$ is universal for metrizable strongly zero-dimensional spaces of weight at most κ. A zero-dimensional metric space has a zero-dimensional metric compactification if and only if it is ***separable***. It is well-known that $B(\omega)$ is homeomorphic to the space $\mathbb{P}$ of irrational numbers.

Under the set theoretic assumption $S(\aleph_0)$, S. Mrówka showed the existence of a zero-dimensional metric space that does not have a zero-dimensional completion [12]. The consistency of $S(\aleph_0)$ follows from a large cardinal assumption, as proved by R. Dougherty [3].

Interrelationship between the spaces of rational and irrational numbers

Trivially, the space of rational numbers $\mathbb{Q}$ is a continuous image of the space of irrational numbers $\mathbb{P}$. It is true that $\mathbb{Q}$ is even a quotient image of $\mathbb{P}$. This follows from the remarkable theorem from 1969: Each metrizable space which is a continuous image of the space of irrational numbers is also a quotient image of the space of irrational numbers (in general, under a different map) [10]. Let us note that both fields $\mathrm{Clop}(\mathbb{Q})$ and $\mathrm{Clop}(\mathbb{P})$ have size 2^ω. The compactifications $\beta\mathbb{Q}$ and $\beta\mathbb{P}$ are the Stone spaces of $\mathrm{Clop}(\mathbb{Q})$ and $\mathrm{Clop}(\mathbb{P})$, respectively. As they have a different number of points with countable character they are not homeomorphic.

Stone duality

The **topological Stone duality** establishes a one-to-one correspondence between compact zero-dimensional spaces and ***Boolean algebra***s. It is a contravariant functor between the category of compact zero-dimensional spaces with continuous maps on one side and Boolean algebras with homomorphisms on the other. This is the reason for the name **Boolean space**s for compact zero-dimensional spaces. The algebra corresponding to a space X is $\mathrm{Clop}(X)$. The space corresponding to an algebra A, the **Stone space** of the algebra A, is the space of all ultrafilters on A, denoted $\mathrm{Ult}(A)$, with the **Stone topology**. The base of the Stone topology is given by all sets $\{U \in \mathrm{Ult}(A)\colon B \in U\}$, where B runs through A. The space $\mathrm{Ult}(A)$ is Boolean and the algebra of all clopen sets on $\mathrm{Ult}(A)$ is isomorphic to A. It is also easily seen that a Boolean space X is homeomorphic with $\mathrm{Ult}(\mathrm{Clop}(X))$, the correspondence $x \mapsto \mathcal{F}_x$ effects the canonical homeomorphism. For an infinite Boolean space X we have $w(X) = |\mathrm{Clop}(X)|$, where $w(X)$ is the ***weight*** of the space X. Compact extremally disconnected spaces are just the Stone spaces of complete Boolean algebras.

Rigidity and homogeneity

An important example can be found in [5]. Let Seq denote the set of all finite sequences of natural numbers, $\mathrm{Seq} = \bigcup_{n\in\omega} {}^n\omega$. For every $s \in \mathrm{Seq}$ let $\mathcal{F}_s$ be a filter on ω containing all cofinite subsets of ω. Let us define a topology τ on Seq by declaring $V \subseteq \mathrm{Seq}$ to be open if for every $s \in V$ the set $\{n \in \omega\colon s^\frown n \in V\}$ belongs to $\mathcal{F}_s$. Of course, the topology τ and its properties depend on the particular choice of filters $\mathcal{F}_s$. Generally, (Seq, τ) is a strongly zero-dimensional space. Moreover, we have the following facts:

(i) Assume that all filters $\mathcal{F}_s$ are ultrafilters. Then (Seq, τ) is extremally disconnected.
(ii) Assume that $\{\mathcal{F}_s\colon s \in \mathrm{Seq}\}$ are pairwise incomparable ultrafilters in the ***Rudin–Keisler order***. Then (Seq, τ) is ***rigid***.
(iii) Assume that $\{\mathcal{F}_s\colon s \in \mathrm{Seq}\}$ are ***weak P-point***s and pairwise incomparable in the Rudin–Keisler order. Then $\beta(\mathrm{Seq}, \tau)$ is a compact rigid extremally disconnected separable space. (The existence of weak P-points in ZFC was proved by K. Kunen in [9].)

There is no direct relation between the topological homogeneity of Boolean spaces and the algebraic homogeneity of their algebras of clopen sets. Many homogeneous Boolean algebras have non-homogeneous Stone spaces; an important example is $\mathcal{P}(\mathbb{N})/fin$, whose Stone space, the ***Čech–Stone remainder*** of $\mathbb{N}$, is not homogeneous. On the other hand, E. K. van Douwen constructed a homogeneous Boolean space X whose algebra $\mathrm{Clop}(X)$ is not homogeneous [4].

The ***hyperspace*** of a Boolean space equipped with the ***Vietoris topology*** is compact zero-dimensional. More generally, for a normal space X its hyperspace is zero-dimensional if and only if X is strongly zero-dimensional. The discrete non-compact space of natural numbers is an example of a strongly zero-dimensional space whose hyperspace is also strongly zero-dimensional and, moreover, it is completely regular but not normal [E, 6.3.22].

There is a remarkable characterization due to Haydon of ***Dugundji compact*** spaces as extensors for compact zero-dimensional spaces. It means that a compact Hausdorff space X is Dugundji if and only if any continuous map from a closed subset of a compact zero-dimensional space into X has a continuous extension to the whole space.

Theorems about three and four sets

An elementary and basic tool for studying ultrafilters on sets is the Lemma on four sets, which says: For any map $f: X \to X$ there is a partition $\{X_0, X_1, X_2, X_3\}$ of X (some of the sets X_i may be empty) such that f is the identity on X_0 and for every $i = 1, 2, 3$ we have $f[X_i] \cap X_i = \emptyset$. This lemma admits generalizations to zero-dimensional spaces but with some restrictions on the map f.

There is a Theorem on three sets due to Krawczyk and Steprāns [8]: Assume that $f: X \to X$ is an arbitrary continuous map of a compact zero-dimensional space into itself without fixed points. Then there is a partition of X into three clopen sets X_1, X_2, X_3 such that $f[X_i] \cap X_i = \emptyset$ for $i = 1, 2, 3$.

There is also a Theorem on four sets for extremally disconnected spaces due to Frolík [6]): Assume that X is an extremally disconnected compact space and $f: X \to X$ is a continuous embedding of X into itself. Then there is a partition of the space into four clopen sets X_0, X_1, X_2, X_3 such that X_0 is the set of fixed points of f and for $i = 1, 2, 3$ we have $f[X_i] \cap X_i = \emptyset$.

The results are optimal in a certain respect. As E. Thümmel showed, there is an example of an extremally disconnected compact space without isolated points, together with a continuous open selfmap having just one fixed point.

When we consider arbitrary continuous selfmaps, we get the following characterization due to J. Vermeer [16]: A subset of an extremally disconnected compact space is the set of all fixed points of some continuous selfmap if and only if it is a retract of the space.

Note that for every infinite extremally disconnected compact space X there is an embedding $f: X \to X$ onto a ***nowhere dense set*** [1]. This fact together with the above mentioned theorem of Frolík immediately proves that an infinite extremally disconnected compact space is not homogeneous.

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f-7 Linearly Ordered and Generalized Ordered Spaces

For any linearly ordered set $(X,<)$, let $\mathcal{I}(<)$ be the topology on X that has the collection of all open intervals of $(X,<)$ as a base. The topology $\mathcal{I}(<)$ is the **open interval topology** (or **order topology**) of the order $<$ and $(X,<,\mathcal{I}(<))$ is a **linearly ordered topological space** or **LOTS**. For a subset $Y \subseteq X$, it can happen that the relative topology $\mathcal{I}(<)|_Y$ on Y does not coincide with the open interval topology $\mathcal{I}(<|_Y)$ induced on Y by the restricted ordering. In some cases, there might be some other ordering of Y whose open interval topology coincides with $\mathcal{I}(<)|_Y$, but in other cases there is not. Perhaps the best known example is the **Sorgenfrey line**, which is homeomorphic to the subspace $Y = \{(x,1)\colon x \in \mathbb{R}\}$ of the lexicographically ordered LOTS $X = \mathbb{R} \times \{0,1\}$. There is no re-ordering of Y whose open-interval topology coincides with the subspace topology that Y inherits from X.

The study of subspaces of LOTS in their own right was pioneered by E. Čech who introduced **generalized ordered spaces** or **GO-spaces**, i.e., triples $(X,<,\mathcal{T})$ where $<$ is a linear ordering of the set X and $\mathcal{T}$ is a Hausdorff topology on X such that each point of X has a $\mathcal{T}$-neighbourhood base consisting of (possibly degenerate) intervals. (Such spaces have also been called **suborderable** spaces.) The class of GO-spaces is known to coincide with the class of topological subspaces of LOTS, and for any GO-space $(X,<,\mathcal{T})$, there is a canonical linearly ordered set $(X^*,<^*)$ whose ordering extends the given ordering of X, and that has the property that $(X,\mathcal{T})$ embeds as a closed subspace of $(X^*,\mathcal{I}(<^*))$. In addition, the extension space X^* "inherits" many, but not all, of the topological properties of X. For example, X is metrizable if and only if $(X^*,\mathcal{I}(<^*))$ is metrizable, but there are examples in which the GO-space X is ***perfect*** (= closed subsets of X are G_δ-sets) while X^* is not.

Historically, LOTS and GO-spaces have been valuable sources of counterexamples in topology. The usual space $[0,\omega_1)$ of all countable ordinals, the Sorgenfrey line, and the **Michael line** (i.e., the usual space of real numbers with each irrational isolated) are perhaps the best known elementary examples. More elaborate examples can be constructed if one is willing to start with more complicated linear orders, e.g., those derived from lexicographic products [6], or various tree-to-line constructions [KV, Chapter 6], including ***Aronszajn trees*** and ***Souslin trees***. In particular, the study of Souslin trees and **Souslin line**s (i.e., non-separable LOTS or GO-spaces that have countable cellularity), whose existence or non-existence is undecidable in ZFC, has been one of the central forces in the study of ordered spaces during the last century [4].

As can be seen in the survey paper [10], the problems of recognizing which topological spaces have the open interval topology of some linear order (the **orderability problem**) and which topological spaces are GO-spaces with respect to some linear order (the **sub-orderability problem**) have been studied since the early years of topology. Topological characterizations of the unit interval (e.g., as the unique compact metric space that is ***connected***, ***locally connected***, and has two non-cut points) can be interpreted as orderability theorems, although it is unlikely that their authors would have seen them as such. In 1941, Eilenberg published one of the first modern orderability theorems, proving that a connected, locally connected space X is orderable if and only if $X^2 - \{(x,x)\colon x \in X\}$ is not connected. Kowalsky proved another orderability theorem for connected spaces: a connected, locally connected space X is orderable if and only if given any three connected proper subsets of X, two of them do not cover X. An orderability theorem for compact connected spaces, given in Theorem 2-24 of [7] asserts that a compact connected space is orderable if and only if it has exactly two non-cut-points. Another orderability theorem for compact connected spaces, based on Michael's Selection Theory, is discussed below.

At the other end of the spectrum from connected spaces are ***zero-dimensional*** spaces. Topological characterizations of spaces of ordinals (**ordinal spaces**) can be viewed as orderability theorems. For metrizable zero-dimensional spaces, Herrlich proved the fundamental orderability theorem in 1965: any strongly zero-dimensional metric space is linearly orderable. (Lynn had previously proved that result for zero-dimensional separable metric spaces.) A nice proof of Herrlich's theorem appears in [E, Problem 6.4.2]. One interesting corollary of Herrlich's theorem is that a certain strange metrizable space described by A.H. Stone [12] is orderable. Stone's metric space is uncountable, strongly zero-dimensional, not a union of countably many relatively discrete subspaces, and has the property that every separable subspace is countable. Knowing that Stone's space is orderable has been the key to several recent examples. Later, in 1977, Purisch gave necessary and sufficient conditions for the orderability of any metric space. His conditions are very technical and are related to the earlier work of Herrlich and to work by Mary Ellen Rudin on orderability of subspaces of the real line.

Special orderability theorems for compact Hausdorff spaces are known. One, due to de Groot and Schnare in 1972, introduced the notion of **nests** (i.e., collections of sets that are linearly ordered by inclusion): a compact Hausdorff space X is orderable if and only if there is a collection of open sets that T_1-separates points of X and that is the union of two nests. Nests played a central role in the solution of the general orderability problem by van Dalen and Wattel in 1973. They first proved that a T_1-space X is sub-orderable (i.e., is a GO-space with respect to some linear ordering) if and only if X has a subbase that consists of two nests. They went further, proving that a T_1-space X is orderable if and only X has a subbase $\mathcal{S}_1 \cup \mathcal{S}_2$ where each $\mathcal{S}_i$ is a nest with the additional property that if $T \in \mathcal{S}_i$ satisfies $T = \bigcap\{S \in \mathcal{S}_i\colon T \subset S \text{ and } T \neq S\}$, then T also satisfies $T = \bigcup\{S \in \mathcal{S}_i\colon S \subseteq T \text{ and } S \neq T\}$. Related characterizations were obtained by Deak and Certanov.

Another type of orderability theorem is based on Selection Theory. Let 2^X be the space of all non-empty closed subsets of X with the Vietoris topology, and let $X(2)$ be the subspace $\{T \in 2^X\colon |T| = 2\}$. In 1951, E. Michael gave an orderability theorem for compact Hausdorff, connected spaces based on the existence of continuous selections, and in 1981 van Mill and Wattel improved Michael's theorem by deleting the hypothesis of connectedness. They proved that the following properties of a compact Hausdorff space X are equivalent: X is orderable; there is a continuous function $f : 2^X \to X$ with $f(T) \in T$ for each $T \in 2^X$; there is a continuous function $f : X(2) \to X$ with $f(T) \in T$ for each $T \in X(2)$. In a subsequent paper, van Mill and Wattel gave a characterization GO-spaces based on selection theory. For more information and references about orderability, see [10] or [8].

In ZFC, all GO-spaces have certain very strong topological properties, e.g., particularly strong normality properties. For example, any GO-space is ***monotonically normal*** and consequently hereditarily ***collectionwise normal***. Another normality-related property of any GO-space is a version of the **Dugundji Extension Theorem**. For any space X, let $C^*(X)$ denote the vector space of all continuous bounded real-valued functions on X, equipped with the sup-norm topology. Heath and Lutzer proved that for any closed subspace A of the GO-space X, there is a linear function $e : C^*(A) \to C^*(X)$ with the property that $e(f)$ extends f and the range of $e(f)$ is contained in the closed convex hull of the range of f for each $f \in C^*(A)$. Thus, e is a norm-1 linear extender. The best recent results on the Dugundji Extension Theorem in GO-spaces are due to Gruenhage, Hattori, and Ohta; see [3, 8]. (The above normality results hold in ZFC. By way of contrast, van Douwen has shown that without the Axiom of Choice, there is a LOTS that is not normal and does not satisfy the the Dugundji Extension Theorem [5].)

In addition to its normality properties, every GO-space has certain strong covering properties. For example, each GO-space is ***countably paracompact***. Furthermore, every GO-space is ω_0**-fully-normal**, i.e., for any open cover $\mathcal{U}$ of a GO-space X, there is an open refinement $\mathcal{V}$ with the property that for any countable subcollection $\mathcal{V}_0 \subseteq \mathcal{V}$ having $\bigcap \mathcal{V}_0 \neq \emptyset$, some member of $\mathcal{U}$ contains $\bigcup \mathcal{V}_0$. Another covering property of any GO-space X is **orthocompactness**, i.e., every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ with the property that $\bigcap \mathcal{V}_1$ is open for every subcollection $\mathcal{V}_1 \subseteq \mathcal{V}$. A final example of a covering property shared by all GO-spaces is the **shrinking property**: each open cover $\mathcal{U}$ of a GO-space has an open refinement $\mathcal{W} = \{W(U)\colon U \in \mathcal{U}\}$ satisfying $\operatorname{cl} W(U) \subseteq U$ for each $U \in \mathcal{U}$. Many of these covering properties can be proved using the "method of coherent collections" explained in [8]. Coherent collections also provide a unified approach to the study of covering properties such as paracompactness in GO-spaces.

The theory of ***cardinal function***s is well-developed for linearly ordered spaces. Souslin's original question, whether a LOTS with countable cellularity must be separable, has been studied at length, and is now known to be undecidable in ZFC. In ZFC we now know that if X is a LOTS, then $c(X) = hL(X) \leqslant c(X \times X) = d(X) = hd(X) \leqslant c(X)^+$, and $|X| \leqslant 2^{c(X)}$. Most of these results were originally proved by Kurepa and have been rediscovered many times by others. A unified presentation appears in Todorčević's paper "Cardinal functions in linearly ordered topological spaces" in [2]. In addition, K. P. Hart showed that $w(X) = c(X) \cdot \psi(X)$, and that result yields yet another proof that the Sorgenfrey line cannot be ordered in such a way that it becomes a LOTS.

Paracompactness [E] is an important covering property that some, but not all, GO-spaces have. The literature contains many weak covering properties (e.g., being ***metacompact***, ***meta-Lindelöf***, θ***-refinable***, weakly $\delta\theta$-refinable, and having the ***D-space*** property) that are equivalent to paracompactness in a GO-space [3, 8]. The first characterization of paracompactness in LOTS was given in the 1950s by Gillman and Henriksen who introduced the notion of a **Q-gap** and who proved in essence that a GO-space X is paracompact if and only if whenever G and H are disjoint open sets that cover X and have the property that $x < y$ whenever $x \in G$ and $y \in H$, then there are closed discrete subsets C and D of X with the property that $C \subseteq G$ is cofinal in G and $D \subseteq H$ is coinitial in H. (This is actually Faber's version [6] of the Gillman–Henriksen theorem.) Another characterization of paracompactness was given by Engelking and Lutzer: a GO-space X is paracompact if and only if no closed subspace of X is homeomorphic to a ***stationary*** subset of a regular uncountable cardinal [8]. (Subsequently, Balogh and Rudin proved that the same result holds for the more general class of monotonically normal spaces.) A typical use of that characterization is to show that if $\mathcal{P}$ is a closed-hereditary topological property that no stationary set in a regular uncountable cardinal can have, then $\mathcal{P}$ implies paracompactness in any GO-space. One example of such a $\mathcal{P}$ is the property "X has a ***point-countable base***" and another is "the space X is perfect".

Metrization theory for LOTS is particularly simple: Lutzer showed that a LOTS is metrizable if and only if it has a G_δ-diagonal. That theorem does not hold for GO-spaces, as the examples of the Sorgenfrey and Michael lines show. (Indeed, the fact that the Sorgenfrey and Michael lines have a G_δ-diagonal and are not metrizable is an easy proof that there is no re-ordering of the set of real numbers whose open interval topology coincides with the Sorgenfrey or Michael line topology.) Metrization theory for GO-spaces is summarized by Faber's metrization theorem [6]: a GO-space $(X, <, \mathcal{T})$ is metrizable if and only if there is a σ-closed-discrete subset $D \subseteq X$ that is dense in $(X, \mathcal{T})$ and has the property that $x \in D$ whenever $x \in X$ has the property that $[x, \rightarrow)$ (respectively $(\leftarrow, x])$ is in $\mathcal{T}$.

Over the years, many topological properties have been studied as components of metrizability, and experience shows that many of these properties, known to be distinct among general topological spaces, are equivalent in the class of GO-spaces. For example, for any GO-space X, the following are equivalent:

(a) X is metrizable;
(b) X has a σ-locally countable base;
(c) X is ***developable***;
(d) X is ***semistratifiable***.

Similarly, for a GO-space the properties of having a σ-disjoint base or having a σ-point finite base are each equivalent to being ***quasi-developable***. Finally, Gruenhage has proved that for GO-spaces, the existence of a point-countable base is equivalent to the Roscoe-Collins property "open-G".

Clearly any metrizable space has a σ-disjoint base, and any space with a σ-disjoint base has a point-countable base. Research has shown that among GO-spaces, there are interesting topological properties that reverse each of those general implications. For example, if a GO-space has a σ-disjoint base and is perfect, or has a σ-disjoint base and is a ***p-space*** in the sense of Arkhangel'skiĭ, then it is metrizable. Finding topological properties that could be added to the existence of a point-countable base to yield a σ-disjoint base is harder. One solution is the following property, called **Property III**: there is a sequence $\langle U_n \rangle$ of open subsets of X, and a sequence $\langle D_n \rangle$ where D_n is a relatively closed-discrete subset of U_n, such that whenever G is open and $p \in G$, then for some $n \geqslant 1$, $p \in U_n$ and $D_n \cap G \neq \emptyset$. See [3] for more details.

As noted above, the basic metrization theorem for GO-spaces is due to Faber. However, there are metrization theorems for GO-spaces that do not seem to follow directly from Faber's result. For example, Bennett and Lutzer showed that metrizability of a GO-space X is equivalent to the statement that every subspace of X is an ***M-space*** in the sense of Morita or that every subspace is a ***p-space*** in the sense of Arhangel'skii (see also [13]), and Balogh and Pytkeev showed that a GO-space must be metrizable if it is hereditarily a ***Σ-space*** in the sense of Nagami. A particularly useful lemma when studying GO-spaces with G_δ-diagonals is an old result of Przymusinski: If $(X, <, \mathcal{T})$ is a GO-space with a G_δ-diagonal, then there is a metrizable topology $\mathcal{M} \subseteq \mathcal{T}$ having the property that $(X, <, \mathcal{M})$ is also a GO-space. See [3] for further details and references.

There is a rough parallelism between the metrization theory for compact Hausdorff spaces and for LOTS. Lutzer's G_δ-diagonal metrization theorem parallels Šneider's much earlier theorem for compact Hausdorff spaces (see [E, 4.2.B]), and both theorems generalize to paracompact subspaces that can be p-embedded in some compact Hausdorff space, or in some LOTS. However, the parallelism is not complete. For example, Juhasz and Szentmiklossy proved that, under the Continuum Hypothesis, any compact Hausdorff space with a ***small diagonal*** must be metrizable. A theorem of van Douwen and Lutzer says that, in ZFC, any Lindelöf LOTS with a small diagonal must be metrizable, but an example constructed from Stone's metric space (above) shows that a LOTS may be paracompact, perfect, Čech-complete, have a small diagonal, and have a σ-closed discrete dense set without being metrizable. See [3] for details.

A great deal of research has been devoted to studying two closely related properties of a GO-space, namely having a σ-closed-discrete dense set, and being perfect. It is easy to see that the first implies the second, and that a Souslin line is a consistent example of a LOTS with the second, but not the first, property. Maarten Maurice asked whether there is a ZFC example of a LOTS that is perfect and yet does not have a σ-closed-discrete dense set. A second question, posed by Heath, asks whether there is a ZFC example of a perfect non-metrizable LOTS with a point-countable base. Heath's question is related to the question of Maurice because it is known that a GO-space with a σ-closed-discrete dense set and a point-countable base must be metrizable, and because (as shown by Bennett and Ponomarev) if there is a Souslin line, then there is a Souslin line with a point-countable base. A third question, posed by Nyikos, asks whether there is a ZFC example of a non-metrizable **non-Archimedean space** (i.e., a space having a base that is a tree when ordered by inclusion) that is perfect. That is a LOTS question because Purisch showed that a perfect non-Archimedean space is a LOTS under some ordering. Once again, if there is a Souslin line, then there is a Souslin line that is a counterexample to Nyikos' question. Qiao and Tall have shown that the questions posed by Maurice, Heath, and Nyikos are equivalent to each other [3]. The crucial lemma proved by Qiao and Tall is that any first-countable LOTS has dense non-Archimedean subspace. That result can be proved for first-countable GO-spaces and it follows that every perfect GO-space has a dense σ-closed-discrete subset if and only if every perfect non-Archimedean space is metrizable. Other results concerning which perfect GO-spaces have dense σ-closed-discrete subsets were obtained by Bennett, Heath, and Lutzer in [1]. For example, they showed that the following properties of a perfect GO-space X are equivalent: X has a σ-closed-discrete dense subset; X has a sequence $\mathcal{G}_n$ of open covers such that for each $p \in X$, $\bigcap\{\mathrm{St}(p, \mathcal{G}_n): n \geqslant 1\}$ is countable; $X = Y \cup Z$ where both Y and Z are GO-spaces with a G_δ-diagonal; there is a continuous ***s*-map** from X into a metrizable space (this is a map with separable fibers).

The questions of Maurice, Heath, and Nyikos are also related to an older question posed by Lutzer, asking whether every perfect GO-space embeds topologically into some perfect LOTS. W. Shi has shown that if the perfect GO-space X has a σ-closed-discrete subset, then X does embed in some perfect LOTS [3]. As mentioned above, if there is a Souslin line, then there is a perfect LOTS that does not have a σ-closed-discrete dense set, but we do not know whether there is a model of ZFC in which each perfect LOTS *does* have a σ-closed-discrete dense set. Special cases of Lutzer's question have been answered in ZFC. Lutzer pointed out that the Sorgenfrey line is a perfect GO-space that cannot be topologically embedded as a closed subset (or as a G_δ-subset) of any perfect LOTS, and Shi, Miwa, and Gao gave a perfect GO space that cannot be topologically embedded as a dense subspace of any perfect LOTS. Other related problems concerning the possibility of monotone embeddings have been solved by Miwa and Kemoto but the topological version of Lutzer's question remains open.

Which topological spaces can be obtained from LOTS or GO-spaces by the action of various kinds of maps? Any topological space is the continuous image of some LOTS, because any topological space is the continuous image of a discrete space, and any discrete space is a LOTS under some ordering. A more interesting result in this direction is due to Hušek and Kulpa who proved that the class of open images of zero dimensional orderable spaces is exactly the class of so-called "butterfly spaces". A much harder question was posed by Nikiel who conjectured that any compact monotonically normal space is the continuous image of some compact LOTS. Mary Ellen Rudin published a sequence of papers that culminated in a proof of Nikiel's conjecture [11].

One particularly important version of the continuous images problem is called the generalized **Hahn–Mazurkiewicz problem** and asks which topological spaces are continuous images of arcs. By an **arc** we mean a compact, connected LOTS. It is known that the usual unit interval is the unique separable arc, and the original Hahn–Mazurkiewicz theorem characterized continuous images of the unit interval $[0, 1]$ as being precisely the compact, connected, locally connected metrizable spaces. A good survey of early work on Hahn–Mazurkiewicz theory is given by Treybig and Ward in [2]. Starting with the work of Mardešić in the early 1960s, topologists have made steady progress in understanding which spaces are continuous images of (non-separable) arcs. Many of the results that they obtained included the hypothesis "Suppose X is the continuous image of some compact LOTS". Rudin's theorem, mentioned above, puts these results in a more natural context. For example, by combining Rudin's theorem with a result of Treybig and Nikiel (Theorem 6.6 of [HvM, Chapter 15]), one would have that a space X is the continuous image of an arc if and only if X is compact, connected, locally connected, and monotonically normal. Similarly, Rudin's theorem combines a theorem of Nikiel and Tymchatyn (Theorem 6.14 of [HvM, Chapter 15]) to say for any homogeneous compact monotonically normal space X, at least one of the following holds: X is metrizable; X is zero-dimensional; or X has finitely many components, each of which is a homogeneous simple closed curve. Other characterizations of continuous images of arcs study the special structure of certain subcontinua. For example, Nikiel showed that if a connected compact space X has the property that every connected, compact subspace Y of X is locally connected, then X is the continuous image of an arc. Ward and Nikiel characterized continuous images of arcs as being those compact connected spaces that can be approximated, in a certain technical sense, by finite ***dendrons***, and as being those compact connected spaces that are locally connected and have the property that each of their true cyclic elements can be approximated by T-sets. See [HvM, Chapter 15] for references and further details. Inverse limit constructions [9] are often key tools in the study of these spaces.

Rim-properties of a space X (by which we mean topological properties of the boundaries of members of some open base for X) have also played an important role in Hahn–Mazurkiewicz theory. For example, Mardešić proved that any continuous image of an arc is rim-metrizable, and while the converse is not true, a theorem of Pearson, Tymchatyn, and Ward shows that each rim-finite connected, compact space is the continuous image of an arc.

Let K be a compact LOTS. The relatively simple structure of such a K has recently been used to study renorming problems in the Banach space $C(K)$ with the sup-norm. Haydon, Jayne, Namioka, and Rogers have shown that for a compact LOTS K, $C(K)$ always has an equivalent Kadec norm, and have characterized when $C(K)$ has an equivalent locally uniformly convex (LUC) norm. For example, they show that if L is the lexicographically ordered product space $[0, 1]^\alpha$, then $C(L)$ has an equivalent LUC norm if and only if $\alpha < \omega_1$ and that if M is a compact, connected Souslin line, then $C(M)$ never has an equivalent LUC norm. See [3] for more details.

Over the last fifty years, many of the most important examples in product theory for normal, Lindelöf, and paracompact spaces have been constructed by fine-tuning certain subspaces of the Sorgenfrey and Michael lines. More recently, researchers have investigated the product theory of various GO-spaces. Alster obtained interesting results about products of GO-spaces with G_δ-diagonals. Kemoto and Yajima extended earier results of B. Scott by showing that if A and B are subspaces of some space $[0, \gamma)$ of ordinal numbers, then $A \times B$ is normal if and only if it is orthocompact. Kemoto, Nogura, Smith, and Tamano, and Fleissner and Stanley obtained definitive results about the equivalence of various covering properties in a finite product X of subspaces of ordinal spaces, and related them to the property that X contains no closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal. See [3] for further details.

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f-8 Unicoherence and Multicoherence

A ***topological space*** X is said to be **unicoherent** provided that it is ***connected*** and for every two ***closed*** connected sets A and B with $X = A \cup B$ the intersection $A \cap B$ is connected. Euclidean spaces $\mathbb{R}^n$, cubes $[0, 1]^n$ for any positive integer n, spheres S^n for $n \geqslant 2$, real projective spaces $P_n(\mathbb{R})$ also for $n \geqslant 2$, Hilbert cube $[0, 1]^\omega$ and ***solenoids*** are unicoherent, while the circle S^1, the **torus** $S^1 \times S^1$ and the Möbius band are not.

The notion of unicoherence (introduced in the middle twenties of the XX century by K. Kuratowski and, independently, by L. Vietoris) is a special case of a general concept of multicoherence (introduced and studied by S. Eilenberg in the thirties). Given a topological space Y, let $b_0(Y)$ denote the number of components of Y less one if this number is finite, and ∞ otherwise. The **multicoherence degree** $r(X)$ of a connected topological space X is defined by: $r(X) = \sup\{b_0(A \cap B)$: A and B are closed connected subsets of X with $X = A \cup B\}$. Thus X is unicoherent if and only if $r(X) = 0$; otherwise X is said to be **multicoherent**.

Unicoherent spaces can be characterized in many ways. The following are equivalent for a space X:

(a) X is unicoherent;
(b) if C and D are disjoint connected subsets of X such that $\operatorname{bd} C \subset \operatorname{bd} D$, then $\operatorname{bd} C$ is connected;
(c) if A is a closed and connected subset of X and R is a component of $X \setminus A$, then $\operatorname{bd} R$ is connected;
(d) if an open subset V of X separates X, then some component of V also separates X;
(e) if C and D are disjoint closed subsets of X which do not separate some two points in X, then $C \cup D$ also does not separate these points;
(f) every (closed) subset of X that separates X between some two points in X has a component that separates X between these points;
(g) every closed subset of X that separates X irreducibly between some two points is connected;
(h) every (closed) subset of X separating X has a component that separates X.

References to all these results can be found in the survey article [5]. Many of them were generalized in [9] using the multicoherence degree; compare also [2, 3, 5] for various characterizations of multicoherent spaces. Let $n \geqslant 3$ be an integer. The **Stone conjecture** $S(n)$ says that a connected, locally connected, ***normal*** space X is multicoherent if and only if it can be represented as the union of n closed connected subsets $A_1, \ldots, A_n$ such that $A_i \cap A_j \neq \emptyset$ iff $|i - j| = 1$ (the suffixes being taken modulo n). It is known that $S(n)$ is true for $n = 3, 4$ and false for $n \geqslant 5$. $S(6)$ holds for compact X, while $S(7)$ is false even for ***Peano continua***.

There are many spaces for which it is difficult to decide, using only definitions or the above intrinsic properties, whether or not they are unicoherent. So, since very beginning of the study of the concept, it was necessary to develop alternative characterizations. The most powerful are those which make use of maps of the space into some "external" spaces. In particular, the use of maps into the circle S^1 to study unicoherence was introduced by K. Borsuk in 1931, developed for metric spaces by S. Eilenberg in 1935–1937 (see [11, Chapter 11]), and extended to topological spaces by T. Ganea in the early fifties of the XX century. A map $f : X \to S^1$ of a topological space X is said to be **equivalent to 1** provided that there exists a map $g : X \to \mathbb{R}$ such that $f = \exp \circ g$. It is known that f is equivalent to 1 if and only if f is ***homotopic*** to a constant map. For a connected, ***locally connected*** and normal space X the following are equivalent: (a) X is unicoherent; (b) each map $f : X \to S^1$ is equivalent to 1; (c) each map $f : X \to S^1$ is homotopic to a constant map. Spaces satisfying (c) are called **acyclic space**s.

Let a map $f : Z \to Y$ between topological spaces be given. A set $K \subset Y$ is **evenly covered** by (Z, f) if $f^{-1}(K)$ is nonempty and each of its components is mapped homeomorphically under f onto K. A **covering space** of Y is a pair (Z, f) such that: the space Z is connected and locally connected, the map $f : Z \to Y$ is a surjection, and each point of Y has a neighbourhood that is evenly covered by (Z, f) (if (Z, f) is a covering space of Y, then f is a ***local homeomorphism*** and Y is also connected and locally connected). A covering space (Z, f) of Y is said to be a **binary covering space** if there exist two open connected subsets U and V of Y whose union is Y and which both are evenly covered by f; and non-trivial if f is not a homeomorphism. Let a space X be connected and locally connected. Then (a) X is unicoherent if and only if for each map $\varphi : X \to Y$ of X into a connected and locally connected space Y and for each binary covering space (Z, f) of Y there exists a map $g : X \to Z$ such that $f \circ g = \varphi$; (b) X is multicoherent if and only if it has a non-trivial binary covering space, [5, Theorems 2.7 and 2.9].

In [5] results are collected about multicoherence (unicoherence) of various ***compactifications***. In particular, definitions of the weak multicoherence degree, the γ-multicoherence degree and the hereditary multicoherence degree are recalled and connections between them are gathered for several compactifications of a ***Tychonoff*** connected space X.

A further group of results discussed in [5] concerns products of topological spaces. We quote some of them. Answering an old question of K. Kuratowski and generalizing results of K. Borsuk and S. Eilenberg, T. Ganea proved that the product of an arbitrary family of locally connected unicoherent spaces is unicoherent. A.H. Stone has shown in [10] that if $\{X_n: n \in \mathbb{N}\}$ is a countable family of metric ***continua***, then $r(\Pi\{X_n: n \in \mathbb{N}\}) = \sup\{r(X_n): n \in \mathbb{N}\}$. A similar result holds for an arbitrary family $\{X_j: j \in J\}$ of connected, locally ***pathwise connected*** normal T_1 spaces X_j, [5, Theorem 5.4].

For a topological space X and any $n \in \mathbb{N}$ let $F_n(X)$ denote the hyperspace consisting of all nonempty subsets of X having at most n elements, with the ***Vietoris topology***. The following results are known for a ***Hausdorff space*** X, [5, Theorems 5.7–5.9]. If the space X is connected, locally connected and unicoherent, then $F_n(X)$ is unicoherent for all n. If X is pathwise connected and locally connected, then $F_n(X)$ is unicoherent for all $n \geqslant 3$. If $F_2(X)$ is normal and X is connected, locally connected and multicoherent, then $r(F_2(X)) = 1$. If X is a metric continuum, then $r(F_2(X)) \leqslant 1$ and $r(F_n(X)) = 0$ (i.e., $F_n(X)$ is unicoherent) for each $n \geqslant 3$.

For unicoherence and multicoherence of other ***hyperspaces*** of a (metric) continuum X, such as the hyperspace 2^X of all nonempty closed subsets of X, $C(X)$ of all subcontinua of X, and $C_n(X)$ of all nonempty closed subsets of X having at most n components (where n is a positive integer) see [6]. It is known that for each (metric) continuum X the above mentioned hyperspaces are all unicoherent.

The following extension of the notion of unicoherence of a topological space S is known. (1) S is (-1)-coherent provided that $S \neq \emptyset$. (2) S is $(n+1)$-coherent if S is n-coherent, locally n-coherent, and whenever $S = A \cup B$, where A and B are closed n-coherent subsets of S, the intersection $A \cap B$ is n-coherent. (3) S is locally n-coherent at a point $p \in S$ provided that if U is an open set containing p, there exists an n-coherent open set V such that $p \in V \subset U$. (4) S is locally n-coherent provided that it is locally n-coherent at each of its points. Note that 0-coherence is connectedness and 1-coherence is unicoherence plus local connectedness.

Since the concepts of unicoherence and multicoherence have proved to be very useful and important tools in continuum theory, a large number of results concerning them are related to (metric) continua. A locally connected continuum X is said to be a **Janiszewski space** provided that if A and B are subcontinua of X whose intersection $A \cap B$ is not connected, the union $A \cup B$ cuts X. Every Janiszewski space is unicoherent, and if it is nondegenerate and contains no ***cut point***, it is homeomorphic to the sphere S^2. A comprehensive information on these spaces is contained in [7, §61, p. 505–542]. A continuum is said to be **hereditarily unicoherent** provided that any of its subcontinua is unicoherent, i.e., provided that the intersection of any two of its subcontinua is connected. The following are equivalent for a metric continuum X: (1) is hereditarily unicoherent; (2) for every two points in X there exists exactly one continuum which is ***irreducible*** between them; (3) each **monotone map** (i.e., with connected preimages of points) defined on X is **hereditarily monotone** (i.e., its restriction to any subcontinuum of X is monotone), [8]. The notions of ***weak cut point order*** and ***closed quasi-order*** have been useful in studying the structure of such continua. The class of hereditarily unicoherent continua contains some important subclasses; in the realm of one-dimensional metric continua, i.e., of ***curves***, we distinguish the following. A locally connected continuum containing no ***simple closed curve*** is called a **dendrite**. A **dendroid** means an ***arcwise connected*** and hereditarily unicoherent continuum. A ***hereditarily decomposable*** and hereditarily unicoherent continuum is named a **λ-dendroid**. A continuum is said to be **tree-like** provided that for each $\varepsilon > 0$ there exist a ***tree*** T and a surjective map $f : X \to T$ such that diam $f^{-1}(y) < \varepsilon$ for each $y \in Y$. These classes are related as follows.

$$\begin{aligned}&\{\text{dendrites}\} \subset \{\text{dendroids}\} \subset \{\lambda\text{-dendroids}\}\\&\quad\subset \{\text{tree-like continua}\} \subset \{\text{acyclic curves}\}\\&\quad\subset \{\text{hereditarily unicoherent curves}\}. \qquad (1)\end{aligned}$$

Important examples of hereditarily unicoherent curves are ***solenoids***. S.B. Nadler Jr has shown that if a (Hausdorff) continuum X is the ***inverse limit*** of an ***inverse system*** $\{X_\lambda, f_{\lambda\mu}, \Lambda\}$ of continua with onto bonding maps, then $r(X_\lambda) \leqslant k$ implies $r(X) \leqslant k$ for any fixed $k < \infty$. In particular, unicoherence of all X_λ implies unicoherence of the limit X, and the same implication holds for hereditary unicoherence. If the bonding maps are additionally monotone, then $r(X_\lambda) = k$ implies $r(X) = k$. These results have many consequences concerning the implication

$$X_\lambda \in \mathcal{K} \text{ for all } \lambda \in \Lambda \implies X \in \mathcal{K},$$

where $\mathcal{K}$ is a class in (1).

The concept of unicoherence has been modified in several ways. A connected space X is said to be **weakly unicoherent** provided that for every two closed connected sets A and B one of which is compact, if $X = A \cup B$ then $A \cap B$ is connected. In [1, 4] characterizations of these (locally connected) spaces are given.

A continuum is said to be **hereditarily unicoherent at a point** p provided that the intersection of any two subcontinua each of which contains p is connected. This localization is used to study ***smoothness*** of continua and their ***decompositions***. A continuum X is said to be **strongly unicoherent** provided that it is unicoherent and for every two proper subcontinua A and B of X with $X = A \cup B$ each of A and B is unicoherent. A continuum is strongly unicoherent if and only if each of its subcontinua with nonempty interior is unicoherent. A continuum is said to be **weakly hereditarily unicoherent** provided that the intersection of any two of its subcontinua with nonempty interiors is connected. This class of continua is intermediate between strongly unicoherent and hereditarily unicoherent ones. For hereditarily decomposable continua the concepts of strong unicoherence and weak hereditary unicoherence do agree, and for arcwise connected continua all three concepts coincide. A continuum X is said to be **unicoherent at a subcontinuum** Y of X provided that for every two proper subcontinua A and B with $X = A \cup B$ the intersection $A \cap B \cap Y$ is connected.

Invariance of multicoherence (of unicoherence) under certain maps is a subject of many results. A map $f : X \to Y$ is said to be an r**-map** if there exists a map $g : Y \to X$ such that $f \circ g = \mathrm{id} \restriction Y$ (every ***retraction*** is an r-map); a **quasi-monotone map** if for each closed connected set $Q \subset Y$ having nonempty interior, $f^{-1}(Q)$ has finitely many components each of which is mapped onto Q under f; a **confluent map** if for each continuum $Q \subset Y$ each component of $f^{-1}(Q)$ is mapped onto Q under f; a **hereditarily confluent map** if for each continuum $C \subset X$ the restriction $f \restriction C$ is confluent; and a **locally monotone map** if each point $p \in X$ has a closed neighbourhood V such that $f(V)$ is a closed neighbourhood of $f(p)$ and the restriction $f \restriction V$ is monotone. Let a surjective map $f : X \to Y$ be given. If either (a) X is connected and locally connected and f is an r-map or (b) X is connected and f is quasi-monotone, then $r(Y) \leqslant r(X)$, [5]. Confluent maps preserve hereditary unicoherence of hereditarily decomposable continua (thus the concepts of a dendrite, a dendroid, and a λ-dendroid are invariants under confluent maps); tree-likeness of continua is also preserved; hereditarily confluent maps as well as locally monotone ones preserve hereditary unicoherence of continua, [8, Chapter 7]. Quasi-monotone compact maps (i.e., with preimages of compact sets being compact) preserve weak unicoherence of locally connected ***locally compact*** connected ***separable*** metric spaces, [4].

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f-9 Topological Characterizations of Separable Metrizable Zero-Dimensional Spaces

A space X is called **zero-dimensional** if it is nonempty and has a base consisting of ***clopen*** sets, i.e., if for every point $x \in X$ and for every neighbourhood U of x there exists a clopen subset $C \subseteq X$ such that $x \in C \subseteq U$. It is clear that a nonempty subspace of a zero-dimensional space is again zero-dimensional and that products of zero-dimensional spaces are zero-dimensional.

We say that a space X is ***totally disconnected*** if for all distinct points x, y in X there exists a clopen set C in X such that $x \in C$ but $y \notin C$. It is clear that every zero-dimensional space is totally disconnected. The question naturally arises whether every totally disconnected space is zero-dimensional. If this were true then checking whether a given space is zero-dimensional would be simpler. However, the answer to this question is in the negative, as was shown by Sierpiński (cf. [KU, Chapter V, §46.VI, Footnote 2]).

In part 1 of this note, we are interested in theorems that state nontrivial and useful topological characterizations of zero-dimensional ***separable metrizable spaces***. So it will be convenient in part 1 to let 'space' denote 'separable metrizable space'. In part 2 we briefly mention a few results for general ***Tychonoff space***s that are in the same spirit.

1. Separable metrizable spaces

It is easy to see that a zero-dimensional space can be embedded in the real line $\mathbb{R}$, and that a nonempty subspace X of $\mathbb{R}$ is zero-dimensional if and only if it does not contain any nondegenerate interval. For example, the **space of rational numbers** $\mathbb{Q}$, the **space of irrational numbers** $\mathbb{P}$, the product $\mathbb{Q} \times \mathbb{P}$, etc., are all zero-dimensional.

A zero-dimensional space X is **strongly homogeneous** provided that all of its nonempty clopen sets are homeomorphic. It is easy to see that every strongly homogeneous space is ***homogeneous***, [5, 1.9.3]. It is tempting to conjecture that all homogeneous zero-dimensional spaces are strongly homogeneous. This is not true however, as was shown by van Douwen.

If $\mathcal{P}$ is a topological property then a space is called **nowhere** $\mathcal{P}$ provided that no nonempty open subset of it has $\mathcal{P}$. The characterizations theorems that we will mention below are all of the following form: up to homeomorphism, there is only one zero-dimensional space which has $\mathcal{P}$ but is nowhere $\mathcal{Q}$. Here $\mathcal{P}$ and $\mathcal{Q}$ can be quite complex topological properties. An interesting consequence will be that all nonempty clopen sets of the spaces considered share the same properties. This means that all these spaces are strongly homogeneous and hence homogeneous. So the characterization theorems give us homogeneity for free. (This phenomenon is not uncommon in topology.)

The following example of a zero-dimensional *compact* space is of particular interest. From $\mathbb{I} = [0, 1]$ remove the interval $(1/3, 2/3)$, i.e., the 'middle-third' interval. From the remaining two intervals, again remove their 'middle-thirds', and continue in this way infinitely often. What remains of $\mathbb{I}$ at the end of this process is called the **Cantor middle-third set**, C. A space homeomorphic to C is called a **Cantor set**.

It is easy to see that C is the subspace of $\mathbb{I}$ consisting of all points that have a triadic expansion in which the digit 1 does not occur, i.e., the set

$$\left\{x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} : x_i \in \{0, 2\} \text{ for every } i\right\}.$$

The Cantor set was introduced by Cantor. It is clearly closed in $\mathbb{I}$, hence is compact. It also has no isolated points and is zero-dimensional because it does not contain any nontrivial interval. Interestingly, the mentioned properties topologically characterize C: up to homeomorphism, C is the only zero-dimensional compact space without isolated points. This is due to Brouwer [5, Theorem 1.5.5]. Observe that the characterization theorem implies that C is homeomorphic to the Cantor cube $\{0, 1\}^{\infty}$. Hence one can look upon Brouwer's Theorem also as a topological characterization of $\{0, 1\}^{\infty}$.

The Cantor set C topologically surfaces 'almost everywhere'. By a result of Souslin, every uncountable ***analytic space*** contains a topological copy of C [4, Exercise 14.13]. This result was partly generalized by van Douwen: if a ***topologically complete*** space X contains an uncountable family of pairwise disjoint homeomorphs of some compact space K then X contains a copy of the product $\mathsf{C} \times K$ [5, Corollary 1.5.15]. Observe that the uncountably many pairwise disjoint homeomorphs of K may be irregularly embedded. Van Douwen's result shows that they can be replaced by a Cantor set of 'regularly' embedded copies of K. Since Souslin's theorem works for analytic spaces, the question naturally arises whether van Douwen's theorem also holds for analytic spaces. This question was considered by Becker, van Engelen and van Mill; it is undecidable.

The Cantor set is a universal object for the class of all zero-dimensional spaces. By a result of Alexandroff and Urysohn, every compact space is a continuous image of C [5, Theorem 1.5.10]. This is also some sort of universal property.

Cantor sets are widely studied in geometric topology. It is easy to prove that all Cantor sets in $\mathbb{R}$ are topologically equivalent. The same result is also true in $\mathbb{R}^2$ but the proof is more difficult. In $\mathbb{R}^3$ there are 'wild' Cantor sets however, the most famous one of which is Antoine's necklace. Similar sets can be constructed in $\mathbb{R}^n$ for $n \geqslant 3$ and in the ***Hilbert cube*** Q. In contrast, in the **countable infinite product of lines** $\mathbb{R}^\infty$, all Cantor sets are tame. For details, and references, see [2, 5].

The rational numbers $\mathbb{Q}$ can also be characterized in topological terms. It is, up to homeomorphism, the only countable space without isolated points. This is due to Sierpiński [5, Theorem 1.9.6]. As is the case with Cantor sets, topological copies of $\mathbb{Q}$ surface 'almost everywhere'. It was shown for example by Hurewicz that if a space X is not a ***Baire space*** then X contains a closed copy of $\mathbb{Q}$ (the simple proof of this which was presented in [5, Theorem 1.9.12] is due to van Douwen).

Alexandroff and Urysohn proved that, up to homeomorphism, $\mathbb{P}$ is the only zero-dimensional topologically complete space which is nowhere compact [5, Theorem 1.9.8]. This implies for example that $\mathbb{P}$ is homeomorphic to $\mathbb{N}^\infty$, where $\mathbb{N}$ is the (discrete) space of natural numbers. They also proved that $\mathbb{Q} \times \mathsf{C}$ is up to homeomorphism the only σ-compact zero-dimensional space which is nowhere compact and nowhere countable. A similar result was obtained by the author: $\mathbb{Q} \times \mathbb{P}$ is the topologically unique zero-dimensional space which is a countable union of closed topologically complete subspaces and which in addition is nowhere topologically complete and nowhere σ-compact. It was subsequently shown by van Engelen [3] that $\mathbb{Q}^\infty$ is the only zero-dimensional absolute $F_{\sigma\delta}$ which is first category and nowhere an absolute $G_{\delta\sigma}$.

Observe that all the spaces considered so far are ***topological groups***. This is clear for $\mathbb{Q}$, being a subgroup of $\mathbb{R}$. It is also clear for C and $\mathbb{P}$ since their characterization theorems imply that they are homeomorphic to the topological groups $\{0, 1\}^\infty$ and $\mathbb{Z}^\infty$, respectively. Since products of topological groups are topological groups, we are also done for $\mathbb{Q} \times \mathsf{C}$, $\mathbb{Q} \times \mathbb{P}$ and $\mathbb{Q}^\infty$. These observations prompted van Douwen to ask whether every zero-dimensional homogeneous space admits the structure of a topological group. He proved that there is a (strongly homogeneous) zero-dimensional space T which is the union of a topologically complete and a countable subspace, and which is nowhere σ-compact and nowhere topologically complete. As to be expected, T is topologically characterized by these properties. Since a topological group containing a dense topologically complete subspace is topologically complete, we have that T is not a topological group. But it *is* homogeneous, being strongly homogeneous. The space T can easily be visualized. Indeed, let D be a countable dense subset of C, and put $P = \mathsf{C} \setminus D$. Then the subspace

$$T = (\mathsf{C} \times \mathsf{C}) \setminus (D \times P)$$

of $\mathsf{C} \times \mathsf{C}$ and T are homeomorphic. Van Douwen did not live long enough to publish his results on T. For details, see [3]. So not all homogeneous zero-dimensional ***absolute Borel sets*** are topological groups. One could revive van Douwen's problem by asking whether every zero-dimensional homogeneous absolute Borel set admits a ***transitive*** action by a topological group. It was recently shown by the author that the answer to this question is in the affirmative.

The difficult problem to topologically characterize *all* zero-dimensional homogeneous absolute Borel sets was solved by van Engelen in [3]. He used the hierarchy of small Borel classes in Δ_3^0 to characterize all homogeneous zero-dimensional absolute Borel sets of ambiguous class 2. However, when extended to the classes Δ_α^0 for $\alpha < \omega_1$, this hierarchy turned out to be too coarse to distinguish between all zero-dimensional homogeneous absolute Borel sets. Instead, he used the so-called Wadge hierarchy of Borel sets developed by Wadge (see [4]) and powerful results of Louveau, Martin, and Steel. He concluded that there are precisely ω_1 homogeneous zero-dimensional absolute Borel sets and that they can all be topologically characterized. The question which of those spaces admits the structure of a topological group is not completely solved. It is known that if a first category zero-dimensional absolute Borel set is homeomorphic to its own square then it is a topological group (van Engelen). The conjecture is that the converse holds.

Call a space ***rigid*** if the identity is its only homeomorphism. It is a natural question related to the results mentioned here whether there exist rigid zero-dimensional absolute Borel sets. This question was posed by van Douwen. The zero-dimensionality is essential for there are simple examples of rigid continua. Van Engelen, Miller and Steel answered it in the negative, [3]. So any zero-dimensional absolute Borel set admits a nontrivial homeomorphism. As far as we know, it is unknown whether the same result can be proved for zero-dimensional analytic spaces.

2. General spaces

There are only a few results known for general topological spaces that are in the same spirit as the results presented in Section 1. An important result to be mentioned is the characterization of $\beta\mathbb{N} \setminus \mathbb{N}$ by Parovičenko, [E, p. 236]. Here $\beta\mathbb{N} \setminus \mathbb{N}$ is the ***Čech-Stone remainder*** of the discrete space $\mathbb{N}$. Parovičenko's characterization states that under the ***Continuum Hypothesis***, $\beta\mathbb{N} \setminus \mathbb{N}$ is the topologically unique zero-dimensional compact ***F-space*** of weight $\mathfrak{c}$ in which nonempty G_δ's have infinite interior. It is known that the assumption of the Continuum Hypothesis is essential in Parovičenko's characterization. There are some alternative and useful characterizations under weaker axioms, but it would lead us too far to go into that. The space $\beta\mathbb{N} \setminus \mathbb{N}$ was and is widely studied, see e.g., the articles on ultrafilters and βX in this volume for more information.

We saw above that C is 'co-universal' for second-countable compact spaces. The space $\beta\mathbb{N} \setminus \mathbb{N}$ has a similar property. It was proved by Parovičenko that every compact space of weight at most ω_1 is a continuous image of $\beta\mathbb{N} \setminus \mathbb{N}$, [E,

p. 236]. Only recently similar results for other classes of spaces were obtained. For example, Dow and Hart proved that every continuum of weight at most ω_1 is a continuous image of the space $\beta\mathbb{H} \setminus \mathbb{H}$; here $\mathbb{H} = [0, \infty)$.

In part 1, we stated Brouwer's characterization of the Cantor cube $\{0, 1\}^\infty$. It is a natural question to ask whether a similar characterization can be found for Cantor cubes of larger weight, i.e., for spaces of the form $\{0, 1\}^\kappa$, where κ is some uncountable cardinal. This is indeed the case. Call a compact space X an ***Absolute Extensor*** in dimension 0 if for each zero-dimensional compactum Z and for each closed subspace Z_0 of Z, any continuous function $f : Z_0 \to X$ can be extended over Z. Ščepin proved that if $\kappa > \omega$ then a zero-dimensional compact space X of weight κ is homeomorphic to $\{0, 1\}^\kappa$ if and only if X is an Absolute Extensor in dimension 0 while moreover all points in X have the same ***character*** [1, Theorem 8.1.6]. This result has nice applications. It can be used for example to prove Sirota's Theorem that $\{0, 1\}^{\omega_1}$ is homeomorphic to its own ***hyperspace***. (It is known that there is no corresponding result for $\{0, 1\}^{\omega_2}$.)

There are similar topological characterizations of various other zero-dimensional spaces. For example, it is possible to generalize the topological characterization of $\mathbb{P}$ stated in part 1 to topological characterizations of all spaces of the form $\mathbb{N}^\kappa$, where $\kappa > \omega$. For this result and others we refer the reader to Chigogidze [1].

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f-10 Topological Characterizations of Spaces

Despite over a century of attention, there are surprisingly few topological spaces that can be given succinct elementary characterizations in purely topological terms. In this article, we list some of the more prominent ones, restricting ourselves to the ***separable metric spaces***, which are characterized by the **Urysohn's Metrization Theorem** as the ***regular Hausdorff*** spaces satisfying the ***second axiom of countability***. The reader should consult the article *Topological characterizations of separable metrizable zero-dimensional spaces* for a deeper treatment of that topic than given here, as well as the articles *CE-mappings, Manifolds, Infinite-Dimensional Topology, Absolute Retracts*, and *Topological Embeddings* (*Universal Spaces*). There is also additional material contained in Chapters 25, 26, and 30 in [vMR] on *Topological Classification of Incomplete Metric Spaces*, by T. Dobrowolski and J. Mogilski, *Finite-Dimensional Manifolds*, by R. Daverman, and *Infinite-Dimensional Topology*, by J. West.

We begin with the simplest and most widely applied, Cantor's 'middle third' set, which may be described as the intersection

$$C = \bigcap_{n=0}^{\infty} A_n$$

where $A_0 = [0, 1]$, and for each $n \geqslant 1$, A_n is the union of the collection of closed sub-intervals $I_{n,m}$ obtained by deleting the open middle third of each of the intervals $I_{n-1,k}$ of A_{n-1}, i.e., $A_n = \bigcup_k I_{n,k}$, where $I_{n,k} = [k/3^n, (k+1)/3^n]$ and k is of the form $\sum_{i=1}^{n} a_i 3^i$ with $a_i \in \{0, 2\}$.

A space is **zero-dimensional** provided that it has a base of open sets with empty boundary; it is a **perfect space** provided that it has no isolated points.

THE CANTOR SET. *A non-void topological space X is homeomorphic with the Cantor set C provided that it is compact, metrizable, zero-dimensional, and perfect.*

Next, we deal with other subsets of the real line. If X is a connected topological space, a subset $S \subset X$ **separates** X provided that $X - S = A \cup B$, where A and B are open, disjoint, and non-empty. X is non-degenerate provided that it has more than one point. A **component** of a space is a maximal connected set.

THE REAL LINE. *Every non-empty locally compact, connected metrizable space X such that each point of X separates X into exactly two components is homeomorphic with the real numbers.*

THE CLOSED INTERVAL. *A compact connected metrizable space X is homeomorphic with the closed interval* $[0, 1]$ *provided that it has exactly two points with connected complements.*

A space homeomorphic with the closed interval is frequently called an **arc**; if the **end point**s, the ones that do not separate the arc, are deleted, the result, homeomorphic with the real numbers, is often termed an **open arc**.

THE RATIONAL NUMBERS. *A separable metrizable space X is homeomorphic with the rational numbers provided that it is infinite, countable and perfect.*

A metrizable space is an **absolute G_δ-set** provided that whenever it is embedded in a metrizable space, Y, its image is the intersection of a countable collection of open subsets of Y. This is equivalent to the existence of a ***complete metric*** for X [E, 4.3].

THE IRRATIONAL NUMBERS. *A separable metrizable space is homeomorphic with the irrational numbers provided that it is a zero-dimensional, perfect, nowhere locally compact, absolute G_δ space.*

A **simple closed curve** is a space homeomorphic with the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$.

THE CIRCLE. *A non-degenerate compact metrizable space is a simple closed curve provided that no point separates it and that each pair of points separates it into precisely two components.*

R.L. Moore gave [12] a system of eight axioms together with a proof that they characterize the **2-sphere**, $S^2 = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$, in the compact case (Chapter VII, p. 362, Theorem 14) and the Plane in the non-compact case (*op. cit.*, p. 356, Theorem 6). They are paraphrased in the following.

THE PLANE AND 2-SPHERE. *Let X be a topological space satisfying the following*:

(1) *There is a sequence $G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$ of open covers of X such that for each open set $U \subset X$ and each two distinct points $x, y \in U$, there is an integer m such that if $V \in G_m$ contains x, then $y \notin \overline{V}$ and $\overline{V} \subset U$. Furthermore, any sequence of closed sets $M_1 \supset M_2 \supset \cdots \supset M_n \supset \cdots$ such that M_n is contained in some element of G_n has a nonempty intersection.*
(2) *X contains at least 3 points, and no point separates X.*
(3) *Each simple closed curve J in X separates it into two connected components of each of which J is the boundary.*

(4) *X is locally connected; moreover, each point has a base of connected neighbourhoods with connected boundaries.*
(5) *If U is a connected open set in X with connected boundary, then no arc that is contained in U except possibly for one of its end points separates U.*
(6) *If $K \subset X$ is compact and connected, then there is at most one component of $X - K$ the closure of which is not compact.*
(7) *X has a countable base for its topology.*

Then X is homeomorphic with the 2-sphere if it is compact and with the Euclidean Plane if it is not compact.

The next theorem, also proved in [12], is the customary succinct reformulation in modern terminology of the preceding one. Although it lacks the philosophically satisfying aspect of a complete characterization, it is the one that is found widely in applications. It marks the beginning of a long development (cf. *Cell-Like Maps, Manifolds*).

R.L. MOORE'S THEOREM. *Let X be a non-degenerate Hausdorff space that is the image of a continuous map $f : S^2 \to X$, where S^2 is the two-dimensional sphere. If each point inverse $f^{-1}(x)$ is connected and does not separate S^2, then X and S^2 are homeomorphic.*

A different statement is the characterization of the 2-sphere conjectured by J.R. Kline and proved by R.H. Bing [4, 13].

KLINE SPHERE CHARACTERIZATION. *A non-degenerate, connected, locally connected, compact metrizable space X is homeomorphic with S^2 provided that each simple closed curve in X separates it, but no pair of points separates it.*

A subset of the plane that is much studied was discovered by B. Knaster [11] in his 1921 Thesis, named the Pseudo-arc by E. Moise, and characterized by R.H. Bing [3]. Terminology for the Bing chaining characterization follows. A connected metric space is **chainable** provided that for each $\varepsilon > 0$ it is the union of a finite sequence $C_1, \ldots, C_n$ of open sets (the links of the chain) of diameter less than ε such that $C_i \cap C_j = \emptyset$ if $|i - j| > 1$. (This is the same thing as saying that there is a map onto an arc with point-inverses of diameter less than δ, for each $\delta > 0$.) A **continuum** K is a compact, connected set. It is ***decomposable*** provided it is the union of two sub-continua, neither of which equals K. It is ***indecomposable*** if it is not decomposable. It is **hereditarily indecomposable** if each of its sub-continua is indecomposable. Bing showed that one recipe for constructing such a continuum in the plane is as the intersection of **crooked chains** in the plane. A chain $\mathcal{C}$ is crooked in another, $\mathcal{D}$, provided that the links of $\mathcal{C}$ are subsets of those of $\mathcal{D}$ and that if $C_i \cap D_h \neq \emptyset \neq C_j \cap D_k$, with $|h-k| > 2$, then there are links C_s, and C_t of C between C_i and C_j, with $i < s < t < j$ or $i > s > t > j$, such that if $h < k$, $C_s \subset D_{k-1}$ and $C_t \subset D_{h+1}$ (and if $h > k$, then $C_s \subset D_{k+1}$ and $C_t \subset D_{h-1}$). One needs pictures to absorb this.

THE PSEUDO-ARC. *Every non-degenerate hereditarily indecomposable, chainable, metric continuum is homeomorphic with the pseudo-arc.*

Another interesting collection of continua admitting an elegant characterization is the solenoids of D. van Dantzig and L. Vietoris. They are most easily defined as ***inverse limits*** of circles. Let C denote the circle viewed as the complex numbers of modulus 1. Then for each positive integer, n, we have the n-th power map $p_n : C \to C$ given by $p_n(z) = z^n$. For a sequence $\sigma = n_1, n_2, \ldots$ of positive integers, the **solenoid** $\Sigma(\sigma)$ is the ***inverse limit*** of C with the sequence of maps p_{n_i} as the ***bonding map***s. C. Hagopian, building on earlier work of Bing, gave the following [9].

SOLENOIDS. *Let X be a non-degenerate homogeneous metric continuum. If every non-degenerate sub-continuum of X is an arc, then X is homeomorphic with a solenoid.*

Proceeding to n-dimensional cubes and $\mathbb{R}^n$, J. de Groot gave a characterization of I^n and of the **Hilbert cube**, $\prod_{n=1}^{\infty}[0, 1]_n$ [8] (cf. [5]). A collection $\mathcal{S}$ of closed sets of a topological space X is a **closed subbase** for X provided that each closed set of X is the intersection of a collection of finite unions of members of $\mathcal{S}$. $\mathcal{S}$ is **comparable** if whenever three members S_1, S_2, and T of $\mathcal{S}$ are given with $S_1 \cap T = S_2 \cap T = \emptyset$, then $S_1 \subset S_2$ or $S_2 \subset S_1$. A subcollection $\mathcal{F}$ of $\mathcal{S}$ is **linked** provided each pair of elements has a non-void intersection. $\mathcal{S}$ is a **binary family** if each linked subcollection has a non-void intersection.

I^n AND I^∞. *A connected T_1 space X of* ***dimension*** *n, $n = 1, 2, \ldots, \infty$ is homeomorphic with the Cartesian product of n closed intervals if and only if it has a countable closed sub-base that is comparable and binary.*

(By ∞, is meant countable infinity.) A more refined version of this theorem, in which the product structure is more clearly apparent, has been given by J. Bruijning [5] following ideas of de Groot.)

A topological space X is called ***n*-connected** provided that for each $k \leqslant n$, each map into X of a k-sphere extends continuously to the $k+1$-ball. It is called **locally *n*-connected** provided that for each point $x \in X$ and neighbourhood U of x, there is a neighbourhood V of x contained in U such that, for each $k \leqslant n$, each map of a k-sphere into V extends to a map of the $k+1$-ball into U.

A different approach to characterizing $\mathbb{R}^n$ and n-manifolds generally is the following theorem of R.D. Edwards, derived from the pioneering work of R.L. Moore given above through work of R.H. Bing, J.W. Cannon, and others. See *CE Maps, Manifolds*. The necessary definitions follow.

(1) X has Cannon's ***disjoint discs property*** (cf. *CE-Mappings*) if for each $\varepsilon > 0$, each two maps $f, g : I^2 \to X$ have ε-approximations $\bar{f}, \bar{g} : I^2 \to X$ with disjoint images.
(2) A space is **CE-resolvable** by an n-manifold M if there is CE-map $f : M \to X$.

(3) A locally compact separable metric space X is called a **generalized n-manifold** (or **ANR homology n-manifold**) provided that it is a finite-dimensional ***Absolute Neighborhood Retract*** with the property that for each point $x \in X$ and each neighbourhood U of x, the pair $(U, U - \{x\})$ is $(n-1)$-connected and the ***relative singular homology*** with integer coefficients $H_n(X, X - \{x\})$ is infinite cyclic.

(The homology of a polyhedron, using integer coefficients is discussed briefly in [K]. Singular homology generalizes the spaces that can be treated by using, instead of actual geometric simplices in the polyhedron, singular simplices, which are the continuous maps from geometric simplices into a space; otherwise, it has the same definition. If $A \subset X$, the **relative homology group** of the pair, $H_k(X, A)$, is defined by setting each simplex lying entirely in A equal to zero in the chain group. The singular homology group, $H_k(S^n)$, of the n-dimensional sphere is the same as that of the pair $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$, and is zero unless $k = n$, in which case it is infinite cyclic.)

n-MANIFOLDS. *Let $n \geqslant 5$. A generalized n-manifold with the Disjoint Disks Property is a topological manifold (of dimension n) provided that it is CE-resolvable by an n-manifold.*

This characterization is profoundly unsatisfying for two reasons: Firstly, there is the restriction $n \geqslant 5$; secondly, it is not strictly a characterization at all, since manifolds are used essentially in the statement. The first objection is somehow connected with the still open Poincaré Conjecture that a compact, connected, 1-connected 3-manifold without boundary is homeomorphic with S^3, the sphere of the unit ball in $\mathbb{R}^4$. Indeed, the techniques used to prove this theorem have been important in settling positively the appropriate generalization of the Poincaré conjecture in higher dimensions (see below), most particularly for $n = 4$.

The second objection has the interesting history that at first it was anticipated that generalized n-manifolds would all be CE-resolvable by n-manifolds, but this is now known not to be true [6] in dimensions six and above. One may replace the resolvability hypothesis by the vanishing of an algebraic-topological index due to F. Quinn, the definition of which is too complex for this article, thus obtaining a truly independent characterization. Quinn's index has the useful property that it does vanish if there is a single point of X (in each component) with a neighbourhood that is homeomorphic with an open subset of $\mathbb{R}^n$. (See *CE-Maps, Manifolds*.)

The Poincaré Conjecture has been affirmed in all higher dimensions. Its importance may be gauged from the fact that two Fields Medals have been awarded for this work (S. Smale, for smooth manifolds of dimension at least six, and M. Freedman, for topological manifolds in dimension four.)

For connected n-manifolds, M, and $n > 1$, there are two possibilities for $H_n(M)$; either it is infinite cyclic, in which case the manifold is **orientable**, or it is the group with two elements, in which case M is not orientable.

S^n. *If $n \neq 3, 0$, a compact connected n-manifold without boundary is homeomorphic with the n-dimensional sphere S^n provided that it is orientable and $(n-1)$-connected.*

There are several characterizations of the Hilbert cube. For example, if X is a topological space, the **cone** of X is the ***quotient space*** $X \times I / X \times \{0\}$. R.M. Schori (unpublished) and independently Lay and Walsh gave the following.

I^∞. *The Hilbert cube is the only (non-empty) compact metrizable **Absolute Retract** that is homeomorphic with its own cone.*

Another, and the most useful, is H. Toruńczyk's characterization, which he gave of both the Hilbert cube and of all infinite-dimensional Hilbert spaces as well as manifolds modeled on them [14, 15]. More terminology about approximating maps is necessary first.

Let X be a metrizable space with metric d, and let $f : Y \to X$ be a function. The function $\hat{f} : Y \to X$ **ε-approximates** f if $d(\hat{f}(y), f(y)) < \varepsilon$ for each $y \in Y$. If instead of a constant ε, a cover $\mathcal{U}$ of X is given, then $\hat{f}$ is said to be **$\mathcal{U}$-close** to f provided that for each $y \in Y$, there is an element $U \in \mathcal{U}$ such that $f(y)$ and $\hat{f}(y)$ are in U. X is said to have the **disjoint k-cells property** provided that for each $n \leqslant k$ and $\varepsilon > 0$ each two maps $f, g : I^n \to X$ may be ε-approximated by maps $\bar{f}, \bar{g} : I^n \to X$ with disjoint images. X has the **discrete n-cells property** provided that for each open cover $\mathcal{U}$, each map $f : I^n \times \mathbb{Z} \to X$ may be $\mathcal{U}$-approximated by a map $\bar{f} : I^n \times \mathbb{Z} \to X$ such that the sets $\bar{f}(I^n \times \{m\})$ are disjoint and discrete (may be enclosed in pairwise disjoint open sets V_m with no limit points other than $\bar{f}(I^n \times Z)$).

I^∞. *A non-empty compact metric Absolute Retract is homeomorphic with the Hilbert cube provided that it has the disjoint n-cells property for all n.*

HILBERT SPACE. *A complete separable metrizable space X is homeomorphic with separable infinite dimensional Hilbert space provided that it is an **Absolute Extensor** for metrizable spaces and satisfies the discrete n-cells property for all n.*

Toruńczyk gave in [15] a characterization of Hilbert spaces of all ***weight***s; one must only replace $\mathbb{Z}$ with a discrete set of appropriate cardinality in the definition of the discrete n-cells property. To obtain a characterization of the manifolds modeled on one of these spaces, it is only necessary to replace the phrase 'Absolute Extensor' by 'Absolute Neighborhood Extensor' in its definition [14, 15].

Following his work, there have been many characterizations given of locally convex but not complete linear metrizable spaces by many authors. See [vMR, Chapter 25].

Another collection of spaces of ever increasing importance are the universal k-dimensional compacta lying in the n-cube given by Sierpiński, Menger, and Lefschetz that are generally called the ***Menger Universal Spaces*** and denoted by M_k^n. (One-dimensional individuals of this class and similar spaces are variously referred to as "the Sierpiński Universal Plane Curve, Carpet or Gasket", and "Menger's Universal curve". M_k^n may be loosely described as constructed analogously to the middle-third Cantor Set by making the same successive subdivisions of the coordinate axes into thirds, taking the product sub cubes, and deleting those not intersecting a k-dimensional face of the previously chosen generation cubes. See *Topological Embeddings* (*universal spaces*), where a careful description is given, together with illustrations.

M. Bestvina [2] characterized Menger manifolds as follows.

UNIVERSAL MENGER SPACES. *A compact metrizable space X is homeomorphic with M_k^n, where $n \geqslant 2k+1$ provided that it is k-dimensional, $(k-1)$-connected, locally $(k-1)$-connected, and satisfies the disjoint k-cells property*; *if X is not assumed globally $(k-1)$-connected, then it is locally homeomorphic with M_k^n, i.e., is a Menger manifold.*

The Nöbeling space N_k^{2k+1} is defined as the set of points $(x_1, \dots, x_{2k+1}) \in \mathbb{R}^{2k+1}$ for which x_i is rational for at most k integers i [7]. Note that N_0^1 is the set of irrational real numbers. The characterization of these spaces with $k > 0$ has been achieved only recently, by K. Kawamura, M. Levin, and E. Tymchatyn for $k = 1$ [10], and by S. Ageev for $k \geqslant 2$ [1].

NÖBELING SPACES. *Let X be a separable, metrizable **k-dimensional** absolute G_δ space that is an Absolute Extensor for the class of metrizable spaces of dimension at most k. Then X is homeomorphic to N_k^{2k+1} if and only if each map $f: Y \to X$ of a separable metrizable space of dimension at most k may be $\mathcal{U}$-approximated by an embedding onto a closed subset for any open cover $\mathcal{U}$.*

Topological Embeddings (*Universal Spaces*) contains a considerable list of extensions of these results to non-metrizable spaces.

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f-11 Higher-Dimensional Local Connectedness

Let n be an integer larger that or equal to 0. The space X is said to be **locally connected in dimension** n at a point $x \in X$ if every neighbourhood U of x contains a neighbourhood V of x such that every map $f : S^n \to V$ is homotopic to a constant map. If X is locally connected in dimension n at each of its points, it said to be **locally connected in dimension** n. Spaces that are locally connected in the dimension 1 are usually said to be **locally simply connected**. For metric case local connectedness in dimension 0 is nothing else but arcwise connectedness.

A space X is said to be **locally n-connected** at a point $x \in X$ iff X is locally connected in dimension m at x for every $m \leqslant n$. We write $X \in \mathrm{LC}^n$ if X is locally n-connected at every point $x \in X$ and $X \in \mathrm{LC}^\infty$ when $X \in \mathrm{LC}^n$ for every n.

A space X is said to be **locally contractible** at a point $x \in X$ iff each neighbourhood U of x contains a neighbourhood V of x which is contractible in U to a point. A space X is said to be **locally contractible** (abbreviated LC) iff it is locally contractible at each of its points.

If $X \in \mathrm{LC}$, then $X \in \mathrm{LC}^\infty$. There are compacta (see [3]) having property LC^n (or respectively LC^∞) but not LC^{n+1} (or not LC).

The properties LC^n and LC appeared in the thirties as consequences of investigating the ANR property (i.e., the property of being an ***Absolute Neighborhood Retract***). The notion of ***polyhedron*** belongs rather to geometry than to topology, but the class of polyhedra plays an important role in topology. In contrast Absolute Neighborhood Retracts are defined in purely axiomatic way, using only topological terms. They are good candidates for replacing polyhedra in many topological considerations.

The determination of whether a particular metric space $X \in \mathrm{LC}$ or $X \in \mathrm{ANR}$ is often a very delicate and difficult problem. Sometimes we can reduce it to the much easier condition $X \in \mathrm{LC}^n$.

Already at the beginning of the theory of retracts it was known [2] that ANR spaces had to possess sufficiently good local properties and that a finite-dimensional compactum is an ANR if and only if it is locally contractible.

In general there exist locally contractible spaces that are not Absolute Neighborhood Retracts. In [3] one finds a compactum $Y \in \mathrm{LC}$ such that Y has nonzero homology group in every dimension. The ***cell-like map***s preserve homology and the image of a compact ANR is LC^∞, but need not to be LC. R.J. Daverman and J. Walsh [5] found cell-like maps $g : X \to Z$ and $f : Z \to Y$ such that $X, Y \in \mathrm{ANR}$, but $Z \in \mathrm{LC}$ is not ANR.

Jan van Mill proved in [17] that there exists a cell-like image H of the Hilbert cube Q such that no non-empty open subset is contractible in H.

This is related to a result of Smale [18]: if $X \in \mathrm{LC}^n$ and all fibers $f^{-1}(y)$ of a map $f : X \to Y$ are AC^n (approximally connected) then $Y \in \mathrm{LC}^n$ and $f_{*i} : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for all $i \leqslant n$.

If X is a metrizable space with $\dim X \leqslant n$, then we have equivalences $X \in \mathrm{LC}^n \iff X \in \mathrm{LC}^\infty \iff X \in \mathrm{LC} \iff X \in \mathrm{ANR}$ (see [3, 6, 12, 13, 10]).

If X is a metrizable space and n is a finite number then the following conditions are equivalent (see [3, 6, 12, 13, 10]): (1) $X \in \mathrm{LC}^n$; (2) If $f : B \to X$ is map defined on a closed subset of Y with $\dim(Y \setminus B) \leqslant n + 1$, then f is extendable over a neighbourhood W of B; (3) For each point $x \in X$ and every neighbourhood U of x, there exists a neighbourhood $V \subset U$ of x such that every map $f : B \to V$ defined on a closed subset B of Y with $\dim(Y \setminus B) \leqslant n + 1$ has an extension $\tilde{f} : Y \to U$; (4) For every $x \in X$ and every neighbourhood U of x there exists a neighbourhood $V \subset U$ of x such that every map $f : Y \to V$ with $\dim Y \leqslant n$ is homotopic in U to a constant map; (5) If X is embedded as a closed subset of Z with $\dim(Z \setminus X) \leqslant n + 1$, then X is a ***neighbourhood retract*** of Z; (6) Every open covering α of X has an open refinement β such that every ***β-near*** maps $f, g : Y \to X$ defined on Y with $\dim Y \leqslant n$ are ***α-homotopic***.

The spaces Y and Z which appear in the above conditions are metrizable. We may even assume that Y is compact. In the case when $n = \infty$ the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold if we assume that $Y \setminus B$ and Y are finite-dimensional.

One can also characterize LC^n spaces in terms of maps defined on polytopes [10, p. 99]. Sometimes these characterizations are useful in verifying that a space is LC^n, despite of their long and rather technical formulation.

For every simplicial complex $\mathcal{K}$ we denote by $|\mathcal{K}|$ the polytope of $\mathcal{K}$ endowed with the CW topology ($U \subset |\mathcal{K}|$ is open if and only if $U \cap \sigma$ is open in the Euclidean topology of σ for each closed simplex σ of $\mathcal{K}$).

Let $\alpha = \{U_\lambda\}_{\lambda \in \Lambda}$ be an open covering of X, $\mathcal{K}$ be a simplicial complex and $\mathcal{L}$ be a subcomplex of $\mathcal{K}$ containing of all its vertices.

A map $f : |\mathcal{L}| \to X$ is called a partial realization of $|\mathcal{K}|$ in X relative to α iff for each closed simplex σ of $\mathcal{K}$ there exists $\lambda \in \Lambda$ such that $f(|\mathcal{L}| \cap \sigma) \subset U_\lambda$. In the case $\mathcal{K} = \mathcal{L}$ we say that f is a full realization of $|\mathcal{K}|$ in X relative to α.

The relation $X \in \mathrm{LC}^n$ is equivalent to each of the following conditions (see [3, 6, 13, 10]): (a) For every open covering α of X there exists a simplicial complex $\mathcal{K}$ of dimension $\leqslant n$ and a map $\psi : |\mathcal{K}| \to X$ such that for every map $f : Y \to X$, defined on a metrizable space Y with $\dim Y \leqslant n$, there exists a map $g : X \to |\mathcal{K}|$ such that f and ψg are α-homotopic; (b) Every open covering α of X has an open refinement β such that for any simplicial complex $\mathcal{K}$ with $\dim \mathcal{K} \leqslant n + 1$, every partial realization of $|\mathcal{K}|$ in X relative to β extends to a full realization of $|\mathcal{K}|$ in X relative to α.

In (a) we may assume that Y is compact and if X is separable (respectively compact) then $\mathcal{K}$ can be assumed to be locally finite (respectively finite).

Next, in (b) we may assume that $\mathcal{K}$ is finite. The condition $X \in \mathrm{LC}^\infty$ is also equivalent to (b) with $\mathcal{K}$ finite-dimensional.

The problem whether $X \in \mathrm{LC}^n$ can be reduced to the problem of extendibility of maps $f: B \to X$ over a neighbourhoods of B in Y, when B is a closed subset of Y and $\dim(Y \setminus B) \leqslant n+1$.

A. Haver has proved [9] that if $X \in \mathrm{LC}$ and X is a ***C-space*** (this class is larger than that of the ***countable-dimensional*** spaces) then for every closed subset B of X and map $f: B \to Y$ there is an extension $F: U_B \to Y$ to a neighbourhood U_B of B.

Generally speaking the discussed characterizations of LC^n spaces are very useful tools in studying their properties. For example using the equivalence (1) $\Leftrightarrow$ (2) one can easily check that the function space $C(X,Y) \in \mathrm{LC}^{n-m}$ if $Y \in \mathrm{LC}^n$ is a metric space and X is compactum with $\dim X \leqslant m$.

Let X and Y be metric spaces, $X_0 \subset X$ closed and $f: X_0 \to Y$ continuous. S. M. Ageev and S. A. Bogatyi observed [1] that the adjunction space $X \cup_f Y \in \mathrm{LC}^n$ if it is a metrizable space and $X, Y \in \mathrm{LC}^n$, $X_0 \in \mathrm{LC}^{n-1}$.

Compacta $X \in \mathrm{LC}^{n-1}$ with $\dim X \leqslant n$ have nice global properties. In particular S. Mardešić showed [16] that such X is ***movable*** and J. Dydak proved [7] that $X \in \mathrm{ANR}$ if and only if the n-dimensional Čech homology group $H_n(X;\mathbb{Z})$ is finitely generated.

The spaces of homeomorphisms of finite polyhedra or manifolds with appropriate topology (compact-open in the compact case) are locally contractible. The first theorem of this type was proved by Černavski in his remarkable paper [4] (see also [8]).

For every continuum X one denotes by 2^X (or respectively $C(X)$) the hyperspace of X, i.e., space whose points are nonempty closed subsets of X (or all subcontinua of X). H. Kato [11] constructed an example of X such that $C(X)$ and 2^X are not locally contractible at X. Recently W. Makuchowski has proved [14] that $C(X)$ and 2^X are LC^∞ at X. Moreover, he has showed that for $C(X)$ and 2^X the local connectedness and local n-connectedness are equivalent at any point.

Sometimes instead of local n-connectedness it suffices to assume that weaker conditions of homological nature are satisfied.

A paracompact Hausdorff space X is said to be lc_s^n if for every $x \in X$ and every open U containing x, there exists an open neighbourhood $V \subset U$ of x such that the inclusion of V into U induces a trivial homorphism $H_m(V;\mathbb{Z}) \to H_m(U;\mathbb{Z})$ of singular homology groups with integer coefficients $\mathbb{Z}$ for $m \leqslant n$. A space X is semi-n-lc_s at x if there exists an open neighbourhood V of x such that the inclusion of V into X induces a trivial homomorphism $H_n(V;\mathbb{Z}) \to H_n(X;\mathbb{Z})$.

There are natural homorphism $v_*: H_q^s(X,A;G) \to H_q(X,A;G)$ from the q-dimensional singular cohomology group $H_q^s(X,A;G)$ to q-dimensional Čech homology group $H_q(X,A;G)$ with coefficients in an Abelian group G and $v^*: H^q(X,A;G) \to H_s^q(X,A;G)$ from the q-dimensional Čech cohomology group to the q-dimensional singular cohomology group. The homomorphisms v_* and v^* are isomorphisms when both X and A are lc_s^n and semi-$(n+1)$-lc_s for all $q \leqslant n+1$ (see [13] and [15]).

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G: Special spaces

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g-1 Extremally Disconnected Spaces

Mainly ***Tychonoff space***s will be discussed here.

A space is called **extremally disconnected** (or **ED** for short) if it is regular and the closure of every open set is open. This term first appeared in print in E. Hewitt's doctoral dissertation [19, Definition 16] in 1943. He gave the credit to a Boolean algebra paper [26] of his teacher M.H. Stone ("extreme of disconnectedness" on p. 267).

ED space is ***zero-dimensional*** and Tychonoff. But it is no ordinary space. It can never be ***metrizable*** unless ***discrete***. In fact it contains no ***convergent sequence*** of distinct points.

1. Connection to Boolean algebra

For each Boolean algebra B, there exists a ***compact*** zero-dimensional space (often called "***Boolean space***") X such that B is isomorphic to the algebra of simultaneously-closed-and-open (***clopen***, for short) sets of X; and such X exists uniquely up to homeomorphism. This was M.H. Stone's discovery (the **Stone Representation Theorem**), and we call the space X the ***Stone space*** of B in his honor. In modern terms here is a ***duality*** (***Stone duality***) between the ***category*** of Boolean algebras and homomorphisms, and the category of compact zero-dimensional spaces and continuous maps. A **complete Boolean algebra** is one in which every subset has both a supremum and an infimum. Translating this into terms of the corresponding Stone space, Stone suggested the present term "extremal disconnectedness". That is, a Boolean algebra is complete iff its Stone space is ED.

Later a few authors noticed that the family of ***regular closed*** sets of any Tychonoff space X is a complete Boolean algebra by the lattice operations: $\bigwedge_\alpha R_\alpha = \mathrm{Cl}(\mathrm{Int}(\bigcap_\alpha R_\alpha))$ and $\bigvee_\alpha R_\alpha = \mathrm{Cl}(\mathrm{Int}(\bigcup_\alpha R_\alpha))$. The Stone space produced by this algebra is often denoted by $\theta(X)$. It is compact ED. The same space is constructed via the algebra of ***regular open*** sets; two algebras are isomorphic to each other. The space $\theta(X)$ is called a **Gleason cover** of X. It has a natural ***irreducible*** map onto X if X is compact (more on this below).

2. Topological properties

Extremal disconnectedness can be characterized in several useful ways. The following are equivalent for every Tychonoff space X: (a) X is ED, (b) every pair of disjoint open sets of X have disjoint closures, (c) for every open sets U, V of X, $\mathrm{Cl}\,U \cap \mathrm{Cl}\,V = \mathrm{Cl}(U \cap V)$, (d) every open subset of X is ***C^*-embedded***, and (e) every dense subset of X is C^*-embedded. These are shown in several places in the book [15].

By the definition, we have: X is ED iff so is its ***Stone–Čech compactification*** βX. Thus $\beta\lambda$ is ED for any infinite cardinal λ. (Throughout this article, discrete space and its cardinal are denoted by the same Greek alphabet.) Naturally: if X is ED and $X \subseteq Y \subseteq \beta X$, then Y is ED.

If X is ED then every countable subset is C^*-embedded and hence X contains no convergent sequence of distinct points. Every infinite compact ED space contains a copy of $\beta\omega$. If X is a compact ED space and has ***cellularity*** $> \lambda$ for an infinite cardinal λ, then, by (d), X contains a copy of $\beta\lambda$.

The above (b) implies: every ***hereditarily normal*** ED space is hereditarily ED (i.e., every subspace is ED). Thus, e.g.: every countable ED space is hereditarily ED. Here "countable" means "consisting of countably many points". A simple non-discrete example is a subspace $\omega \cup \{p\}$ of $\beta\omega$ with any $p \notin \omega$. There are countable ED spaces without isolated points (e.g., [4, 10.1]).

From (b) also follows a useful fact: every irreducible ***closed map*** of a Hausdorff space onto an ED space is one-one (and hence a homeomorphism). Some of the corollaries are: if a compact space X is continuously mapped onto a compact ED space Y, then X contains a copy of Y. And: if a compact space X is continuously mapped onto a ***Tychonoff cube I^λ***, then X contains copies of all compact ED spaces of ***weight*** $\leqslant \lambda$. Hence, e.g.: every compact ED space of weight $\leqslant 2^\tau$ is embedded in $\beta\tau$. This implies: every ED space of ***density*** $\leqslant \tau$ is embedded in $\beta\tau$. In particular, every separable ED space is embedded in $\beta\omega$.

On the other hand, every separable subspace of an ED space is ED; more generally, every ***ccc*** subspace of an ED space is ED. Hence we can say: a separable space is embedded in $\beta\omega$ iff it is ED.

Both (d) and (e) tell us: an open or dense subspace of an ED space is also ED. But a closed subspace of an ED space is not necessarily ED, as witnessed by the remainder $\omega^* = \beta\omega \setminus \omega$; it is not ED while $\beta\omega$ is. Note that a ***retract*** of an ED space is always ED.

If X is ED and $f: X \to Y$ is an onto ***open map***, then Y is ED and the Stone–Čech extension of f, $\beta X \to \beta Y$, is an open map also.

Note also E.K. van Douwen's result [4, 13.1]: a certain ED space can have a connected compactification (or compact ***connectification***).

A question was posed more than twenty years ago whether every ***locally compact normal*** ED space is ***paracompact***. Apparently a negative answer was expected, and is known to be provided partially by a certain space X satisfying $\kappa \subset X \subset \beta\kappa$ for a ***weakly compact cardinal*** κ, which is by the way quite big if exists (see [KV, p.743] for the definition of this cardinal). See [vMR, Problems 93 and 250] for more on this.

The following points deserve mention perhaps. If X is a zero-dimensional space, then the family of its clopen sets is a Boolean algebra. Let, say, S denote the corresponding Stone space. Then S is a compactification of X called ***Banaschewski compactification*** (i.e., maximal zero-dimensional compactification, denoted by $\beta_0 X$ in [24, 4.7]) and $S = \beta X$ iff X is ***strongly zero-dimensional***. This is one point. Another is: S is ED iff X is ED ("only if" because every dense subspace of an ED space is ED). That is, when X is not necessarily compact, X is ED iff X is zero-dimensional and the Boolean algebra of its clopen sets is complete.

3. Related spaces

Closely related to ED space, **basically disconnected** (or **BD** for short) space is defined by: the closure of every ***cozero set*** (often called "***functionally open*** set") is open. In terms of Stone's duality, it is equivalent to "σ-completeness" of the Boolean algebra of clopen sets; that is, a compact zero-dimensional space is BD iff every countable subset of the Boolean algebra has a supremum and an infimum.

F**-space** is also related to ED space. A space is called an F-space if it satisfies either of the equivalent conditions: (a) every disjoint cozero sets are ***completely separated***, or (b) every cozero set is C^*-embedded. It can also be stated in terms of the ring $C(X)$ of real-valued continuous functions on X, (c) every finitely generated ideal of $C(X)$ is principal, or (d) for every $p \in \beta X$, the set $\{f \in C(X)\colon \operatorname{Int}_{\beta X} \operatorname{Cl}_{\beta X} f^{\leftarrow}[\{0\}] \ni p\}$, which was denoted by O^p in [15], is a prime ideal. The condition (a) may be paraphrased: X is an F-space iff $\operatorname{Cl}_{\beta X}[U \cap V] = \operatorname{Cl}_{\beta X} U \cap \operatorname{Cl}_{\beta X} V$ for every cozero sets U, V of X (or, equivalently, of βX). Note that this F-space has nothing to do with the F-space defined in [vMR, Chapter 30, 9.1], which is another name of complete linear metric space.

In terms of Stone's duality, we have: a compact zero-dimensional space is an F-space iff, for every countable subsets P, Q of the Boolean algebra of its clopen sets such that $(\forall p \in P)(\forall q \in Q)(p \leqslant q)$, there is an s such that $(\forall p \in P)(\forall q \in Q)(p \leqslant s \leqslant q)$. This is called "countable separation property" [21, 5.26] or "weak countable completeness" [5] of the Boolean algebra.

The basic relation between these spaces together with other spaces like ***P-space*** and strongly zero-dimensional space is summarized in the diagram below in which $A \Rightarrow B$ denotes that A implies B (see Figure 1). We have: X is BD iff βX is BD. The same applies to F-space and strongly zero-dimensional space. Note that for compact spaces, strongly zero-dimensional, zero-dimensional and ***totally disconnected*** are all synonymous.

Every countable subset of an F-space is C^*-embedded. Hence no F-space contains a convergent sequence of distinct points. Also: every infinite compact F-space contains a copy of $\beta\omega$.

Every C^*-embedded subspace of an F-space is an F-space. Hence ω^* is an F-space, although not BD.

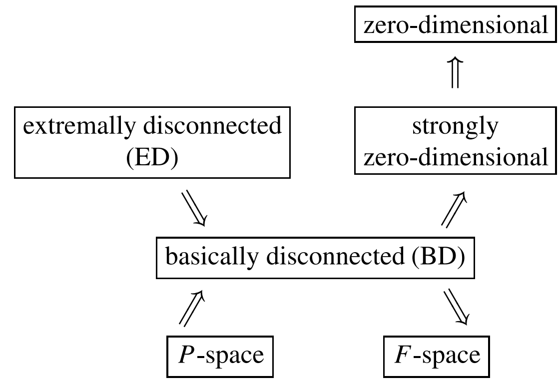

Fig. 1.

It is pointed out that every ccc F-space is ED (see, e.g., [24, 6L(8)]).

Some F-spaces are connected, as seen by: the remainder of the space of non-negative real numbers is an F-space. This is because: the remainder X^* of every ***locally compact*** ***σ-compact*** space X is an F-space [15, 14.27].

4. Further discussions

4.1. As noted above, the Gleason cover $\theta(X)$ of X has an irreducible map onto X if X is compact. According to J. Flachsmeyer, S. Iliadis and others, a certain subspace of $\theta(X)$ works just as well for a (not-necessarily-compact) Tychonoff space X, that is: there is a subspace $E(X)$ of $\theta(X)$ such that (1) $E(X)$ is ED and has an irreducible ***perfect*** onto map $\varphi\colon E(X) \to X$, and (2) if P is ED and has an irreducible perfect onto map $f\colon P \to X$, then there is a homeomorphism $h\colon P \to E(X)$ such that $\varphi \circ h = f$. $E(X)$ is called the ***absolute*** (or **Iliadis cover**) of the space X. Obviously, if X is compact then $E(X) = \theta(X)$.

J. Vermeer and others re-formulated this: the ***full subcategory*** of ED spaces is ***coreflective*** (in the sense of [23, IV]) in the category of Tychonoff spaces and irreducible perfect onto maps, and the ***covariant functor*** E is the right adjoint for the inclusion functor. And it led to the generalization: the full subcategories of BD spaces and of quasi-F-spaces are both coreflective (a **quasi-F-space** is a space in which every dense cozero set is C^*-embedded). Thus, in topological terms, we have the following. Let $\mathcal{P}$ be the class of BD spaces or of quasi-F-spaces. Then to each Tychonoff space X corresponds a space $P(X)$ such that (1) $P(X)$ is in $\mathcal{P}$ and has an irreducible perfect onto map $\varphi\colon P(X) \to X$ and (2) if Q is in $\mathcal{P}$ and has an irreducible perfect onto map $f\colon Q \to X$, then there is an irreducible perfect onto map $h\colon Q \to P(X)$ such that $\varphi \circ h = f$. Note that, if $f, g\colon X \to Y$ and $h\colon Y \to Z$ are irreducible closed onto maps, and if Y is Hausdorff and $h \circ f = h \circ g$, then $f = g$.

Also note that the notion of ***reflectiveness***, which is dual to that of coreflectiveness, is related to ***extension***s of spaces, that is, e.g., Stone–Čech compactification gives us that the full subcategory of compact spaces is reflective in the category of Tychonoff spaces and continuous maps.

For more on these and related matters, the readers are referred to other parts of the Encyclopedia or literatures such as [18, 24] or [27].

4.2. Compact ED spaces are precisely the **projective objects** (see [23, V.4] or [E, 6.3.19]) in the category of compact spaces and continuous maps, that is, a compact space X is ED iff, for any continuous onto map $g: P \to Q$ between compact spaces and any continuous map $f: X \to Q$, there is a continuous map $h: X \to P$ such that $f = g \circ h$. As is well-known, in the category of Hausdorff spaces and continuous maps, projective objects are precisely discrete spaces.

4.3. The notion of extremal disconnectedness is localized. A space X is called **extremally disconnected at a point** x if $x \notin \mathrm{Cl}\, U \cap \mathrm{Cl}\, V$ for every disjoint open sets U, V. A space is ED iff it is ED at every point. According to E.K. van Douwen [4, 12.1]: if X is dense in Y and Y is ED at every point of $Y \setminus X$, then X is C^*-embedded in Y. His famous theorem in the same paper was: βX is ED at each ***remote point*** of X. This helped him prove that the remainders $\mathbb{Q}^*$, $\mathbb{P}^*$ and $\mathcal{S}^*$ are all non-homogeneous, where $\mathbb{Q}$ and $\mathbb{P}$ are the spaces of rational, irrational numbers, respectively, and $\mathcal{S}$ is the ***Sorgenfrey line***.

4.4. The following 1955 theorem of J.R. Isbell has been often re-discovered: each ***P-point*** of an ED space of ***Ulam nonmeasurable*** cardinal (see [15, 12.1] or [KV, Chapter 4, 1.13]) is isolated. R.L. Blair [3] gave an alternative proof of this. Since an Ulam measurable cardinal, if it exists, is quite big, this means that in most cases each point of an ED space is either isolated or non-P. The strength of this theorem is seen by one of its consequences (see, e.g., [24, 6P(2)]): if the product $X \times Y$ is ED and of Ulam non-measurable cardinal, then X or Y is discrete (non-P-points of the factor spaces would produce a point at which the product space fails to be ED).

Recently A.V. Arhangel'skiĭ [1, 1.1] has made Isbell's theorem more precise by showing: if X is ED at its P-point x, $\mathcal{U}$ is any family of open sets containing x such that the cardinality $|\mathcal{U}|$ is Ulam non-measurable, then x belongs to the interior of the intersection of sets of $\mathcal{U}$.

4.5. E. Hewitt pointed out [19, Theorem 32]: if a topology τ on a set X is ***dense-in-itself*** (i.e., no point is isolated in the space (X, τ)) and is maximal in the family of dense-in-itself topologies on X and if τ is regular, then the space (X, τ) is ED. This is because: if $\mathrm{Cl}_\tau U \notin \tau$ for some $U \in \tau$ then the family $\{W \cap \mathrm{Cl}_\tau U\colon W \in \tau\}$ generates a dense-in-itself topology strictly finer than τ. Hewitt called the space (X, τ) with the maximal τ a **maximal space**. Clearly, any dense-in-itself topology is extended to a maximal one, but not necessarily of the same separation axiom. E.K. van Douwen [6, 3.3] constructed a countable regular maximal space. His and Hewitt's papers contain many additional information. See also [1] and other parts of the Encyclopedia.

4.6. Let R be an arbitrary commutative ring with unity and $m(R)$ denote the set of minimal prime ideals of R, where a minimal prime ideal means a proper prime ideal that contains no smaller prime ideals. We can introduce what is commonly called **hull-kernel topology** (or **Zariski topology**) into $m(R)$ by defining (see [15, 7M]): $N \in \mathrm{Cl}\,\mathcal{A}$ iff $\bigcap\{M\colon M \in \mathcal{A}\} \subseteq N$. In the 1960s M. Henriksen and M. Jerison [17] (as well as other authors) showed: $m(R)$ is a zero-dimensional Tychonoff space.

They went on to consider the ring $C(X)$ of real-valued continuous functions on an arbitrary Tychonoff space X, and showed: $m(C(X))$ and $m(C(\beta X))$ are homeomorphic to each other. This meant: it suffices to consider $m(C(X))$ only for compact spaces X.

$m(C(X))$ is ***countably compact*** although not necessarily compact. By [15, 14G] $m(C(\omega + 1))$ is homeomorphic to $\beta\omega$. According to [17, 5.3], when X is compact, there is a natural irreducible onto map $\iota: m(C(X)) \to X$, because every prime ideal of $C(X)$ is contained in a unique maximal ideal [15, 14.12]. The map ι is bijective iff X is an F-space, and a homeomorphism iff X is BD. Hence: $m(C(X))$ is BD if X is BD, and, since ω^* is an F-space but not BD, $m(C(\omega^*))$ is not compact. Henriksen and Jerison showed that if $m(C(X))$ is locally compact then it is BD, and that $m(C(\omega^*))$ is ***nowhere locally compact*** (i.e., no point has a compact neighbourhood). But they did not determine whether $m(C(\omega^*))$ is BD or not, nor even whether $m(C(X))$ can ever be non-BD. The questions remained open (still do partially) more than twenty-five years untill the publication of [10]. In fact, [10] used the notion of a P-set and proved a theorem that included: (1) for some compact F-space X, $m(C(X))$ is not an F-space (hence not BD), and (2) [MA] $m(C(\omega^*))$ is not an F-space. See [vMR, Problem 11] for more on this.

Note the following. Suppose X is a compact F-space. Then the minimal prime ideals of $C(X)$ are precisely the ideals O^p with $p \in X$ (for the symbol O^p see (d) in the above definition of F-spaces), and the map ι gives us: $m(C(X))$ may be viewed as the set X re-topologized by basic open sets of the form $\mathrm{Cl}_\tau[X \setminus f^{\leftarrow}[\{0\}]]$ where $f \in C(X)$ and τ denotes the original topology of X.

4.7. E.K. van Douwen [5] showed, among others: for every infinite compact F-space X, (1) if λ is the number of points of X then $\lambda = \lambda^\omega$ and $\lambda \geqslant 2^{\mathfrak{c}}$, and (2) if λ is the weight of X then $\lambda = \lambda^\omega$.

This means: no infinite compact F-space can consist of λ many points or have weight λ unless $\lambda = \lambda^\omega$ holds. In a reverse direction, for every cardinal λ satisfying $\lambda = \lambda^\omega \geqslant 2^{\mathfrak{c}}$, there is a compact BD space X so that $|X| = w(X) = \lambda$. This was also shown by van Douwen in the same paper. It can be described briefly. He added an imaginary point ∞ to the discrete space λ and constructed a ***Lindelöf*** space $\lambda \cup \{\infty\}$ in which ∞ is the only non-isolated point. It is a P-space and hence BD. Then $X = \beta(\lambda \cup \{\infty\})$ is BD, consists of the point ∞ and λ^ω many clopen copies of $\beta\omega$, and serves our purpose. Note that no compact ED space X satisfies $|X| = w(X)$, as the B. Balcar and F. Franek's result shows below. However, for every λ with $\lambda = \lambda^\omega$, there are compact ED spaces of weight λ; the absolute of the Cantor cube of weight λ is one of such.

Van Douwen also showed: for every infinite compact ED space, the cardinal functions ***spread***, ***character*** and weight are equal to each other.

4.8. B. Balcar and F. Franek [2] showed in Boolean algebra terms: for every infinite compact ED space X of weight λ, (1) X consists of exactly 2^λ many points, (2) X contains 2^λ many points, at each of which the character of the space X is λ, and (3) X is continuously mapped onto the Cantor cube of weight λ.

Naturally (1) follows from (2). Because of (3), we have: if both X and Y are compact ED spaces with weight $w(X) \geqslant w(Y)$, then Y is embedded in X. This can be re-phrased: every compact ED space of weight λ is a ***universal space*** (in the sense of [E, p. 83]) for the class of compact ED spaces of weight $\leqslant \lambda$.

4.9. In 1963, I.I. Parovičenko solved a question of P. Alexandroff by showing: [CH] every compact space of weight $\leqslant \mathfrak{c}$ is the continuous image of ω^*. Basic to the argument was: ω^* is what is nowadays called a **Parovičenko space**, that is, a compact zero-dimensional F-space of weight $\mathfrak{c}$ in which every nonempty ***G_δ-set*** has infinite interior. Later authors noticed that Parovičenko's result was actually split into two, one without CH, another with CH: (1) every compact space of weight $\leqslant \omega_1$ is the continuous image of an arbitrary Parovičenko space, and (2) [CH] ω^* is the only Parovičenko space. According to E.K. van Douwen and J. van Mill, (2) is equivalent to CH. See [E, 3.12.18] or [KV, Chapter 11] for details.

A recent result of A. Dow and K.P. Hart [9] was: under a set-theoretic assumption called ***Open Colouring Axiom***, if a non-compact space X is locally compact and σ-compact, and if X^* is the continuous image of ω^*, then X is the disjoint union of ω and a compact space. This means: we can neither prove nor disprove in ZFC that the remainder of, say, the real line is the continuous image of ω^*.

4.10. In 1973, A. Louveau [22] showed under CH: a compact space is embedded in ω^* iff it is a compact zero-dimensional F-space of weight $\leqslant \mathfrak{c}$. Later A. Dow, R. Frankiewicz and P. Zbierski [8] showed that the statement is consistent with $\mathfrak{c} = \omega_2$ (and hence not equivalent to CH).

E.K. van Douwen and J. van Mill considered two problems: (1) whether every compact zero-dimensional F-space is embedded in a BD space, and (2) whether every compact BD space is embedded in an ED space. Partial solutions are now known: neither of them can be true in ZFC, in other words, it is consistent with ZFC that each of them is false (see [7, 11], [KV, Chapter 11, 2.4] or [vMR, Problems 9 and 217]).

4.11. A. Dow and J. Vermeev [12] showed, among others: a compact zero-dimensional space X is embedded in an ED space iff X is projective with respect to the class of ***Absolute Retract***s, or equivalently, with respect to all retracts of the Cantor cubes. As a corollary they showed: if a compact zero-dimensional space X is embedded in an ED space Y, then Y can be chosen so that $w(Y) = w(X)$. In the same paper they also constructed, for each infinite cardinal λ, a compact BD space of weight λ which is universal for the class of compact BD spaces of weight $\leqslant \lambda$.

4.12. Towards his ultimate solution of the homogeneity problem (see below), Z. Frolík [14] considered ***fixed point***s of embeddings on compact ED spaces and showed: for every continuous one-one map f of a compact ED space X into itself, (1) the fixed-point set $F = \{x\colon f(x) = x\}$ is clopen and (2) if $X \setminus F$ is nonempty, it is decomposed into three mutually disjoint nonempty clopen sets B_1, B_2, B_3 so that $f[B_i] \cap B_i = \emptyset$ for each i. He also showed: if f is a continuous map of an ED space into itself and has a fixed point x, then x has an arbitrarily small clopen neighbourhood U so that $f[U] \subseteq U$. Later J. Vermeer [28, 6.3] pointed out: for every continuous map of a compact ED space X into itself, the fixed-point set F is, if nonempty, a retract of X, and $X \setminus F$ is ***pseudocompact***.

K.P. Hart and J. Vermeer [16, 2.1] showed: the fixed-point set of a continuous one-one map of a ***finite-dimensional*** compact F-space into itself is a P-set. They further noted: (1) this is not true for an ***infinite-dimensional*** compact F-space, and (2) [CH] its converse is true for ω^*, that is, a closed subset of ω^* is a P-set iff it is the fixed-point set of an embedding of ω^* into itself. Here (2) was shown on the base of the Parovičenko's theorem (see above).

4.13. Z. Frolík showed: no infinite compact ED space is ***homogeneous***. We know more now: no infinite compact F-space is homogeneous. Not only that, but also: no product of infinite compact F-spaces is homogeneous. More precisely, R. Frankiewicz, K. Kunen and P. Zbierski [13] showed: if A is any index set, finite or infinite, and if each X_α, $\alpha \in A$, is either an infinite compact F-space or a ***first-countable*** compact space and at least one of X_α is not first-countable, then the product space $\prod_{\alpha \in A} X_\alpha$ is not homogeneous. To the proof, the ***Rudin–Keisler order*** of points of ω^* and the existence of ***weak P-point***s in ω^* were essential.

Note that the topological homogeneity does not exactly translate, by the Stone's duality, into the algebraic homogeneity of the corresponding Boolean algebra as described in [21, 9.12].

It is also known that every ***topological group*** which is a ***k-space*** and an F-space is discrete. Hence every locally compact BD topological group is discrete. For further details, the readers are referred to other parts of the Encyclopedia or W.W. Comfort's expository article in [KV, Chapter 24].

4.14. B.D. Šapirovskiĭ's theorem [25, Theorem 6] was: if a compact ED space Z of weight λ is embedded in the product of τ many compact spaces X_α and $2 \leqslant \tau < \mathrm{cf}(\lambda)$, then Z is embedded in some X_α. In particular, if $\beta\lambda$ is embedded in the product of τ many compact spaces X_α and $2 \leqslant \tau < \mathrm{cf}(2^\lambda)$, then $\beta\lambda$ is embedded in some X_α.

According to S. Negrepontis [KV, Chapter 23, 3.6], M. Talagrand also obtained the latter result. For $\lambda = \omega$ it was first pointed out by V.I. Malyhin.

Šapirovskiĭ proved his theorem, as a corollary, in a series of arguments which included: (1) a compact space X is continuously mapped onto the Tychonoff cube I^λ of weight $\lambda \geqslant \omega$ iff X contains a closed set F so that ***π-character*** at each point of F in F is $\geqslant \lambda$, and (2) if the product of τ many compact spaces X_α is continuously mapped onto I^λ and $2 \leqslant \tau < \mathrm{cf}(\lambda)$, then some X_α is continuously mapped onto I^λ. Note that (1) is deep and important. I. Juhász [20, 3.18] discussed it at length. So did S. Koppelberg [21, 10.16] albeit in a zero-dimensional version.

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g-2 Scattered Spaces

1. Basic definitions and notions

All ***topological space***s will be assumed to be ***Hausdorff*** in this entry.

A topological space is called **scattered** (**dispersed**, **clairsemé**) if its every non-empty subspace has an isolated point. A space is scattered iff it is ***right separated***, so $|X| \leqslant w(X)$ for scattered spaces.

Given a topological space X, for each ordinal number α, the α**th derived set** of X, $X^{(\alpha)}$, is defined as follows: $X^{(0)} = X$, $X^{(\alpha+1)}$ is the ***derived set*** of $X^{(\alpha)}$, i.e., the collection of all limit points of $X^{(\alpha)}$, and if α is limit then $X^{(\alpha)} = \bigcap_{\nu<\alpha} X^{(\nu)}$. Since $X^{\alpha} \supseteq X^{\beta}$ for $\alpha < \beta$ we have a minimal ordinal α such that $X^{\alpha} = X^{\alpha+1}$. This ordinal α, denoted by $\mathrm{ht}(X)$, is called the **Cantor–Bendixson height**, or the **scattered height** of X. Clearly the subspace $X^{(\alpha)}$ does not have any ***isolated points***, it is ***dense-in-itself***. The derived sets are all closed, so $X^{(\alpha)}$ is ***perfect***. Moreover, $Y = X \setminus X^{(\alpha)}$ is scattered and so it has cardinality $\leqslant w(X)$. This yields the classical Cantor–Bendixson theorem: every space of countable weight can be represented as the union of two disjoint sets, of which one is ***perfect*** and the other is countable. G. Cantor and I. Bendixson proved this fact independently in 1883 for subsets of the real line.

Historically, the investigation of scattered spaces was started by G. Cantor. He proved, in [3], that if the partial sums of a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converge to zero except possibly on a set of points of finite scattered height, then all coefficients of the series must be zero.

Denote by $I(Y)$ the isolated points of a topological space Y. For each ordinal α define the αth **Cantor–Bendixson level** of a topological space X, $I_\alpha(X)$, as follows:

$$I_\alpha(X) = I\Big(X \setminus \bigcup\{I_\beta(X) \colon \beta < \alpha\}\Big).$$

Clearly $I_\alpha(X) = X^{(\alpha)} \setminus X^{(\alpha+1)}$ and $\mathrm{ht}(X) = \min\{\alpha\colon I_\alpha(X) = \emptyset\}$.

Observe that X is scattered iff $X = \bigcup\{I_\alpha(X) \colon \alpha < \mathrm{ht}(X)\}$ iff $X^{\mathrm{ht}(X)} = \emptyset$.

Given a scattered space X, define the **width** of X, $wd(X)$, as follows: $wd(X) = \sup\{|I_\alpha(X)| \colon \alpha < \mathrm{ht}(X)\}$. The **cardinal sequence** of a scattered space X, $CS(X)$, is the sequence of the cardinalities of its Candor–Bendixson levels, i.e.,

$$CS(X) = \big\langle \big|I_\alpha(X)\big| \colon \alpha < \mathrm{ht}(X)\big\rangle.$$

Axioms of separation

The coarsest refinement of $\beta\omega$ in which $\beta\omega \setminus \omega$ is discrete is a scattered Hausdorff space which is not ***regular***. There are regular, but not completely regular scattered spaces: see, e.g., [E, Example 1.5.9]. A ***completely regular*** scattered space is not necessarily ***zero-dimensional***, counterexamples were constructed, e.g., by M. Rajagopalan, R.C. Solomon and J. Terasawa. A scattered space is ***hereditarily disconnected***. Since in a compact space the ***component*** and the ***quasi-component*** of a point are the same, locally compact scattered spaces are ***totally disconnected*** and so they are ***zero-dimensional***. A zero-dimensional scattered space need not be normal. If $\mathcal{A} \subset [\omega]^\omega$ is a ***Lusin family*** then the ***Ψ-space*** $X(\mathcal{A}) = \langle \omega \cup \mathcal{A}, \tau\rangle$ is an uncountable, locally compact scattered space of height 2, which is not normal.

Dual algebras

A ***Boolean space*** is a compact, zero-dimensional, Hausdorff space. The ***Stone duality*** establishes a 1–1 correspondence between Boolean spaces and ***Boolean algebra***s.

A Boolean algebra B is **superatomic** iff every homomorphic image of B is atomic. B is superatomic iff every quotient algebra of B is atomic iff every subalgebra of B is atomic.

Write $\mathrm{Clop}(X)$ for the Boolean algebra of the ***clopen*** subsets of a Boolean space X. Under Stone duality, ***closed subspaces*** of X correspond to quotient algebras of $\mathrm{Clop}(X)$, and isolated points of a closed subspace correspond to atoms of the corresponding quotient algebra of $\mathrm{Clop}(X)$. Since a space is scattered iff every closed subspace of X has an isolated point, a Boolean space X is scattered iff $\mathrm{Clop}(X)$ is superatomic.

Let B be a Boolean algebra; then the Boolean algebra B^* is a **free complete extension** of B if B^* is ***complete*** and B can be embedded in B^* in such a way that homomorphisms of B into complete Boolean algebras can be extended to complete homomorphisms on B^*. G. Day, in [4], proved that a Boolean algebra is superatomic iff it has a free complete extension. He also showed that every Boolean algebra generated by a finite number of superatomic subalgebras is superatomic.

Preservation theorems

Subspaces of scattered spaces are scattered. The product of finitely many scattered spaces is scattered, but 2^ω is dense in itself. In [13], V. Kannan and M. Rajagopalan proved that a closed continuous image of a scattered zero-dimensional Hausdorff space need not be scattered although it is zero-dimensional and Hausdorff as well. Under CH they constructed a locally compact, locally countable, scattered, first countable, sequentially compact, separable Hausdorff space, that admits a closed continuous map onto the closed unit interval [0, 1].

Scattered compactification
The first example of a completely regular scattered space with no scattered compactification was given by P. Nyikos. For more details see entry Compactification.

Variations of scatteredness
A topological space is **C-scattered** if for every closed subspace $F \neq \emptyset$ there is a point x with a compact neighbourhood contained in F. The notion of C-scatteredness is a simple simultaneous generalization of scatteredness and of local compactness. A **rim-scattered** space has a base each of whose elements has a scattered boundary. A space X is called **σ-scattered** if $X = \bigcup\{X_n : n \in \omega\}$, where each X_n is scattered. A space is **G_ω-scattered** if every subspace contains a point which is a (relative) G_δ. A space is **N-scattered** iff nowhere dense subsets are all scattered.

2. Theorems on scattered spaces

Mrówka, Rajagopalan and Soudararajan proved in [22] that compact scattered spaces are ***pseudoradial*** (***chain-net***) spaces. A completely regular, scattered, ***countably compact*** space need not be pseudoradial. They also proved that a space is compact and scattered iff it is a **chain compact space**; i.e., every chain-net (= net with a totally ordered directed set) has a convergent cofinal subnet.

Gerlits and Juhász, in [6], proved that a left-separated compact T_2 space is both scattered and sequential. Tkačenko generalized this theorem showing that every countably compact Hausdorff space which is a union of ω left-separated subspaces is scattered.

Let X be a compact space. Then X is scattered iff $C_p(X)$ is Fréchet–Urysohn (Gerlits and Nagy [7]) iff the ***Pixley–Roy hyperspace*** of X is normal (Przymusiński [23]) iff the tightness of the product $C(X) \times Y$ is countable for every Y of countable tightness (Uspenskiĭ [27]). V.I. Malykhin showed in [16] that if X is compact and $C_p(X)$ is **subsequential** (i.e., a subspace of a ***sequential space***), then X is scattered.

Nyikos and Purisch characterized monotonically normal compact scattered spaces as continuous images of compact ordinal spaces. There are ***monotonically normal*** (MN) spaces which are not ***acyclically monotonically normal*** (AMN) as was shown by M.E. Rudin in 1992. P.J. Moody proved that any scattered MN space is AMN.

A topological space X **omits** κ if $|X| > \kappa$ and $|F| = \kappa$ for no closed subset F of X. Juhász and Nyikos proved that scattered regular spaces do not omit κ with $\kappa = \kappa^{<\kappa}$. On the other hand, there is a model of set theory in which there is a Lindelöf scattered space Y of cardinality $2^{\omega_1} > 2^\omega$ which has no closed (nor even Lindelöf) subset of cardinality 2^ω.

In [2] Y. Bregman, A. Shostak and A. Shapirovskiĭ, proved that under some additional set-theoretical assumption (e.g., $V = L$), every space X can be represented as the union of two subspaces X_1 and X_2 in such a way that, if $Y \subset X$ is compact and either $Y \subset X_1$ or $Y \subset X_2$, then Y is scattered. Using large cardinal assumptions, S. Shelah showed that it is consistent that there is a space X such that if $X = X_1 \cup X_2$ then X_1 or X_2 contains a copy of the ***Cantor set***.

S. Banach raised the following question: determine the topological spaces admitting coarser compact T_2 topology. Katětov, in [14], showed that a countable regular space possesses the above property iff it is scattered. I. Juhász and Z. Szentmiklóssy, in [10], proved that there is a Tychonov scattered space of size ω_3 with no smaller compact T_2 topology. They also gave a consistent (counter)example of cardinality ω_1.

3. Cardinal sequences of scattered spaces

Regular spaces
Since $|X| \leqslant 2^{|I(X)|}$ for a regular, scattered space X, we have $\mathrm{ht}(X) < (2^{|I(X)|})^+$. S. Shelah, in [1], constructed, for each $\gamma < (2^\omega)^+$ a zero-dimensional, scattered space of height γ and width ω.

For an infinite cardinal κ, let S_κ be the following family of sequences of cardinals: $S_\kappa = \{\langle \kappa_i : i < \delta \rangle : \delta < (2^\kappa)^+, \kappa_0 = \kappa$ and $\kappa \leqslant \kappa_i \leqslant 2^\kappa$ for each $i < \delta\}$.

Building on the method of Shelah, it was observed by Juhász and Soukup [8] that $s = CS(X)$ for some regular scattered space X iff $s = CS(X)$ for some zero-dimensional scattered space X iff for some natural number m there are infinite cardinals $\kappa_0 > \kappa_1 > \cdots > \kappa_m$ and sequences $s_i \in S_{\kappa_i}$ such that $s = s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_m$ or $s = s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_m^\frown \langle n \rangle$ for some natural number $n > 0$.

Compact spaces
The question concerning the cardinal sequences of (locally) compact scattered spaces is much harder. If X is a compact scattered space, then $\mathrm{ht}(X)$ should be a successor ordinal, $\mathrm{ht}(X) = \beta + 1$, and $I_\beta(X)$ is finite. The subspace $Y = X \setminus I_\beta(X)$ is a locally compact, noncompact, scattered space and $CS(X) = CS(Y)^\frown \langle |I_\beta(X)| \rangle$. On the other hand, if Y is an arbitrary locally compact, noncompact scattered space, then $X = \alpha Y$, the one-point compactification of Y, is a compact scattered space and $CX(X) = CS(Y)^\frown \langle 1 \rangle$. Hence, instead of compact, scattered spaces we can study the cardinal sequences of locally compact, scattered (**LCS**) spaces.

It is a classical result of S. Mazurkiewicz and J. Sierpiński, [19], that a countable compact scattered space is determined completely by the ordinal β, where $\beta + 1 = \mathrm{ht}(X)$, and by the natural number $n = |I_\beta(X)|$: X is homeomorphic to the compact ordinal space $\omega^\beta \cdot n + 1$.

An LCS space X is called **ω_1-thin thick** iff $\mathrm{ht}(X) = \omega_1 + 1$, $|I_\alpha(X)| = \omega_1$ for $\alpha < \omega_1$, but $|I_{\omega_1}(X)| = \omega_2$. It is **very thin thick** iff $\mathrm{ht}(X) = \omega_1 + 1$, $|I_\alpha(X)| = \omega$ for $\alpha < \omega_1$, but $|I_{\omega_1}(X)| = \omega_2$. It is **thin tall** iff $\mathrm{ht}(X) = \omega_1$

and $wd(X) = \omega$. It is **thin very tall** iff $\mathrm{ht}(X) = \omega_2$ and $wd(X) = \omega$.

The following problem was first posed by R. Telgarsky in 1968 (unpublished): Does there exist a thin tall LCS space? After some consistency results Rajagopalan, in [24], constructed such a space in ZFC. In [11] Juhász and Weiss showed that for each $\alpha < \omega_2$ there is an LCS space with height α and width ω.

W. Just proved in [12] that the result of [11] is sharp in the following sense. Add ω_2 ***Cohen real***s to a model of ZFC satisfying CH. Then, in the generic extension, CH fails and there are neither thin very tall nor very thin thick LCS spaces. The first part of this result was improved in [9] by I. Juhász, S. Shelah, L. Soukup and Z. Szentmiklóssy: if we add Cohen reals to a model of set theory satisfying CH, then, in the new model, every LCS space has at most ω_1 many countable levels.

Let $f : [\omega_2]^2 :\to [\omega_2]^{\leqslant\omega}$ be a function with $f(\{\alpha, \beta\}) \subset \alpha \cap \beta$ for $\{\alpha, \beta\} \in [\omega_2]^2$. We say that two finite subsets x and y of ω_2 are *good for* f provided that for $\alpha \in x \cap y$, $\beta \in x \setminus y$ and $\gamma \in y \setminus x$ we always have (i) if $\alpha < \beta, \gamma$ then $\alpha \in f(\{\beta, \gamma\})$, (ii) if $\alpha < \beta$ then $f(\{\alpha, \gamma\}) \subset f(\{\beta, \gamma\})$, (iii) if $\alpha < \gamma$ then $f(\{\alpha, \beta\}) \subset f(\{\gamma, \beta\})$. We say that f is a ***Δ*-function** if every uncountable family of finite subsets of ω_2 contains two sets x and y which are good for f. The notion of Δ-function was introduced in [1]. Baumgartner and Shelah proved that (a) the existence of a Δ-function is consistent with ZFC, (b) if there is a Δ-function then a thin very tall LCS space can be obtained by a natural ccc forcing. In this way they obtained the consistency of the existence of a thin very tall LCS space.

The consistency of the existence of an LCS space of height ω_3 and width ω_1 is open. Step (a) of the proof of Baumgartner and Shelah can be generalized: a 'Δ-function' $f : [\omega_3]^2 :\to [\omega_3]^{\leqslant\omega_1}$ can be constructed. However, it is unclear how to step up in part (b) without collapsing cardinals.

J. Roitman [25] proved that the existence of a very thin thick LCS spaces is consistent with ZFC. On the other hand, J. Baumgartner and S. Shelah in [1] proved that in Mitchell's model there are no ω_1-thin thick LCS space.

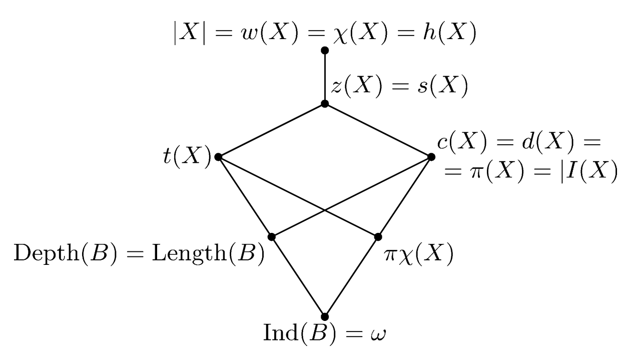
La Grange in [15] characterized the countable cardinal sequences of LCS spaces: a countable sequence $\langle \kappa_\alpha : \alpha < \delta \rangle$ of infinite cardinals is the cardinal sequence of an LCS space iff $\kappa_\beta \leqslant {\kappa_\alpha}^\omega$ for each $\alpha < \beta < \delta$. Juhász and Weiss proved that LaGrange's characterization holds even for $\delta = \omega_1$. For $\delta > \omega_1$ we need more assumptions because of the consistency result of Just. The strongest known theorem is the following (unpublished) result of Juhász and Weiss: If $\delta < \omega_2$ and $s = \langle \kappa_\alpha : \alpha < \delta \rangle$ is a sequence of infinite cardinals such that $(*)$ $\kappa_\beta \leqslant {\kappa_\alpha}^\omega$ for each $\alpha < \beta < \delta$, and $(**)$ $\kappa_\alpha \in \{\omega, \omega_1\}$ for each $\alpha < \delta$ with $\mathrm{cf}(\alpha) = \omega_1$, then there is an LCS space X with cardinal sequence s.

The cardinal sequences do not characterize uncountable LCS-spaces. An autohomeomorphism of a scattered space is called trivial if it is the identity on some (infinite) Cantor–Bendixson level (and then also on all higher levels). The quotient of the full autohomeomorphism group of a scattered space X by its normal subgroup of trivial autohomeomorphisms is denoted $G(X)$. A. Dow and P. Simon in [5] showed in ZFC that for each countable group G, there are 2^{ω_1} pairwise nonisomorphic thin-tall LCS space X such that $G(X)$ is isomorphic to G.

The above mentioned estimate $\mathrm{ht}(X) < (2^{|I(X)|})^+$ is sharp for LCS spaces with countably many isolated points: it is easy to construct an LCS space with countable "bottom" and of height α for each $\alpha < (2^\omega)^+$ (see [9]). Much less is known about LCS spaces with ω_1 isolated points, for example it is a long standing open problem whether there is, in ZFC, an LCS space of height ω_2 and width ω_1. In fact, as was noticed by Juhász in the mid eighties, even the much simpler question if there is a ZFC example of an LCS space of height ω_2 with only ω_1 isolated points, turned out to be surprisingly difficult. Z. Szentmiklóssy proved in 1983 that if GCH holds in V and we add $\aleph_{\omega_1}$ Cohen reals to V then in the generic extension there is no compact scattered space X such that $|X| = 2^{\omega_1} = \aleph_{\omega_1+1}$ and $I(X) = \omega_1$. Martínez in [17] proved that it is consistent that for each $\alpha < \omega_3$ there is a LCS space of height α and width ω. In [9] I. Juhász, S. Shelah, L. Soukup and Z. Szentmiklóssy gave an affirmative answer to the above question of Juhász: they construct, in ZFC, an LCS space of height ω_2 with ω_1 isolated points. However, it is unknown whether there is, in ZFC, an LCS space X of height ω_3 having ω_2 isolated points.

4. Cardinal functions on compact scattered spaces

The results mentioned in this section can be found in [20]. Let X be a compact, scattered spaces and $B = \mathit{clopen}(X)$ be the corresponding superatomic Boolean algebra. The following diagram summarizes the relationship between cardinal functions.



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g-3 Dowker Spaces

A normal Hausdorff space X is said to be a **Dowker space** if $X \times [0,1]$ is not normal. The problem of the existence of such a space originally arose from homotopy extension considerations in the early 50s, but soon became a hard problem that propelled the development of set-theoretic topology.

It is easy to prove [KV, Chapter 17] that a ***normal space*** X is a Dowker space if and only if it has an increasing open cover $\langle U_n \rangle_{n\in\omega}$ such that there is no closed cover $\langle F_n \rangle_{n\in\omega}$ of X with $F_n \subset U_n$ for every $n \in \omega$. The family $\mathcal{I} = \{Y \subset X\colon \exists n \in \omega \text{ with } \overline{Y} \subset U_n\}$ is an **ideal** on X, i.e., closed under the taking of subsets and finite unions. Since X cannot be covered by any countable subfamily of $\mathcal{I}$ we even obtain a proper **σ-ideal** called the **Dowker ideal**. The difficult thing is to construct this Dowker ideal in the presence of normality.

The following two topics will be dealt with in our article.

(1) The construction of Dowker spaces in ZFC. There have been basically two methods to do so which we describe in detail. Some applications will be mentioned.
(2) The construction of Dowker spaces in special models of ZFC. Surprisingly, in spite of the wide variety and long history of these constructions, there has been, in essence, only one method used to do this.

1. Dowker spaces in ZFC

Rudin's ZFC Dowker space [KV, Chapter 17] was found in 1971. It is relatively easy to describe explicitly. Consider the ***ordinal spaces*** ω_n with the interval topology. Take the ***box product*** $\Box_{1<n<\omega}\omega_n$, i.e., the ***Cartesian product*** equipped with the topology generated by all sets of the form $\prod_{n\in\omega} U_n$, with each U_n an open subset of ω_n. Rudin's space is the subspace $X = \{f \in \Box_{1<n<\omega}\omega_n\colon \text{there is a } k \in \omega \text{ such that } \omega_1 \leqslant \text{cofinality of } f(n) < \omega_k \text{ for every } 1 < n < \omega\}$.

Using the unequal lengths of the factors Rudin gives a proof that the open sets $U_n = \{f \in X\colon \exists i < n \text{ such that } f(i) < \omega_i\}$ form an increasing cover which cannot be followed by closed sets $F_n \subset U_n$. The proof that X is ***normal*** is difficult. It can be pulled off because each ω_n is a ***linearly ordered topological space*** and therefore it is "very" normal. Now, box products of a small number of large ordinal spaces retain enough of the structure of the ordinal spaces (including versions of closed unbounded set combinatorics) that normality is retained.

The generalized ***stationary set*** combinatorics in Rudin's proof made it plausible that Shelah's PCF theory can be applied. Indeed, Kojman and Shelah [8] found a closed Dowker subspace F of Rudin's space which is of size $\aleph_{\omega+1}$. It uses Shelah's theorem that there is an infinite subset $B \subset \omega$ and a sequence of $\bar{f} = \langle f_\alpha \rangle_{\alpha<\aleph_{\omega+1}}$ of functions on $\prod_{n\in B}\omega_n$ such that (a) $\alpha < \beta$ implies $f_\alpha(n) < f_\beta(n)$ for all but finitely many $n \in B$; (b) for every $f \in \prod_{n\in B}\omega_n$ there is an f_α such that $f(n) < f_\alpha(n)$ for all but finitely many $n \in B$. The space of Kojman and Shelah now is the subspace F of Rudin's space X in $\Box_{n\in B}(\omega_n + 1)$ consisting of all $f \in X$ such that (a) there is an α with $f(n) = f_\alpha(n)$ for all but finitely many $n \in \omega$; (b) $\mathrm{cf}(f(n))$ is constant on a cofinite subset of B.

The space F is valuable because its ***cardinality*** $\aleph_{\omega+1}$ is small on the aleph-scale in ZFC. Later in this article Dowker spaces of cardinality $\mathfrak{c} = 2^\omega$ will be discussed, but 2^ω may or may not be smaller than $\aleph_{\omega+1}$, depending on one's set theory.

The other method in ZFC can be described as induction through all initially open 2-covers using ***elementary submodels*** to control the outcome. In the most basic version of the method (only published as part of more complex constructions) one takes $X = \mathfrak{c} \times \omega$ as the underlying set of points with the complement $X \setminus \{x\}$ of each point $x \in X$ and the sets $W_n = \mathfrak{c} \times n$ for each $n = \{0, 1, \ldots, n-1\} \in \omega$ declared open from the start. $\mathcal{W} = \{W_n\colon n \in \omega\}$ will be designed to be the open cover that cannot be followed by closed sets. Next, we make a list $\langle S_\xi \rangle_{\xi<2^\mathfrak{c}}$ of all covers $S_\xi = \langle U_\xi^0, U_\xi^1 \rangle$ of X by two sets mentioning each $2^\mathfrak{c}$ times. At step ξ of the construction we go through the following checklist.

(1) Is S_ξ an open cover in the topology generated by the initial open sets $X \setminus \{x\}$ and W_n as well as open sets defined at steps $\eta < \xi$ as a subbase?

If no then we have nothing to do at step ξ.
If yes then we ask the following question.

(2) Is ξ the smallest ordinal for which Question 1 has a positive answer?

If not, then we have taken care of S_ξ at an earlier step, so there is no reason to act at step ξ. However, if ξ is smallest, then we need to define a partition $\{B_\xi^0, B_\xi^1\}$ of X with $B_\xi^0 \subset U_\xi^0$, $B_\xi^1 \subset U_\xi^1$ and declare B_ξ^i $(i = 0, 1)$ (subbasic) open sets.

The topology τ generated by all sets $X \setminus \{x\}$, W_n and B_ξ^i is obviously a normal T_1 topology. All variations come from what partition B_ξ^0, B_ξ^1 we take at step ξ, if we need to act at that step. For each point $x \in X$ we need to make a decision whether x will belong to B_ξ^0 or B_ξ^1. The decisions have to be made so that there be no closed sets $F_n \subset W_n$ such that $\bigcup_{n\in\omega} F_n = X$. This is a hard job. The simple structure of the construction above is a frame for a lot of hard details. The key is to define a list of countable substructures that reflect a potential sequence $\langle F_n \rangle_{n\in\omega}$ (to be destroyed). Since there are only $\mathfrak{c}^\omega = \mathfrak{c}$ many such substructures, we can assume each point $x \in X$ codes one of them. To decide whether x goes to B_ξ^0 or B_ξ^1 we use the countable substructure x codes. The goal is to kill the potential $\langle F_n \rangle_{n\in\omega}$ the substructure coded by x reflects.

At the end of the proof we assume indirectly that there is a counter-example $\langle F_n \rangle_{n\in\omega}$. We take a countable elementary submodel M containing X, τ, and $\langle F_n \rangle_{n\in\omega}$ and we look at the trace of $\langle F_n \rangle_{n\in\omega}$ on M as a countable substructure. In the basic case, a contradiction already arises at the point x coding that substructure with the help of a technique called complete neighbourhoods.

Using countable substructures as described above was first applied to normality type problems by Rudin [11]. The problem of Dowker she solves there is not related to Dowker spaces. The first Dowker space built by using these ideas by Balogh [3] plugs into a machine invented by Watson [13]. The space is hereditarily normal and it is the union of countably many discrete subspaces. Using Watson's machine simplifies the proof but it also obscures the general ideas described above.

[2] and [1] contain full-fledged versions of the general method, but they aim at getting peculiar Dowker spaces which makes the construction much more complex. In [2] a screenable Dowker space is constructed. It solves a problem of Bing and Nagami from 1953 whether there is a normal space which is **screenable** (i.e., every open cover has a ***σ-disjoint*** open refinement) yet not paracompact. In [1] a hereditarily ***collectionwise normal***, ***realcompact***, ***meta-Lindelöf*** Dowker space is constructed, answering questions of Rudin and Hodel from the early 70s. Again, the construction of the basic Dowker space with this method without any extras has not appeared in print.

ZFC Dowker spaces have various applications. Hart, Junnila and van Mill construct a Dowker ***topological group*** based on the existence of a Dowker space which is a ***P*-space**, i.e., in which the intersection of countably many open sets is open. Rudin's space as well as a version of Balogh's space are such spaces. In [6] a normal space is constructed whose normality is destroyed by adding a single Cohen real. Again, the existence of a Dowker P-space is needed to carry out the proof.

The methods to construct ZFC Dowker spaces can also be applied to other problems. The Rudin–Balogh method was used by the latter to prove Morita's conjectures on normality of products. The construction of a ***Q*-set space** (a space in which every subset is a G_δ-set, but the space is not the union of countably many discrete subspaces) is another example.

2. Dowker spaces in various models of set theory

A large number of Dowker spaces have been constructed under various set-theoretic assumptions each of which holds in only some models of ZFC set theory. Perhaps the most interesting observation we can make about these constructions is that all of them use a single method of building topologies, namely defining a topology inductively as a ***direct limit*** of open subspaces with the help of countable sets as ***neighbourhood bases*** or ***weak neighbourhood bases***.

Two apparent exceptions are Watson's hereditarily normal σ-relatively discrete Dowker space from the existence of a very large (strongly compact) cardinal [13], and M.E. Rudin's screenable Dowker space [10] from the combinatorial consequence $\diamondsuit^{++}$ of the axion of constructibility $V = L$. However, as is described in the section about ZFC examples of Dowker spaces, both of these types of Dowker spaces have been constructed in ZFC. The technique and ideas of both Watson's and Rudin's example above are still worth studying. There is, in fact, a possibility that one can start with the ladder systems of Rudin and construct a consistent example of a Dowker space with a σ-disjoint base the existence of which is one of the relevant open problems.

All other examples in special models of set theory are defined with the single inductive method described above. Depending on the set-theoretic assumptions made they yield fascinating Dowker spaces with a variety of nice properties. They include M.E. Rudin's ***first-countable*** Dowker spaces of size ω_1 from a ***Souslin tree***, P. de Caux's ***σ-discrete***, size ω_1 Dowker space from $\diamondsuit$, M.E. Rudin's hereditarily separable Dowker space from CH and its ***locally compact*** modification by Juhasz, the first-countable, locally compact example from $\mathsf{MA}(\omega_1) + \diamondsuit(\omega_2)$ of Weiss, Bell's first countable Dowker space from $\mathfrak{p} = \mathfrak{c}$, the Dowker ***product*** of Beslagic from $\diamondsuit$, Szeptycki's Dowker space from a ***Lusin set*** and C. Good's first-countable Dowker space from the existence of an E-set with $\diamondsuit(E)$. For references see [12] or, partly [KV, Chapter 17]. After the appearance of [12], Beslagic constructed two more Dowker products [5, 4] and M.E. Rudin constructed a Dowker manifold [9] with the inductive method.

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H: Connection with other structures

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h-1 Topological Groups

In order to keep the size of this article and its bibliography to a minimum we give, whenever possible, references to monographs [2, 6, 11, 12, 14, 19, 20] and surveys [4, 5, 21, 23], [KV, Chapter 24], [HvM, Chapter 2] rather than to original articles and we omit several quite interesting topics (e.g., ***cardinal invariant***s of topological groups, see the surveys [1], [KV, Chapter 24], [24]).

A ***topology*** τ on a group G is a **group topology**, if the group operations $(x, y) \mapsto xy$ (multiplication) and $x \mapsto x^{-1}$ (inverse) are ***continuous***, when $G \times G$ carries the ***product topology***. If H is a subgroup of G, then every group topology on G induces a group topology on H. A **topological group** is a pair (G, τ) consisting of a group G and a group topology τ on G. When the ***topological space*** (G, τ) is ***compact***, one says briefly **compact group**. This applies to other topological properties as well. In the presence of good properties of the topology τ the separate continuity of the multiplication alone yields τ is a group topology (for ***locally compact Hausdorff*** topology τ, this is Ellis' theorem [10]). Two topological groups H, N are **isomorphic**, denoted by $H \cong N$, if there exists a group isomorphism $H \to N$ that is also a ***homeomorphism*** of the underlying spaces.

For a topological group G the family $\mathcal{N}_G$ of all ***open*** subsets of G containing the identity element e is a ***filter base*** with the following properties:

(a) for every $U \in \mathcal{N}_G$ there exists $V \in \mathcal{N}_G$ such that $VV \subseteq U$,
(b) for every $U \in \mathcal{N}_G$ there exists $V \in \mathcal{N}_G$ such that $V^{-1} \subseteq U$,
(c) for every $U \in \mathcal{N}_G$ and each $a \in G$ there exists $V \in \mathcal{N}_G$ such that $aV \subseteq Ua$.

Conversely, if $\mathcal{N}$ is a filter base of a group G satisfying properties (a)–(c), then the family

$$\tau = \{V \subseteq G: \forall a \in V\ \exists U \in \mathcal{N}(aU \subseteq V)\}$$

is a group topology on G such that $\mathcal{N}$ is a ***local base*** for τ at e. Hence a group topology is completely determined by the filter base $\mathcal{N}_G$.

The **quotient group** G/N of a topological group G with respect to a normal subgroup N of G is usually equipped with the ***quotient topology***. It is always a group topology, G/N is Hausdorff iff N is ***closed***, G/N is ***discrete*** iff N is open, G/N is ***indiscrete*** iff N is ***dense***. The canonical map $G \to G/N$ is ***open***. The topological version of the "isomorphism theorem" was proved by H. Freudenthal: if $f: G \to H$ is a continuous open surjective homomorphism, then $H \cong G/\ker f$ [11].

The direct product of a family of topological groups equipped with the product topology is a topological group. For topological groups G, H and a homomorphism $\theta: H \to Aut(G)$ one can define the **semidirect product** $G \rtimes_\theta H$. This is the Cartesian product $G \times H$ equipped with the following group operation:

$$(g, h) \cdot (g_1, h_1) = (g\theta_h(g_1), hh_1),$$
$$\text{for } g, g_1 \in G,\ h, h_1 \in H.$$

Then the product topology of $G \times H$ is a group topology for $G \rtimes_\theta H$ iff the map $G \times H \to G$ defined by $(g, h) \mapsto \theta_h(g)$ is continuous [11].

Some relevant examples of topological groups are: the (additive) group of real numbers $\mathbb{R}$, the circle group $\mathbb{T}$ (usually identified with the quotient group $\mathbb{R}/\mathbb{Z}$), the linear groups (subgroups of $GL_n(\mathbb{C})$ equipped with the topology induced by the canonical embedding of the $n \times n$ matrices over $\mathbb{C}$ in $\mathbb{C}^{n^2}$ equipped with the ***Euclidean topology***), the symmetric group $S(X)$ consisting of all permutations of a discrete set X equipped with the ***topology of pointwise convergence*** on X, the **Bohr topology** of an Abelian group G (this is the smallest topology which makes all homomorphisms $G \to \mathbb{T}$ continuous), the group G equipped with the Bohr topology is denoted by $G^\#$.

In 1944 A. Markov posed the problem of existence of an infinite group that admits only the discrete topology as a Hausdorff group topology. In 1980 S. Shelah constructed such a group of cardinality ω_1 under the assumption of CH (G. Hasse showed later that CH can be removed from Shelah's construction). At the same time A. Ol'shanskij found an elegant example of a countable group G with this property. It has a subgroup C of order m, such that every element $x \in G \setminus C$ satisfies one of the $m-1$ equations $x^m = a$ with $a \in C$, $a \neq e$. Hence, for every Hausdorff group topology τ on G the cofinite set $G \setminus C \neq \emptyset$ is closed, so τ is discrete (see [6, §2.9.3] and [HvM, Chapter 2] for more details).

A ***pseudometric*** d on a group G is **left** (**right**) **invariant** if $d(ax, ay) = d(x, y)$ (respectively, $d(xa, ya) = d(x, y)$) for all $a, x, y \in G$. A **pseudonorm** on a topological group G is a non-negative real-valued function ρ on G with $\rho(e) = 0$, $\rho(x^{-1}) = \rho(x)$ and $\rho(x \cdot y) \leqslant \rho(x) + \rho(y)$ for all $x, y \in G$. Every pseudonorm defines a left (right) invariant pseudometric d by putting $d(x, y) = \rho(x^{-1}y)$ (respectively, $d(x, y) = \rho(xy^{-1})$). On the other hand, every left (right) invariant pseudometrics d gives rise to a pseudonorm defined by $\rho(x) = d(x, e)$. S. Kakutani and G. Birkhoff proved the following ***metrization theorem*** for topological groups. A Hausdorff topological group G is ***metrizable*** if and only if it has a countable ***base*** at identity. The proof is based on the construction of a pseudonorm ρ corresponding to a decreasing chain of symmetric members $\{U_n: n \in \mathbb{N}\}$ of $\mathcal{N}_G$

with $U_n^3 \subseteq U_{n-1}$ for all $n \in \mathbb{N}$, so that the 2^{-n}-ball B_n of ρ around e satisfies $U_n \subseteq B_n \subseteq U_{n-1}$. Consequently, every ***first-countable*** Hausdorff topological group admits left invariant and right invariant ***compatible metric***s.

The distinct separation axioms satisfied by topological groups are fewer than those appearing for all topological spaces. Every T_0 topological group is ***Tychonoff*** (to build continuous real-valued functions one can exploit the continuous pseudonorms as above). Moreover, for an arbitrary topological group G the ***closure*** N of e is a closed normal subgroup of G and the quotient G/N is a T_0 group that shares most of the topological properties as G. So from now on we assume that all topological groups are T_0. There are many examples of non-***normal*** groups. The simplest ones are $\mathbb{Z}^\alpha$ for any uncountable cardinal α and $G^\#$ for any uncountable Abelian group G [HvM, Chapter 2].

In general, a topological group G has four ***compatible uniformities***. The **left uniformity** $\mathcal{U}_l$ of G has as a base of its ***entourages*** the sets

$$U_l = \{(x, y) \in G \times G: x^{-1}y \in U\},$$

where U runs over $\mathcal{N}_G$. The **right uniformity** $\mathcal{U}_l$ is defined analogously with typical entourages

$$U_r = \{(x, y) \in G \times G: xy^{-1} \in U\},$$

where U runs over $U \in \mathcal{N}_G$. The **two-sided uniformity** of G is the supremum of $\mathcal{U}_l$ and $\mathcal{U}_r$, while their infimum is called **lower uniformity** of G. The relation $\mathcal{U}_l = \mathcal{U}_r$ defines the **balanced groups**, known also as **SIN-groups** (since for such a group G the ***filter*** $\mathcal{N}_G$ has a base of invariant, under conjugation, ***neighbourhoods***). Abelian topological groups and compact groups are balanced, but locally compact (briefly, LC) nilpotent groups need not be balanced. A group G is **functionally balanced** if the ***uniformly continuous*** real valued functions w.r.t. $\mathcal{U}_l$ and $\mathcal{U}_r$ coincide. Clearly balanced groups are functionally balanced. The question of whether the inverse implication is also true was raised in [13] and seems to be still open.

A subset B of a group G is **left big** (or **left syndetic**, or **left relatively dense**) if $FB = G$ for some finite subset F of G. Analogously, one defines **right big** (**right syndetic**, etc.). These two concepts need not coincide in non-Abelian groups. A topological group G is **precompact** if every non-empty open set of G is left big (this is equivalent to ask "every non-empty open set of G is right big"). Precompact groups are balanced. For every Abelian group G the Bohr topology of G is the finest precompact topology on G.

A topological group G is a **complete topological group** (or **Weil complete**) if one of the underlying ***uniform spaces*** $(G, \mathcal{U}_l)$ or $(G, \mathcal{U}_r)$ is ***complete*** (in such a case both are complete). A **completion** (or **Weil completion**) of G is a complete group containing G as a dense subgroup. The completion is uniquely determined up to isomorphism (keeping the points of G fixed), but need not exist in general (e.g., $S(X)$ has no completion when X is infinite). The completion of G exists iff the ***Cauchy filter***s with respect to $\mathcal{U}_l$ and $\mathcal{U}_r$ coincide [26] (in particular, when G is balanced; nevertheless LC groups are complete, but need not be balanced). Note that G is precompact iff G has a compact completion.

The group operation of a topological group G can be extended to its completion $\overline{G}$ w.r.t. the two-sided uniformity, so that $\overline{G}$ becomes a topological group that is complete in its two-sided uniformity. The two-sided completion $\overline{G}$ is known also as **Raĭkov completion** of G, the group G is said to be **Raĭkov complete** if $G = \overline{G}$. Clearly, Weil complete groups are Raĭkov complete. In general the group operation of G need not be extendable to a group operation of the completion of $(G, \mathcal{U}_l)$ that makes it a topological group. This is the reason why the Weil completion need not exist in general. When it exists, it coincides with the Raĭkov completion [2].

Let G be a topological group and $B(G)$ denote all bounded complex-valued functions on G. Then $f \in B(G)$ is **almost periodic** if the set $\{f_a: a \in G\}$ is **relatively compact** (i.e., its closure is compact) in the ***uniform topology*** of $B(G)$, where $f_a(x) = f(xa)$ for all $x \in G$ and $a \in G$. According to J. von Neumann, a group G is **maximally almost periodic** (briefly, **MAP**), if the continuous almost periodic functions of G separate the points of G. The continuous almost periodic functions of a group G are related to the so called **Bohr compactification** $b_G : G \to bG$ of G, where b_G is a continuous homomorphism with dence image and bG is a compact group, such that for every continuous homomorphism $f : G \to K$ into a compact group K there exists a (unique) continuous homomorphism $f' : bG \to K$ with $f' \circ b_G = f$. It turns out that every continuous almost periodic function $f : G \to \mathbb{C}$ admits an 'extension' to bG, i.e., the continuous almost periodic function of G are precisely the compositions of b_G with continuous functions of the compact group bG. Then G is MAP iff b_G is injective. Every discrete Abelian group G is MAP and bG coincides with the completion of $G^\#$. In case b_G is a singleton, G is called **minimally almost periodic**. According to a classical result of E. Følner, an Abelian group G is MAP iff for every $a \neq e$ in G there exists a big set B such that a does not belong to the closure of $BB^{-1}BB^{-1}$. On the other hand, R. Ellis and H. Keynes proved that an Abelian group G is minimally almost periodic iff for every three big sets A, B, C of G the set $AB^{-1}C$ is dense in G [6, Chapter 1].

The nice structure theory of LC groups is due to the Haar measure and Haar integral in LC groups. Every LC group G admits a **right Haar integral**, i.e., a positive linear functional I defined on the space $C_0(G)$ of all continuous real-valued functions on G with compact support that is **right invariant** (in the sense that $I(f_a) = I(f)$ for every $f \in C_0(G)$ and $a \in G$). Moreover, if J is another right Haar integral of G then there exists a positive $c \in \mathbb{R}$ such that $J = cI$. The measure m induced by a right Haar integral on the family of all ***Borel set***s of G is called a **right Haar measure**. The group G has finite measure iff G is compact. In such a case the measure m is determined uniquely by the additional condition $m(G) = 1$. Analogously, a LC group admits a unique, up to a positive multiplicative constant, **left Haar integral**. Every compact group G admits a unique Haar integral that is right and left invariant, such that its Haar measure satisfies $m(G) = 1$. The representations of LC groups are based on the Haar integral. According to Gel'fand–Raĭkov's theorem for every LC group G and $a \in G$, $a \neq e$, there exists a continuous irreducible representation V of G by unitary operators of some ***Hilbert space*** H, such that $V_a \neq e$ [11, 22.12]. If G is compact, H can be chosen finite dimensional. It was proved by Freudenthal and Weil that the ***connected*** LC groups with the last property have the form $\mathbb{R}^n \times G$, where G is compact. The case of Gel'fand–Raĭkov's theorem with compact group G, known as Peter–Weyl–van Kampen theorem, can be stated as follows: for every $a \in G$, $a \neq e$, there exists a continuous homomorphism f of G into the group $U(n)$ of $n \times n$ unitary matrices over $\mathbb{C}$, such that $f(a) \neq e$ (n may depend on a). In particular, a topological group G is MAP iff the continuous homomorphisms $G \to U(n)$ (with n varying in $\mathbb{N}$) separate the points of G. In the case of Abelian groups the continuous irreducible unitary representations are simply the continuous characters $G \to \mathbb{T}$ (so that an Abelian topological group G is MAP iff the continuous homomorphisms $G \to \mathbb{T}$ separate the points of G). One can obtain a proof of Peter–Weyl–van Kampen theorem in the Abelian case without any recourse to Haar integration (for an elementary approach in this line, based on Følner's theorem mentioned above and ideas of Iv. Prodanov, see [6, Chapter 1]).

A topological group G is **locally Euclidean** if the underlying space of G is an ***n-manifold***, G is a **Lie group** if the underlying space of G admits a structure of an **analytic n-manifold**, i.e., such that the multiplication and the inversion are ***analytic functions***. Hilbert's fifth problem is the question: Is every locally Euclidean topological group a Lie group? This problem was solved in the affirmative in 1952 by D. Montgomery and L. Zippin using essentially results of A. Gleason. It turned out that a LC group is a Lie group iff G is locally Euclidean iff G has no small subgroups (i.e., there exists $U \in \mathcal{N}_G$ such that $\{e\}$ is the only subgroup of G contained in U [14]). An important by-product of these efforts was the proof of Iwasawa's conjecture (every LC group, that is compact modulo the ***connected component*** of e, is an ***inverse limit*** of Lie groups).

Pseudocompact topological groups admit a rather transparent description, due to Comfort–Ross' theorem, as the ***G_δ-dense*** subgroups of the compact groups. Consequently, any product of pseudocompact topological groups is again pseudocompact. Furthermore, pseudocompact groups are precompact and their compact completion coincides with their ***Čech–Stone compactification*** [HvM, Chapter 24].

For a topological Abelian group G consider the group $\widehat{G}$ of all continuous characters $G \to \mathbb{T}$. It is a topological group when equipped with the **compact-open topology** having as base of $\mathcal{N}_{\widehat{G}}$ the family of all sets of the form

$$W(K, U) = \{\chi \in \widehat{G} \colon \chi(K) \subseteq U\},$$

where K is an arbitrary compact subset of G and $U \in \mathcal{N}_{\mathbb{T}}$. Then $\widehat{G}$ is discrete (compact) iff G is compact (respectively, discrete); $\widehat{G}$ is LC whenever G is a LC Abelian (briefly, **LCA**) group. Let G and H be Abelian topological groups and $\varphi : G \to H$ be a continuous homomorphism. The **dual homomorphism** $\hat{\varphi} : \widehat{H} \to \widehat{G}$, defined by $\hat{\varphi}(\chi)(g) = \chi(\varphi(g))$ for all $\chi \in \widehat{H}$ and $g \in G$, is continuous. Moreover, there is a **canonical homomorphism** $\omega_G : G \to \widehat{\widehat{G}}$ given by $\omega_G(g)(\chi) = \chi(g)$ for $\chi \in \widehat{G}$ and $g \in G$. According to Pontryagin–van Kampen's duality theorem, ω_G is a topological isomorphism for every LCA group G [19]. This theorem enables us to identify a LCA group G with $\widehat{\widehat{G}}$. Some non-LCA groups admit duality (as $\mathbb{R}^\omega$, some ***free Abelian topological group***s etc.). In categorical terms, the assignment $G \mapsto \widehat{G}$ defined by the duality is a ***contravariant functor*** of the ***category*** $\mathcal{L}$ of LCA groups in itself such that the composition $\widehat{\ }\circ\widehat{\ }$ is naturally equivalent to the identity functor of $\mathcal{L}$ (i.e., $\widehat{\ }$ is an **involutive functor**). D. Roeder established a uniqueness theorem showing that every other involutive contravariant functor of the category $\mathcal{L}$ in itself must coincide, up to natural equivalence, with $\widehat{\ }$. Iv. Prodanov extended this approach to LC modules over commutative rings, where uniqueness may fail in general [6, §§ 3.4, 3.7.2].

It was proved independently by L. Ivanovskiĭ and V. Kuz'minov that the underlying space of every compact group is ***dyadic*** (B. Pasynkov showed later that it is actually, ***Dugundji compact***). Consequently, compact groups have many non-trivial ***convergent sequences***. E. van Douwen [9] proved, under the assumption of MA, that ***countably compact*** groups may have no non-trivial convergent sequences and used this fact to produce two countably compact groups whose product is not countably compact.

A topological group (G, τ) is **minimal** if every Hausdorff group topology on G ***coarser*** than τ coincides with τ; G is **totally minimal** if every Hausdorff quotient of G is minimal. A minimal group need not be totally minimal, while compact groups are totally minimal. It was pointed out by S. Dierolf and U. Schwanengel that the totally minimal group $S(X)$ is not even precompact for infinite X. Later Iv. Prodanov and L. Stoyanov proved that the minimal Abelian groups are precompact (hence complete minimal Abelian groups are compact). The unitary group of every Hilbert space is totally minimal [22]. Recently V. Uspenskij found a new proof of this fact (based on ***compactifications*** w.r.t. the lower uniformity) and proved that every topological group is isomorphic to a subgroup of a complete minimal group without proper closed normal subgroups. More details and references can be found in [6] (for the recent developments see [4]). Another impressive application of the lower uniformity was given in [25], where a negative answer was given to a long standing open problem of K.-H. Hofmann on whether the ***epimorphisms*** in the category of Hausdorff topological groups and continuous homomorphisms have dense range.

Not every group admits a compact group topology. In 1944 P. Halmos posed the problem of characterization of those Abelian groups that admit a compact group topology. This question brought to a significant development of the theory of abstract Abelian groups and was resolved by 1956 by H. Hulanicki and D. Harrison [11]. For the counterpart of Halmos' problem for pseudocompact topologies see [7] and [HvM, Chapter 2], for minimal topologies see [4, 6].

The connected component $c(G)$ (quasi-component $q(G)$) of e in a topological group G is a closed normal subgroup of G, called briefly **connected component** (respectively **quasi-component**) of G. The connected component (respectively, quasi-component) of an arbitrary point $x \in G$ is the coset $xc(G)$ (respectively, $xq(G)$). The quotient $G/c(G)$ is ***hereditarily disconnected***, while $G/q(G)$ is ***totally disconnected***. The arc-component of e is again a normal subgroup, but it need not be closed even in a LCA group G, where it coincides with the subgroup generated by the images of all continuous homomorphisms $\mathbb{R} \to G$. The following proposition can neither be proved nor disproved in ZFC: every compact ***arcwise connected*** Abelian group is isomorphic to a power of $\mathbb{T}$ [12].

The connected component and the quasi-component coincide for LC or countably compact groups, but this fails to be true even for pseudocompact totally minimal groups [5]. The LC hereditarily disconnected groups have a local base at e consisting of open subgroups (in particular they are ***zero-dimensional***). In the compact case these open subgroups can be chosen normal. Analogously, countably compact hereditarily disconnected groups have a local base at e consisting of open normal subgroups [5] (in particular they are zero-dimensional). This yields that every compact totally disconnected group is an inverse limit of finite groups, so that these groups are known also as **profinite groups**. They are related to Galois Theory as follows. A field extension E/K (where the field K is a subfield of E) is a **Galois extension** if E/K is normal and separable (i.e., E/K is an algebraic extension such that every irreducible polynomial $f(x) \in K[x]$ having a root in E has all its roots in E and they are simple). The **Galois group**

$$G(E/K) = \{\sigma \in Aut(E) \colon \sigma|_K = \mathrm{id}_K\}$$

of a (possibly infinite) Galois extension E/K carries the **Krull topology** [20] (it coincides with the topology of pointwise convergence on the discrete set E). This makes $G(E/K)$ a compact totally disconnected group. Moreover, every profinite group coincides with the Galois group of some Galois extension E/K. Further information on profinite groups can be found in [20].

The theory of compact connected groups can be easily reduced to that of Abelian ones and compact Lie groups in the following way. For every compact connected group G the derived group G' (this is the subgroup of G generated by all commutators $[a, b] = a^{-1}b^{-1}ab$ with $a, b \in G$) is a closed connected group, such that $G'/Z(G')$ (where $Z(G')$ is the center of G') is a product of algebraically simple compact connected Lie groups L_i, $i \in I$. Moreover, if $\tilde{L}_i$ denotes the universal covering group of each L_i (cf. [12, Definition A2.19]), then G' is isomorphic to a quotient of the product $L = \prod_i \tilde{L}_i$ with respect to a closed totally disconnected central subgroup N of L (i.e., N is contained in $Z(L)$). On the other hand, $c(Z(G)) \cap G'$ is a totally disconnected subgroup of $Z(G')$. Hence, G is isomorphic to a quotient of the product $c(Z(G)) \times \prod_i \tilde{L}_i$ w.r.t. some closed totally disconnected central subgroup. Finally, for any compact group G there is a compact totally disconnected subgroup D of G providing a "sandwich situation"

$$\begin{aligned} c(G) \rtimes_\iota D &\xrightarrow{m} G \xrightarrow{q} G/\bigl(c(G) \cap D\bigr) \\ &\cong c(G)/\bigl(c(G) \cap D\bigr) \rtimes D/\bigl(c(G) \cap D\bigr), \end{aligned}$$

where the action ι of D on $c(G)$ is defined by conjugation, m is simply the restriction of the multiplication map $G \times G \to G$ and q is the quotient map, so that both morphisms are surjective and have totally disconnected kernel isomorphic to $c(G) \cap D$ [12, Theorem 9.42] (note that $c(G) \cap D$ is central since every totally disconnected normal subgroup of a connected group is central [11, (7.17)]).

V. Uspenskij proved that the group of all auto-homeomorphisms of the ***Hilbert cube*** in the compact-open topology is a universal object for the class of ***second-countable*** topological groups (i.e., every second-countable groups is isomorphic to some of its subgroups). Later he found another example of a universal second-countable group, namely the group $\mathrm{Is}(U)$ of isometries of Urysohn's universal complete separable metric space U onto itself endowed with the topology of pointwise convergence [HvM, Chapter 2] (V. Pestov proved recently that these two universal groups are not ***homeomorphic***). Recently S. Shkarin showed that also the class of all Abelian second-countable groups has a universal object. Actually, he shows that under the assumption of GCH, this result can be extended to the larger classes of Abelian topological groups of ***weight*** less than or equal to a given cardinal τ.

A positive answer to the following question of A. Kechris in the Abelian case is given in [18]: Is there a projectively universal Polish group (that is, a Polish group G such that every Polish group is a quotient group of G)? A **Polish group** means a topological group which is a ***Polish space***. It is shown that there is a projectively universal complete metric Abelian group of an arbitrary weight. Furthermore, in the non-Abelian case, there is a projectively universal group for those complete groups which admit ***metrics*** that are both left and right invariant.

A topological group G is said to be **topologically generated** by a subset X if X generates a dense subgroup of G. Tate [8] proved that every profinite group can be generated by **supersequence** (i.e., a compact set with at most one non-isolated point). Later this was generalized by K.-H. Hofmann and S. Morris, they showed that every locally compact group G is topologically generated by a **suitable set**, namely a relatively discrete set X having at most one non-isolated point x, such that $\{x\} \cup X$ is closed. More on this topic, including examples of groups without a suitable set, can be found in [12, 23].

Every topological group has a plenty of self-homeomorphisms of the underlying space, e.g., the left and the right translations. An example of a topological group whose only self-homeomorphisms are the translations can be found in [15]. Moreover, under CH, there exists a topological group such that each continuous self-map is either constant or a group translation [15].

Inquiring further about the connection between homeomorphisms and homomorphisms let us mention that sometimes the topology alone determines the structure of a topological group. S. Scheinberg proved that two connected compact Abelian groups are isomorphic as topological groups whenever they are homeomorphic (but homeomorphic countably compact connected Abelian groups need not be isomorphic even as abstract groups [5]). In other cases, the algebraic structure determines completely the topology: it is a classical theorem of B. van der Waerden that every homomorphism of a compact Lie group with finite center into a compact group is continuous. Resolving negatively a problem of E. van Douwen, K. Kunen proved that if V_p is the countable vector space over the finite field $\mathbb{Z}/p\mathbb{Z}$, then $V_p^\#$ and $V_q^\#$ are not homeomorphic as topological spaces for distinct primes p and q.

B. Pasynkov [17] proved that every LC topological group G satisfies

$$\operatorname{ind} G = \operatorname{Ind} G = \dim G$$

and this was extended by M. Tkachenko to locally ***pseudocompact*** groups. The first example of a normal topological group with non-coinciding ***dimensions*** was constructed by D. Shakhmatov [21]. The dimension of a compact Abelian group G coincides with the free rank of the discrete Abelian group $\widehat{G}$ (i.e., the maximum rank of a free subgroup of $\widehat{G}$). Since every LCA group has the form $\mathbb{R}^n \times T_0$ for some integer $n \geqslant 0$ and a group T_0 having an open compact subgroup, the dimension theory of LCA groups can be brought to purely algebraic issues. For a general LC group G the equality

$$\dim G = \dim H + \dim G/H$$

holds for every closed subgroup H of G (this formula is proved in [27] modulo a property of finite-dimensional LC groups proved in [16]). In particular, $\dim G = \dim c(G)$ for every LC group G. Since, every connected LC group G is homeomorphic to $\mathbb{R}^n \times K$, where K is a connected compact subgroup of G, this reduces the dimension theory of LC groups to that of compact connected groups K. The derived group K' is either ***strongly infinite-dimensional***, or a Lie group. In the latter case $\dim K'$ coincides with the dimension of K' as a locally Euclidean space and

$$\dim K = \dim K' + \dim Z(K)$$

since $K/K' \cong Z(K)/Z(K) \cap K'$ and $\dim Z(K) \cap K' = 0$. For further information on dimension theory of topological groups see the survey [21].

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h-2 Topological Rings, Division Rings, Fields and Lattices

The theory of topological rings and fields is lucky enough to have good introductory textbooks [2, 13, 16–18]. In order to keep the bibliography of this article to a minimum we include, whenever possible, references to one of these books rather than to the original articles where the referenced result first appeared in print. References to the original articles are provided only when the proof cannot be found in one of the above five textbooks.

A **binary operation** $\circ$ on a set R is a map $(r,s)\mapsto r\circ s$ from $R\times R$ to R. A binary operation $\circ$ on R is **associative** provided that $r\circ(s\circ t)=(r\circ s)\circ t$ for $r,s,t\in R$. A **ring** is a set R together with two associative binary operations $+$ (**addition**) and $\cdot$ (**multiplication**), and a fixed element $0\in R$ (called **zero**) such that: (i) $(R,+,0)$ is an Abelian (= commutative) group, and (ii) $a\cdot(b+c)=a\cdot b+a\cdot c$ and $(b+c)\cdot a=b\cdot a+c\cdot a$ for $a,b,c\in R$ (the distributive laws). An element $1\in R$ is called the **unitary element of** R provided that $r\cdot 1=1\cdot r=r$ for all $r\in R$. An element a of a ring R with 1 is called an **invertible element** if there exists $b\in R$ such that $a\cdot b=b\cdot a=1$. If a is invertible, then the element b as above is unique, is called the **inverse element of** a and is denoted by a^{-1}. If all elements $a\neq 0$ of a ring R are invertible, then R is called a **division ring** or a **skew field**. A ring R is **commutative** if $r\cdot s=s\cdot r$ for $r,s\in R$. A **field** is a commutative division ring. A minimal positive integer p such that $p\cdot 1=1+\cdots+1=0$ (the sum is taken $p-1$ many times), if such p exists, is called the **characteristic** of a division ring. If no such p exists, then the division ring has **zero characteristic**. If $a,b\in R$, $a\cdot b=0$ but $a\neq 0$ and $b\neq 0$, then each of a and b is called a **zero divisor**. For subsets A and B of a ring R we define $A+B=\{a+b\colon a\in A,\ b\in B\}$, $-A=\{-a\colon a\in A\}$, $A\cdot B=\{a\cdot b\colon a\in A,\ b\in B\}$; furthermore, if R is a division ring, then $A^{-1}=\{a^{-1}\colon a\in A\}$. If $a\in R$, then we write $a+A$ and $a\cdot A$ instead of $\{a\}+A$ and $\{a\}\cdot A$.

Let R be a ring. A ***topology*** $\mathcal{T}$ on a set R is called a **ring topology** provided that: (1) the additive group $(R,+,0)$ is a ***topological group*** when equipped with the topology $\mathcal{T}$; that is, both the addition map $(r,s)\mapsto r+s$ from $(R,\mathcal{T})\times(R,\mathcal{T})$ to $(R,\mathcal{T})$ and the additive inverse map $r\mapsto -r$ from $(R,\mathcal{T})$ to $(R,\mathcal{T})$ are ***continuous***, and (2) the multiplication map $(r,s)\mapsto r\cdot s$ from $(R,\mathcal{T})\times(R,\mathcal{T})$ to $(R,\mathcal{T})$ is continuous. A **topological ring** is a pair $(R,\mathcal{T})$ consisting of a ring R together with some ring topology $\mathcal{T}$ on R. If R is a division ring, then a topology $\mathcal{T}$ on R is called a **division ring topology** or a **skew field topology** on R provided that $\mathcal{T}$ is a ring topology on R and the multiplicative inverse map $r\mapsto r^{-1}$ from $(R\setminus\{0\},\mathcal{T}')$ to $(R\setminus\{0\},\mathcal{T}')$ is continuous, where $\mathcal{T}'$ is the ***subspace topology*** on $R\setminus\{0\}$ induced by $\mathcal{T}$. A **topological division ring** or a **topological skew field** is a division ring equipped with some division ring topology. A division ring topology on a field R is called a **field topology** on R, and a field equipped with some field topology is called a **topological field**.

From (i) and (1) it follows that a topological ring is a particular type of an Abelian topological group, and thus all results about (Abelian) topological groups also hold in topological rings. Conversely, every Abelian topological group G can be turned into a topological ring via defining the **trivial multiplication** on G by $g\cdot h=0$ for $g,h\in G$. Therefore, if one is searching for specific topological results that hold for topological rings but do not hold for general topological groups, it is necessary to eliminate this trivial multiplication, for example, by considering only rings without zero divisors (this latter class includes division rings and fields).

Both the ***discrete topology*** and the **trivial topology** (also called the **indiscrete topology**) $\{\emptyset,R\}$ on a ring (field) R are ring (field) topologies. It is not yet known if every infinite ring admits a non-discrete ***Hausdorff*** ring topology. However, the positive answer to this question is known for countable (not necessarily even associative) rings [2, Theorem 5.3.8] and for commutative rings [2, Theorem 5.5.9]. There is an example of a non-associative commutative ring without a non-discrete Hausdorff ring topology [2, Chapter 5.6]. Every infinite field admits a Hausdorff non-discrete field topology. Moreover, if R is a field, then there exists exactly $2^{2^{|R|}}$ pairwise non-isomorphic field topologies on R, the maximal number possible [18, Theorem 1.4.6]. It is not known if every infinite division ring admits a non-discrete Hausdorff division ring topology. The answer is not known even for countable division rings.

It follows from (1) that if a topological ring is T_0***-space*** (in particular, if a topological ring is Hausdorff) then it is automatically a ***Tychonoff space***. There exists a topological field whose topology is Tychonoff but is not ***normal*** [9], a normal topological field that is not ***hereditarily normal*** [9] and a hereditarily normal topological field that is not ***perfectly normal*** [8]. A non-trivial ring topology on a division ring is automatically Hausdorff [16, Corollary 4.7]. Moreover, a non-trivial ring topology on a division ring is either ***connected*** or ***totally disconnected*** [16, Corollary 4.7].

Similar to the case of topological groups, the most general method for introducing ring topologies on a ring is by specifying ***neighbourhood***s of zero. Let R be a topological ring. Then the family $\mathcal{B}_0$ of all ***open*** subsets of R containing 0 satisfies the following properties:

(3) $0 \in \bigcap \mathcal{B}_0$, (4) if $U, V \in \mathcal{B}_0$, then there exists $W \in \mathcal{B}_0$ with $W \subseteq U \cap V$, (5) for every $U \in \mathcal{B}_0$ there exists $V \in \mathcal{B}_0$ such that $V + V \subseteq U$, (6) for every $U \in \mathcal{B}_0$ there exists $V \in \mathcal{B}_0$ such that $-V \subseteq U$, (7) for every $U \in \mathcal{B}_0$ there exists $V \in \mathcal{B}_0$ such that $V \cdot V \subseteq U$, (8) for every $U \in \mathcal{B}_0$ and each $r \in R$ there exists $V \in \mathcal{B}_0$ such that $r \cdot V \subseteq U$ and $V \cdot r \subseteq U$. Conversely, if $\mathcal{B}_0$ is a family of subsets of a ring R satisfying properties (3)–(8), then the family $\mathcal{T} = \{V \subseteq R: \forall r \in V\ \exists U_r \in \mathcal{B}_0\ (r + U_r \subseteq V)\}$ is a ring topology on R such that $\mathcal{B}_0$ is a ***local base*** for $\mathcal{T}$ at 0 [2, Theorem 1.2.5]. If R is a topological division ring, then the family $\mathcal{B}_0$ of all open subsets of R containing 0 satisfies properties (3)–(8) together with the following property: (9) for every $U \in \mathcal{B}_0$ there exists $V \in \mathcal{B}_0$ such that $((1 + V) \setminus \{0\})^{-1} \subseteq 1 + U$. Conversely, if $\mathcal{B}_0$ is a family of subsets of a division ring R satisfying properties (3)–(9), then the same family $\mathcal{T}$ as above is a division ring topology on R such that $\mathcal{B}_0$ is a local base for $\mathcal{T}$ at 0 [2, Theorem 1.2.13].

Historically, the traditional way of introducing ring topologies on a ring is via norms and absolute values. A real-valued function φ defined on a ring R is called a **norm** provided that: (10) $\varphi(r) \geqslant 0$ for $r \in R$, (11) $\varphi(r) = 0$ if and only if $r = 0$, (12) $\varphi(a - b) \leqslant \varphi(a) + \varphi(b)$ for $a, b \in R$, (13) $\varphi(a \cdot b) \leqslant \varphi(a)\varphi(b)$ for $a, b \in R$. (The notation $|r|$ is often used instead of $\varphi(r)$.) An **absolute value** on a ring R is a norm φ on R satisfying the stronger condition (13′) $\varphi(a \cdot b) = \varphi(a)\varphi(b)$ for $a, b \in R$. An absolute value φ on a ring R is called a **non-Archimedean absolute value** provided that $\varphi(a + b) = \max(\varphi(a), \varphi(b))$ for $a, b \in R$, and is called **Archimedean** otherwise. Every norm φ on a ring R defines a ***metric*** d_φ on R via $d_\varphi(x, y) = \varphi(x - y)$ for $x, y \in R$. A norm φ is a **complete norm** if d_φ is a ***complete metric***. The topology on R generated by the metric d_φ is a ring topology called the **norm topology** on R generated by φ. A norm is a **proper norm** if it generates a non-discrete topology. Two norms on a ring are **equivalent** if they generate the same topology. A norm topology on a division ring generated by an absolute value is automatically a division ring topology (combine (15) and (16) below). An absolute value on a division ring of prime characteristic is non-Archimedean [17, Theorem 18.15].

The function $x \mapsto |x|$ is an absolute value on both the field $\mathbb{Q}$ of rational numbers and the field $\mathbb{R}$ of real numbers. The function that maps a complex number $z = a + bi$ to $|z| = \sqrt{a^2 + b^2}$ is an absolute value on the field $\mathbb{C}$ of complex numbers. Let $\mathbb{H}$ be the 4-dimensional vector space over the field $\mathbb{R}$ with the basis $\{e, i, j, k\}$. Define the multiplication on the basis as follows: $e \cdot e = e$, $i \cdot i = j \cdot j = k \cdot k = -e$, $e \cdot i = i$, $e \cdot j = j$, $e \cdot k = k$, $i \cdot j = -j \cdot i = k$, $j \cdot k = -k \cdot j = i$, $k \cdot i = -i \cdot k = j$. This multiplication can be extended from the basis over $\mathbb{H}$ using distributivity laws and the commutativity of real numbers with the elements of the basis. As a result we get a division ring that is called the **division ring of quaternions**. The function which maps a quaternion $h = h_e e + h_i i + h_j j + h_k k \in \mathbb{H}$ to $|h| = \sqrt{h_e^2 + h_i^2 + h_j^2 + h_k^2}$ is an absolute value on $\mathbb{H}$.

The importance of the above fields can be best illustrated by two fundamental results. The first result is the famous 1931 Pontryagin's description of connected ***locally compact*** division rings: The only locally compact connected division rings are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ [17, Theorem 27.2]. Furthermore, a non-discrete locally compact division ring of characteristic 0 is topologically isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or a finite-dimensional division algebra over the p-adic number field $\mathbb{Q}_p$ (to be defined later in (22)) for some prime p [17, Theorem 27.1]. The second result is a beautiful description of Archimedean absolute values on division rings found in 1915 by Ostrowski: If φ is an Archimedean absolute value on a division ring (field) R, then there exists $r \in (0, 1]$ and an isomorphism σ from R to a subring of the division ring of quaternions $\mathbb{H}$ (the field $\mathbb{C}$ of complex numbers, respectively) such that $\varphi(x) = |\sigma(x)|^r$, where $|\cdot|$ is the absolute value on $\mathbb{H}$ (respectively, on $\mathbb{C}$) defined above [17, Theorem 26.14].

If φ is a norm on a ring R and $r > 0$ is a real number, then φ^r is the function on R defined by $\varphi^r(x) = \varphi(x)^r$ for $x \in R$. If φ is an absolute value on a division ring R and $J(\varphi)$ is the set of all positive reals r such that φ^r is an absolute value on R, then $(0, 1] \subseteq J(\varphi)$; moreover, $J(\varphi) = \{r \in \mathbb{R}: r > 0\}$ if and only if φ is non-Archimedean [17, Theorem 18.5]. (14) Two absolute values φ_1 and φ_2 on a division ring are equivalent if and only if $\varphi_2 = \varphi_1^r$ for some $r > 0$ [17, Theorem 18.4].

We are going to describe all absolute values on $\mathbb{Q}$. If φ is a proper Archimedean absolute value on $\mathbb{Q}$, then $\varphi = |\cdot|^r$ for some $r \in (0, 1]$ [13, Theorem 12.3]. Thus (14) implies that the usual topology on $\mathbb{Q}$ is the only non-discrete topology on $\mathbb{Q}$ that can be generated by an Archimedean absolute value. Fix a prime number p. Define $|0|_p = 0$. For a non-zero rational number q there exists a unique integer k such that $q = (p^k a)/b$, where a and b are integers and both a and b are not divisible by p. Define $|q|_p = 1/2^k$. The function $|\cdot|_p$ is a non-Archimedean absolute value on $\mathbb{Q}$ called the ***p*-adic absolute value**. There are essentially no other non-Archimedean absolute values on $\mathbb{Q}$: If φ is a proper non-Archimedean absolute value on $\mathbb{Q}$, then $\varphi = |\cdot|_p^r$ for some prime number p and $r > 0$ [17, Theorem 18.18]. The topology generated by $|\cdot|_p$ is called the ***p*-adic topology** on $\mathbb{Q}$. From the above result and (14) it follows that a non-discrete topology on $\mathbb{Q}$ generated by a non-Archimedean absolute value coincides with the p-adic topology for some prime p.

A subset B of a topological ring R is called **left bounded** (**right bounded**) provided that for every open set U containing zero 0 there exists an open set V containing zero such that $V \cdot B \subseteq U$ ($B \cdot V \subseteq U$, respectively). A subset of a topological ring is a **bounded subset** if it is both left bounded and right bounded. Clearly, a ***compact*** subset of a topological ring is bounded. A topological ring R is **left bounded** (**right bounded**) if R is a left bounded (right bounded) subset of R. A non-trivial topological division ring is (left or right) bounded if and only if it is discrete. A topological ring is left (right) bounded if and only if it has a basis of neighbourhoods of zero consisting of right (left) ideals of its multiplicative semigroup. A topological ring is **locally bounded** if it has a bounded open neighbourhood of zero. In particular, locally compact rings are locally bounded. A locally compact totally disconnected ring has a basis of neighbourhoods of zero consisting of its open subrings. A bounded locally compact totally disconnected ring has a basis of neighbourhoods of zero consisting of open compact ideals. In particular, a compact totally disconnected ring has a basis of neighbourhoods of zero consisting of open compact ideals. These results are directly applicable to many locally compact rings due to the following fact: A locally compact left (right) bounded ring with 1 is totally disconnected; in particular, a compact ring with 1 is totally disconnected. A Hausdorff compact ring R is embeddable into a compact ring with 1 if and only if R is totally disconnected [2, Corollary 4.4.10]. A compact division ring is finite [16, Corollary 12.21].

A topological division ring R is a **locally retrobounded ring** or a **ring of type** V provided that R is Hausdorff and for every open set U containing 0 the set $(R \setminus U)^{-1}$ is bounded in R. (15) A locally retrobounded ring topology on a division ring is a locally bounded division ring topology [16, Theorem 13.7]. (16) A division ring topologized by an absolute value is locally retrobounded [16, Theorem 13.6]. Locally retrobounded topologies are the most general type of ring topologies on a division ring for which the following so-called "**generalized approximation theorem**" is known to hold: If R is a division ring, $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n$ is a finite family of pairwise distinct non-discrete Hausdorff division ring topologies on R such that $\mathcal{T}_1, \ldots, \mathcal{T}_n$ are locally retrobounded and each $\mathcal{T}_i$, $i = 1, \ldots, n$, is not coarser than $\mathcal{T}_0$, then the topologies $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n$ are independent, i.e., $\bigcap_{i=0}^{k} U_i \neq \emptyset$ for any choice of non-empty sets $U_0 \in \mathcal{T}_0, U_1 \in \mathcal{T}_1, \ldots, U_n \in \mathcal{T}_n$ [17, Theorem 28.13]. Clearly, a finite family $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n$ of topologies on R is independent if and only if the "diagonal" $\Delta(R) = \{(r, r, \ldots, r)\colon r \in R\} \subseteq R^{n+1}$ is ***dense*** in the ***product space*** $(R, \mathcal{T}_0) \times (R, \mathcal{T}_1) \times \cdots \times (R, \mathcal{T}_n)$. From (16) we get the classical **approximation theorem**: If $\varphi_1, \ldots, \varphi_n$ is a finite family of pairwise non-equivalent proper absolute values on a division ring R, then for every finite sequence $a_1, \ldots, a_n$ of elements of R and each $\varepsilon > 0$ there exists $x \in R$ such that $\varphi_i(x - a_i) < \varepsilon$ for $i = 1, \ldots, n$ [17, Corollary 28.14].

A subset S of a topological ring R is called **topologically nilpotent** provided that for every open set U containing 0 one can find a positive integer n_0 such that $S^n = \{s_1 \cdot \cdots \cdot s_n\colon s_1 \in S, \ldots, s_n \in S\} \subseteq U$ for $n \geqslant n_0$. An element $a \in R$ is said to be **topologically nilpotent** if the one-element set $\{a\}$ is topologically nilpotent, or equivalently, if for every open set U containing 0 there exists a positive integer n_0 such that $a^n \in U$ for $n \geqslant n_0$. An element a of a topological ring R with 1 is **neutral** provided that a is invertible and neither a nor a^{-1} is topologically nilpotent. An element a of a topological ring R is topologically nilpotent if and only if all its positive powers a^n are topologically nilpotent if and only if some positive power a^k is topologically nilpotent [2, Corollary 1.8.36]. If φ is a norm on a ring R and $\varepsilon < 1$, then the set $\{r \in R\colon \varphi(r) < \varepsilon\}$ is topologically nilpotent [2, Proposition 1.8.18]. If a locally bounded topological ring R with 1 has a topologically nilpotent invertible element, then R has a topologically nilpotent open set containing 0 [2, Proposition 1.8.38]. A locally bounded connected topological division ring has an open topologically nilpotent set containing 0 [2, Corollary 1.8.40]. A non-discrete locally compact Hausdorff topological ring without zero divisors has a topologically nilpotent element [2, Theorem 1.8.41]. It follows that a non-discrete locally compact topological division ring contains a topologically nilpotent open neighbourhood of zero [2, Corollary 1.8.41]. The set of all topologically nilponent elements of a division ring equipped with a locally compact ring topology is open (and non-empty) [2, Corollary 1.8.43]. If $\mathcal{T}$ is a ring topology on a division ring R such that there exists a topologically nilponent set $U \in \mathcal{T}$ containing 0, then $\mathcal{T}$ is automatically a division ring topology on R [2, Theorem 1.8.44]. In particular, both a connected locally bounded ring topology and a locally compact ring topology on a division ring are division ring topologies [2, Corollaries 1.8.45 and 1.8.46].

We say that a ring topology $\mathcal{T}$ on a ring R is a **normalized topology** or can be generated by a norm if there exists a norm φ on R such that $\mathcal{T}$ conincides with the norm topology generated by φ. We are now ready to present various criteria of normalizability of ring topologies. A ring topology $\mathcal{T}$ on a ring R is normalized if and only if R is a union of a countable family of its bounded subsets and there exists a sequence $\{U_i\colon i \in \omega\} \subseteq \mathcal{T}$ of neighbourhoods of 0 such that $U_{i+1} + U_{i+1} \subseteq U_i$ and $U_i \cdot U_j \subseteq U_{i+j}$ for $i, j \in \omega$ [2, Theorem 2.1.4]. In this result the condition "R is a union of a countable family of its bounded subsets" can be replaced by the following alternative condition: For every $a \in R$ there exists $n \in \omega$ such that $a \cdot U_{i+n} \subseteq U_i$ and $U_{i+n} \cdot a \subseteq U_i$ for all $i \in \omega$ [2, Theorem 2.1.4]. If a topological ring R with 1 contains an invertible topologically nilpotent element d and a bounded open neighbourhood U of 0 such that $d \cdot U = U \cdot d$, then R is normalized [2, Theorem 2.1.11]. As a corollary, one gets a nice criterion for normalizability of topological fields: A topological field is normalized if and only if it is locally bounded and possesses a non-zero topologically nilpotent element [2, Corollary 2.1.12]. The topology of a topological division ring R can be generated by an absolute value if and only if the set T of topologically nilpotent elements of R is open and right bounded in R, and $(T \cup N) \cdot T \subset T$, where N is the set of all neutral elements of R [2, Theorem 2.2.3]. The topology of a topological division ring R can be generated by an absolute value if and only if the set T of topologically nilpotent elements of R is open, and $T \cup N$ is bounded in R, where N is the set of all neutral elements of R [2, Theorem 2.2.4]. In particular, the topology of a locally compact field can always be generated by an absolute value [2, Corollary 2.2.5]. A field topology on a field R can be generated by some absolute value if and only if the set of topologically nilpotent elements of R is open and R is locally retrobounded [2, Theorem 2.2.6].

Norms and absolute values represent a special case of a more general notion. A **prevaluation** on a field R is a surjection v of R onto a directed Abelian group G augmented by a least element 0, such that the following holds: (17) $v(x)=0$ if and only if $x=0$; (18) $v(xy)=v(x)v(y)$; (19) there exists $\lambda \in G$ (called a **dominator**) such that $v(x), v(y) \leqslant g$ implies $v(x+y) \leqslant \lambda g$. If $v(-1)=1$ also holds, then a prevaluation v is called a **near valuation**. If G is linearly ordered, we have $v(-1)=1$, and (19) may be replaced by $v(x+y) \leqslant \lambda \max(v(x), v(y))$. In this case, v is called a **valuation**. A valuation with $\lambda = 1$ is called a **non-Archimedean valuation** or a **Krull valuation**. When G is the multiplicative group of positive real numbers and λ is arbitrary, we get the definition of **real valuation**, which for $\lambda \leqslant 2$ turns out to be an absolute value defined above, i.e., (19) may be replaced by (19′) $v(x+y) \leqslant v(x)+v(y)$ [13, Theorem 15.2.3]. In fact, v is a real valuation on a field R if and only if, for a suitable $r > 0$, its power v^r (defined by $v^r(x) = v(x)^r$ for $x \in R$) is an absolute value [13, Theorem 12.5.2]. The notion of a prevaluation v turns out to be strong enough to yield a ring topology $\mathcal{T}_v$ on R by taking the family of balls $\{\{x \in R \mid v(x) < \varepsilon\} \colon \varepsilon \in v(R \setminus \{0\})\}$ as neighbourhoods of zero. There is a one-to-one correspondence between all non-trivial, non-discrete locally bounded ring topologies on a field R and prevaluations on R [13, Theorem 4.4.2]. This result indicates that the notion of prevaluation is a proper tool to describe locally bounded ring topologies on a field.

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X. Recall that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$ and $\mathcal{T}'$ is **finer** than $\mathcal{T}$ provided that $\mathcal{T} \subseteq \mathcal{T}'$. Every ring topology on a field has a coarser field topology [2, Theorem 1.7.8]. However, it is an open question whether every ring topology on a division ring has a coarser division ring topology. Let R be a division ring equipped with a Hausdorff ring topology $\mathcal{T}$ such that $(X, \mathcal{T})$ has a non-discrete set S. Then there exists a Hausdorff ring topology $\mathcal{T}'$ on R such that $\mathcal{T}' \subseteq \mathcal{T}$ and $\mathcal{T}'$ has a basis $\mathcal{B}$ of neighbourhoods of zero with $|\mathcal{B}| \leqslant |S|$ [2, Proposition 1.7.9]. In particular, if a Hausdorff ring topology $\mathcal{T}$ on a division ring contains an infinite countable non-discrete subset, then $\mathcal{T}$ has a coarser metrizable ring topology. It follows easily from this fact that (20) every ***countably compact*** subset of a topological division ring is metrizable. Let $\mathcal{P}$ be a class of topologies on a ring R. A ring topology $\mathcal{T}$ on R is called **minimal among topologies from** $\mathcal{P}$ provided that $\mathcal{T}' = \mathcal{T}$ whenever $\mathcal{T}'$ is a topology from the class $\mathcal{P}$ coarser than $\mathcal{T}$. A non-discrete locally retrobounded ring topology on a division ring R is minimal among all Hausdorff ring topologies on R (combine Theorems 13.3 and 13.8 in [16]). Since a division ring topologized by an absolute value is locally retrobounded [16, Theorem 13.6], a non-discrete Hausdorff ring topology on a division ring generated by an absolute value is minimal among all Hausdorff ring topologies on this ring [2, Theorem 1.7.11]. It is a well-known open problem to prove or disprove that every non-trivial topological field whose topology is minimal among all Hausdorff ring topologies is locally bounded.

A topological ring R is a **complete topological ring** if the additive topological group $(R, +)$ is ***complete*** in its ***left uniformity***. (Note that since the additive topological group $(R, +)$ is Abelian, its left uniformity, ***right uniformity*** and ***two-sided uniformity*** coincide.) For every topological ring R there exists a complete topological ring $\widehat{R}$ that contains R as its dense subring and which is unique up to an isomorphism; the ring $\widehat{R}$ is called the **completion** of R. In fact, $\widehat{R}$ coincides with the **topological group completion** (i.e., the completion with respect to its uniformity) of the additive topological group $(R, +)$, and the multiplication operation has a natural continuous extension from $(R, +)$ to $\widehat{R}$ making $\widehat{R}$ into a ring. If the topology of a topological ring R is defined by a norm or an absolute value, then the topology of $\widehat{R}$ can also be defined by a norm or an absolute value; moreover, the norm or the absolute value on R has a unique continuous extension over $\widehat{R}$ [2, Theorem 3.2.37].

In general, the completion of a topological field may have zero divisors. Even when the completion of a topological field is algebraically a division ring, it still need not be a topological field. It is therefore an important question as to when the completion of a topological ring is a field. An element a of a topological ring R is called a left (right) **topological divisor of zero** if there exists a subset $S \subseteq R$ such that 0 is not in the closure of S but belongs to the closure of $a \cdot S$ (of $S \cdot a$ respectively). For a Hausdorff topological ring R the following conditions are equivalent: (i) $\widehat{R}$ has 1 and every non-zero element of $\widehat{R}$ is invertible in $\widehat{R}$, and (ii) R does not contain either left or right topological divisors of 0 and every non-zero one-sided ideal of R is dense in R [2, Theorem 3.5.1]. Let R be a Hausdorff topological ring with 1 which has a topologically nilpotent neighbourhood of 0. If every non-zero one-sided ideal of R is dense in R, then every non-zero element of $\widehat{R}$ is invertible in $\widehat{R}$ [2, Theorem 3.5.2]. Let $(R, \mathcal{T})$ be a commutative Hausdorff topological ring such that $\mathcal{T}$ is minimal for the class of all Hausdorff ring topologies on R. Then the completion of $(R, \mathcal{T})$ is a topological field if and only if the ring $(R, \mathcal{T})$ is **topologically simple**, i.e., all non-zero ideals of $(R, \mathcal{T})$ are dense in $(R, \mathcal{T})$ [2, Theorem 3.5.3]. Since every topological field is topologically simple, from the above result we get the following important corollary: (21) If $(R, \mathcal{T})$ is a Hausdorff topological field such that the topology $\mathcal{T}$ is minimal among all Hausdorff ring (equivalently, field) topologies on R, then the completion of $(R, \mathcal{T})$ is a topological field [2, Corollary 3.5.4]. From (21) we obtain that the completion of a non-discrete absolute valued field is a (absolute valued) topological field. Using this fundamental fact we conclude that (22) the completion of the field of rational numbers $\mathbb{Q}$ equipped with the p-adic topology is a topological field $\mathbb{Q}_p$ called the **field of p-adic numbers**. The field of p-adic numbers is locally compact.

Every topological ring can be embedded as a ***closed*** subring into an ***arcwise connected*** topological ring [1]. Every discrete division ring (field) can be embedded as a closed subring (subfield) into an arcwise connected topological division ring (field) [7]. In particular, there exists a (arcwise) connected field of arbitrary characteristics. It is unknown if every topological field (or division ring) can be embedded into a connected topological field (division ring, respectively).

For topological rings there is an analogue of the notion of a ***free topological group***. Let $(X, \mathcal{T})$ be a Tychonoff space. A Hausdorff topological ring $(R, \mathcal{T}^*)$ is called a **free topological ring** generated by $(X, \mathcal{T})$ if the following conditions are satisfied: (i) $(X, \mathcal{T})$ is a ***subspace*** of $(R, \mathcal{T}^*)$, (ii) R does not contain any proper subrings containg the set X, (iii) for any topological ring $(R', \mathcal{T}')$ and an arbitrary continuous map $f:(X, \mathcal{T}) \to (R', \mathcal{T}')$ there exists a continuous ring homomorphism $f':(R, \mathcal{T}^*) \to (R', \mathcal{T}')$ extending f. For every Tychonoff space $(X, \mathcal{T})$ the free topological ring generated by $(X, \mathcal{T})$ exists and is unique up to an isomorphism; moreover, $(X, \mathcal{T})$ is a closed subspace in its free topological ring [2, Theorem 6.2.5]. A free topological ring does not have zero divisors. As a corollary, we have that every Tychonoff space is ***homeomorphic*** to a closed subspace of some topological ring without zero divisors. This is the best "universal embedding theorem" as one can get. Indeed, (20) implies that a non-***metrizable*** compact space cannot be embedded into a topological division ring [11]. It makes interesting a question of classifying Tychonoff spaces that are homeomorhic to a subspace of a topological division ring or a topological field.

A ***pseudocompact*** subspace of a topological field is metrizable [11]. If a Tychonoff space X has a coarser metric topology, then X is homeomorphic to a subspace of some topological field R (which in addition can be choosen to have an arbitrary characteristic) [9]. For a wide class of topological spaces there is a complete solution to the topological field embedding question: If a Tychonoff space X has at least one infinite non-discrete subspace, then X can be embedded as a subspace into a topological field if and only if X has a coarser metric topology [9]. However no criteria is known outside of this class: It is unknown when a Tychonoff space in which all countable subsets are closed (and discrete) is homeomorphic to a subspace of some topological field. If a Tychonoff space has a coarser compact metric topology, then it is homeomorphic to a closed subspace of some topological field (which in addition can be choosen to have an arbitrary characteristic) [9]. This result demonstrates that subspaces of topological fields can have many "pathological properties" found in general topological spaces. For example, one can get examples, under the ***Continuum Hypothesis*** CH, of topological fields which are strong S-spaces or strong L-spaces [10]. (A topological space X is called a **strong S-space** if all finite powers X^n of X are ***hereditarily separable*** but X is not ***Lindelöf***, and X is called a **strong L-space** if all finite powers X^n of X are ***hereditarily Lindelöf*** but X is not ***separable***; equivalently, X is a strong S-space (strong L-space) if and only if each finite power X^n of X is an ***S-space*** (an ***L-space***, respectively).) It is not known, however, if there exists a topological field that is normal but not ***countably paracompact***, i.e., if a topological field can be a ***Dowker space***. It is an open question if every topological field is ***Dieudonné complete*** [11]. For an uncountable cardinal τ let $L(\tau)$ be the so-called one-point Lindelöfication of a discrete space of size τ, i.e., the space of size τ with a single non-isolated point x whose filter of open neighbourhoods consists of all subsets of $L(\tau)$ that contain x and have at most countable complement in $L(\tau)$. If $\tau \geqslant \omega_2$, then $L(\tau)$ cannot be embedded into a topological field as a subspace [11]. On the other hand, $L(\omega_1)$ is homeomorphic to a subspace of some topological field [11]. Therefore, the statement "$L(2^\omega)$ can be embedded as a subspace in some topological field" is both consistent with and independent of ZFC [11].

Topological fields have some additional relationships between their ***cardinal invariant***s which need not hold in topological rings. We present a sample of results from [11]. Let X be a space. Recall that $iw(X)$ is the smallest infinite cardinal τ such that X admits a one-to-one continuous map onto a space of weight $\leqslant \tau$. We use $c(X)$, $d(X)$, $l(X)$ and $\psi(X)$ to denote the ***Souslin number***, the ***density***, the ***Lindelöf number*** and the ***pseudocharacter*** of X, respectively. If R is a topological field, then $iw(R) \leqslant c(R)$ and $iw(R) \leqslant l(R)$; in particular, if R is a non-trivial topological field, then $|R| \leqslant 2^{c(R)}$ and $|R| \leqslant 2^{l(R)}$. If X is a subspace of a topological field, then $\psi(X) \leqslant c(X)$, $iw(X) \leqslant \sup\{c(X^n): n \in \omega\} \leqslant d(X)$ and $iw(X) \leqslant l(X)^+$. It is not known if $c(R \times R) = c(R)$ and $c(R) = d(R)$ hold for a topological field R.

For every natural number $n \geqslant 1$ there exists a totally disconnected subfield of $\mathbb{R} \times \mathbb{C}^n$ of ***dimension*** n [15]. On every infinite countable field there exists a non-discrete field topology in which all ***convergent sequence***s are eventually constant [14].

Lists of open problems about topological rings and fields can be found in the end of both [2] and [18], as well as in [12].

A **lattice** is a triple $(X, \cup, \cap)$ consisting of a set X and two binary operations on X, **join** $\cup$ and **intersection** $\cap$, satisfying the following axioms: (23) $x \cup y = y \cup x$ and $x \cap y = y \cap x$ for all $x, y \in X$, (24) $x \cup (y \cup z) = (x \cup y) \cup z$ and $x \cap (y \cap z) = (x \cap y) \cap z$ for $x, y, z \in X$, (25) $x \cup (y \cap x) = (x \cup y) \cap x = x$ for all $x, y \in X$. Every lattice satisfies the following condition (26) $x \cup x = x \cap x = x$ for $x \in X$. A lattice $(X, \cup, \cap)$ is **distributive** if one of the following two equivalent conditions holds: (27) $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$, (28) $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.

A topology $\mathcal{T}$ on a lattice $(X, \cup, \cap)$ is **join-continuous** (**meet-continuous**) if the map $(x, y) \mapsto x \cup y$ (the map $(x, y) \mapsto x \cap y$, respectively) is a continuous map from the space $(X, \mathcal{T}) \times (X, \mathcal{T})$ to $(X, \mathcal{T})$. A **lattice topology** is a topology on a lattice that is both joint-continuous and meet-continuous. A **topological lattice** $(L, \mathcal{T})$ is a lattice L together with some lattice topology $\mathcal{T}$ on it.

A **partially ordered set**, or **poset**, is a set X together with a binary ralation $\leqslant$ on X (called **partial order**) satisfying the following conditions: (i) $x \leqslant x$ for all $x \in X$, (ii) if $x, y \in X$, $x \leqslant y$ and $y \leqslant x$, then $x = y$, (iii) if $x, y, z \in X$, $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$. Let A be a subset of a poset $(X, \leqslant)$. An element $x \in X$ is called an **upper bound** (a **lower bound**) of A provided that $a \leqslant x$ for all $a \in A$ ($x \leqslant a$ for all $a \in A$, respectively). A lower bound of A that belongs to A is always unique and is called the **minimal element** of A or the **least element** of A. Similarly, an upper bound of A

that belongs to A is unique as well and is called the **maximal element** of A or the **greatest element** of A. If the set of all lower bounds of A has a maximal element, then this element is called the **infimum** or the **greatest lower bound** of A and is denoted by $\inf A$. Similarly, if the set of all upper bounds of A has a minimal element, then this element is called the **supremum** or the **least upper bound** of A and is denoted by $\sup A$. A poset $(X, \leqslant)$ is a **linearly ordered set** provided that for every pair $x, y \in X$ either $x \leqslant y$ or $y \leqslant x$ holds.

Lattices can be viewed as a special kind of partially ordered sets. Indeed, if x and y are two elements of a lattice $(X, \cup, \cap)$, then conditions $x \cup y = y$ and $x \cap y = x$ are equivalent, and we can define $x \leqslant y$ if and only if (one of) the above conditions hold(s). With this definition $(X, \leqslant)$ becomes a poset such that $x \cap y = \inf\{x, y\}$ and $x \cup y = \sup\{x, y\}$ for every two elements $x, y \in X$. Conversely, if $(X, \leqslant)$ is a poset such that $\inf\{x, y\}$ and $\sup\{x, y\}$ exist for all $x, y \in X$, then $(X, \cup, \cap)$ becomes a lattice if one defines $x \cap y = \inf\{x, y\}$ and $x \cup y = \sup\{x, y\}$. In particular, linearly ordered sets can be viewed as lattices. From now on we will use this interpretation of a lattice as a partially ordered set.

Let $(L, \leqslant)$ be a poset (in particular, a lattice). For a subset $A \subseteq L$ we define $\uparrow A = \{x \in L\colon \exists a \in A\ a \leqslant x\}$ and $\downarrow A = \{x \in L\colon \exists a \in A\ x \leqslant a\}$. If $A = \downarrow A$, then A is called a **lower set**, and if $A = \uparrow A$, then A is called an **upper set**. For $x \in X$, we write $\uparrow x$ and $\downarrow x$ instead of $\uparrow\{x\} = \{y \in L\colon x \leqslant y\}$ and $\downarrow\{x\} = \{y \in L\colon y \leqslant x\}$. The topology $\mathcal{T}_l$ on L that has the family $\{L \setminus \uparrow x\colon x \in L\}$ as its ***subbase*** is called the **lower topology** on L, and the topology $\mathcal{T}_u$ on L that has the family $\{L \setminus \downarrow x\colon x \in L\}$ as its subbase is called the **upper topology**. The topology on L generated by the family $\mathcal{T}_l \cup \mathcal{T}_u$ as its subbase is called the **interval topology** on L.

A subset D of a poset $(L, \leqslant)$ is called **upward directed** provided that for all $d_0, d_1 \in D$ there exists $d \in D$ with $d_0 \leqslant d$ and $d_1 \leqslant d$. A subset U of $(L, \leqslant)$ is called **Scott open** provided that U is an upper set satisfying the following condition: If D is an upward directed subset of L, $\sup D$ exists and $\sup D \in U$, then $D \cap U \neq \emptyset$. The collection of all Scott open subsets of a poset $(L, \leqslant)$ forms a topology $\mathcal{T}_\sigma$ on L called the **Scott topology**. A complement of a Scott open set is called **Scott closed**. The topology on L generated by the family $\mathcal{T}_l \cup \mathcal{T}_\sigma$ as a subbase is called the **Lawson topology**. It can be easily seen that each set $L \setminus \downarrow x$ is Scott open, and so the upper topology is coarser than the Scott topology. This implies that the interval topology is always coarser than the Lawson topology. For linearly ordered sets, the upper topology and the Scott topology coincide, and thus the interval topology coincides with the Lawson topology.

Let $(L, \leqslant)$ be a poset (in particular, a lattice). We say that an element $x \in L$ is **way below** an element $y \in L$, and we write $x \ll y$, provided that for every upward directed set $D \subseteq L$ such that the supremum $\sup D$ exists and satisfies $y \leqslant \sup D$ there exists some $d \in D$ with $x \leqslant d$. An element $x \in L$ satisfying $x \ll x$ is called a **compact element**. A subset B of L is called an **order-theoretic basis** for $(L, \leqslant)$ provided that for every element $y \in L$ one has $y = \sup\{x \in B\colon x \text{ is way below } y\}$ and the latter set is upward directed. A poset having an order-theoretic basis is called **continuous**. In particular, a continuous poset that is a lattice is called a **continuous lattice**. A poset $(L, \leqslant)$ is **algebraic** if the set of its compact elements is an order-theoretic basis for $(L, \leqslant)$. If each upward directed subset of $(L, \leqslant)$ has a supremum, then $(L, \leqslant)$ is called a **directed complete partial order** or **DCPO**. A continuous DCPO is called a **continuous domain**, and an algebraic DCPO is called an **algebraic domain**. The notions defined in this paragraph together with topologies defined in the previous paragraph play a crucial role in applications of ***topology*** in computer science, in particular, in domain theory.

For additional information related to topological and continuous lattices, as well as topologies on them, the reader is refered to [6] and [4]. Extensive bibliography on this topic can be found in [5].

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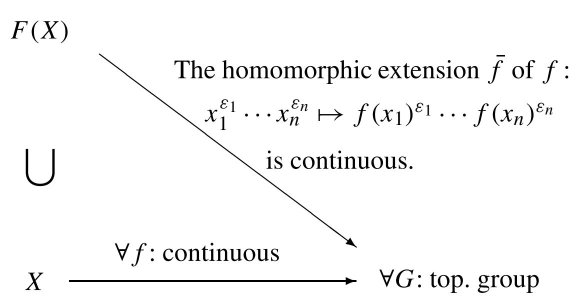
h-3 Free Topological Groups

Markov [9] defined the **free topological group** and the **free Abelian topological group** on an arbitrary ***Tychonoff*** space X. He proved the existence and the uniqueness of these groups. In this note, all ***topological space***s (we call them just spaces) are assumed to be Tychonoff.

1. Free topological groups and free Abelian topological groups

Let $F(X)$ and $A(X)$ be, respectively, the **free topological group** and the **free Abelian topological group** on a Tychonoff space X in the sense of Markov [9]. As an abstract group, $F(X)$ is free on X and it carries the finest ***group topology*** that induces the original topology of X; every ***continuous*** map from X to an arbitrary ***topological group*** G lifts in a unique fashion to a continuous homomorphism from $F(X)$. Similarly, as an abstract group, $A(X)$ is the free Abelian group on X, having the finest group topology that induces the original topology of X, so that every continuous map from X to an arbitrary Abelian ***topological group*** G extends to a unique continuous homomorphism from $A(X)$.



For each $n \in \mathbb{N}$, $F_n(X)$ stands for the subset of $F(X)$ formed by all words whose reduced length is less than or equal to n. For $n = 0$, we put $F_0(X) = \{e\}$, where e is the unit element of $F(X)$. Then, it is well known that X and each $F_n(X)$ are ***closed*** in $F(X)$, and $F(X) = \bigcup_{n=0}^{\infty} F_n(X)$. $A_n(X)$ is defined in a same fashion. It is also well known that X and each $A_n(X)$ are closed in $A(X)$, and $A(X) = \bigcup_{n=0}^{\infty} A_n(X)$. For each $n \in \mathbb{N}$ we denote the natural map from $(X \oplus X^{-1} \oplus \{e\})^n$ onto $F_n(X)$ by i_n, where the topological space X^{-1} means the set $\{x^{-1}: x \in X\}$ with the topology

$$\{U^{-1} = \{x^{-1}: x \in U\}: U \text{ is } \textit{\textbf{open}} \text{ in } X\}$$

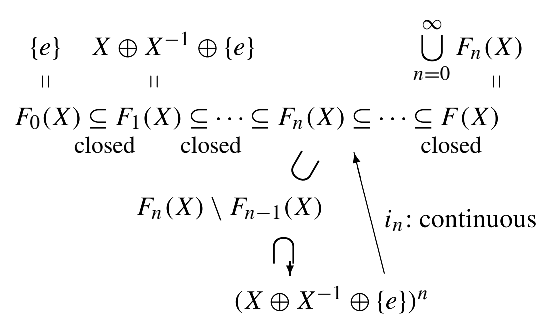
and $X \oplus X^{-1} \oplus \{e\}$ is the ***topological sum*** of X, X^{-1} and $\{e\}$. We also use the same symbol i_n in the Abelian case, that is, i_n means the natural map from $(X \oplus (-X) \oplus \{0\})^n$ onto $A_n(X)$, where 0 means the unit element of $A(X)$. Clearly the map i_n is continuous for each $n \in \mathbb{N}$. Moreover, Arhangel'skiĭ [1] and Joiner [5] showed that for each space X and each $n \in \mathbb{N}$ the map

$$i_n|_{i_n^{-1}(F_n(X)\setminus F_{n-1}(X))} : i_n^{-1}\big(F_n(X) \setminus F_{n-1}(X)\big) \to F_n(X) \setminus F_{n-1}(X)$$

is a ***homeomorphism***. Also, with the same argument, we can prove that the map

$$i_n|_{i_n^{-1}(A_n(X)\setminus A_{n-1}(X))} : i_n^{-1}\big(A_n(X) \setminus A_{n-1}(X)\big) \to A_n(X) \setminus A_{n-1}(X)$$

is an ***open*** and ***closed*** map, which is $n!$-to-one.



2. The topologies for $F(X)$ and $A(X)$

The topological structures of $F(X)$ and $A(X)$ are rather complicated even for very simple spaces X. For example, let C be the space consisting of a ***convergent sequence*** with its ***limit***. Then $F(C)$ is not ***first-countable***. Indeed, if a space X is not ***discrete***, then neither $F(X)$ nor $A(X)$ is first-countable [3]. The structures of $F(X)$ and $A(X)$ are very simple from the algebraic point of view; every subgroup of $F(X)$ ($A(X)$) is a free (Abelian) group, in particular for every subset Y of a set X, the subgroup of $F(X)$ ($A(X)$) generated by Y is free (Abelian) group on Y. For free topological groups, these are not true. Graev [3] showed that even a closed subgroup of the free topological group is not necessarily topologically isomorphic to some free topological group. Then, the following natural question arises; in which cases the free (Abelian) topological group on a space X can be ***embedded*** as a topological subgroup into the free (Abelian) topological group on a space Y? This problem has been considered for a long time. We denote by $G \leqslant H$ a topological group G embedding into a topological group H. Mack, Morris and Ordman [8] proved that for a ***compact*** space X, $F(X) \leqslant F(I)$ if and only if X is ***metrizable*** and ***finite-dimensional***, where I means the closed unit interval [0, 1]. Therefore, from the result, we have that $F(I^n) \leqslant F(I)$ and $F(S^n) \leqslant F(I)$, where I^n is the unit n-cube and S^n is the unit n-sphere (see [E, 2.3.9]). However, things were much less clear in Abelian case, until Leiderman, Morris and Pestov [7] gave the following complete description of the spaces X such that $F(X)$ ($A(X)$) embeds into $F(I)$ ($A(I)$) as a topological subgroup. They proved that for a space X the following are equivalent:

(1) $A(X) \leqslant A(I)$;
(2) $F(X) \leqslant F(I)$;
(3) X is ***homeomorphic*** to a closed subspace of $A(I)$;
(4) X is homeomorphic to a closed subspace of $F(I)$;
(5) X is homeomorphic to a closed subspace of $\mathbb{R}^\infty$;
(6) X is ***k_ω-space*** such that every compact subspace of X is metrizable and finite-dimensional;
(7) X is a ***submetrizable*** k_ω-space such that every compact subspace of X is finite-dimensional.

Therefore, we have that $A(I^n) \leqslant A(I)$, $A(S^n) \leqslant A(I)$, and furthermore $A((0, 1)) \leqslant A(I)$ (see also [6]) and $F((0, 1)) \leqslant F(I)$ (see also [11]), because the unit open interval (0, 1) can be embedded as a closed subspace of $A(I)$ and $F(I)$.

On the other hand, Hunt and Morris [4] showed that if we regard (0, 1) as a subspace of I, and hence of $F(I)$ (so, (0, 1) is not closed in $F(I)$), then the topological subgroup of $F(I)$ generated by (0, 1) is not topologically isomorphic to any free topological group. More generally, Sipacheva [19] proved quite recently that for a subspace Y of a space X, the topological subgroup of $F(X)$ generated by Y is topological isomorphic to $F(Y)$ if and only if every bounded continuous pseudometric on X can be extended to a continuous pseudometric on X ("only if" part was proved by Pestov [15]). In the Abelian case, Tkačenko [20] obtained the same result.

3. M-equivalence

The topological spaces X and Y are called ***M*-equivalent** if the free topological groups $F(X)$ and $F(Y)$ are topologically isomorphic. Markov [9] asked the following question: are M-equivalent spaces homeomorphic? Graev [3] gave a negative answer to the question, when he showed that the closed interval and the triod are M-equivalent, where the **triod** means the space which is homeomorphic to the subspace

$$\{(x, 0):\ -1 \leqslant x \leqslant 1\} \cup \{(0, y): 0 \leqslant y \leqslant 1\}$$

of the plane $\mathbb{R}^2$. This result raised the following question: which topological properties are preserved by M-equivalent? Graev [3] showed that compactness is preserved by M-equivalence. Afterwards, several mathematicians gave the properties which are preserved by M-equivalence and the properties which are not preserved by M-equivalence (see [2] and [HvM, §4 in Chapter 2]). Okunev [12] developed Graev's example to a general method of construction of examples of M-equivalent spaces. He called the ***retractions*** r_1 and r_2 of the space X **parallel** if $r_1 \circ r_2 = r_1$ and $r_2 \circ r_1 = r_2$. The images of the space X under parallel retractions are called **parallel retracts** of X. If F_1 and F_2 are parallel retracts of a space X, then the ***quotient spaces*** X/F_1 and X/F_2 obtained from X by identifying the sets F_1 and F_2 to a point are M-equivalent. This method yields a number of examples of M-equivalent spaces, and hence topological properties which are not preserved by M-equivalent, for example, ***tightness***, ***normality***, the property of being a ***k-space*** and ***local connectedness***. Tkachuk [24] proposed a general method for the construction of M-equivalent spaces before the publication of Okunev's paper. A particular case of Tkachuk's construction turned out to be a useful variant of Okunev's construction; if X is an infinite compact space of ***cardinality*** τ, then the ***Alexandroff duplicate*** of X is M-equivalent to the topological sum of X and the ***Alexandroff compactification*** of the discrete space of cardinality τ (see [E, 3.5.14]). Hence, it follows that a first-countable compact space may be M-equivalent to a compact space which is not first-countable.

4. Open sets in $F(X)$ and $A(X)$

The definitions of $F(X)$ and $A(X)$ say nothing about any constructive form of open sets. Consequently, it is difficult to investigate topological properties of $F(X)$ and $A(X)$. Graev [3] showed that the topologies of $F(X)$ and $A(X)$ on a compact space X have a simple description. Let $\mathcal{U}$ be a ***cover*** of a topological space X. Then X is said to be the **inductive limit** of $\mathcal{U}$ if a subset V of X is open whenever $V \cap U$ is open in $\mathcal{U}$ for each $U \in \mathcal{U}$. If $F(X)$ is the inductive limit of $\{F_n(X): n \in \mathbb{N}\}$, we say that $F(X)$ has the **inductive limit topology**. This concept is defined for $A(X)$ in the same fashion. Graev [3] showed that both $F(X)$ and $A(X)$ on every compact space X have the inductive limit topology. If a space X is compact, then the space $(X \oplus X^{-1} \oplus \{e\})^n$ is also compact for each $n \in \mathbb{N}$. So, every natural map i_n is closed [E, 3.1.12], and hence it is a ***quotient*** map. Therefore, if a space X is compact, then every open subset of $F(X)$ can be described as follows: a subset U of $F(X)$ is open if and only if $i_n^{-1}(U \cap F_n(X))$ is open in $(X \oplus X^{-1} \oplus \{e\})^n$ for each $n \in \mathbb{N}$. Later Mack, Morris and Ordman [8] generalized the Graev's result to an arbitrary k_ω-space. Apart from that, it is easy to see that $F(X)$ is a P-space ($\equiv$ every G_δ-set is open) whenever X is such a space (as was observed in [13, Lemma 5.6]), and hence has the inductive limit topology [22, Theorem 8(b)]. The same is valid for $A(X)$. Recently, Tkačenko [23], Sipacheva [18] and Pestov and Yamada [17] came up with new classes of spaces X such that $F(X)$ or $A(X)$ have the inductive limit topology. Tkačenko [23] proved that for each ***pseudocompact*** space X, $F(X)$ has the inductive limit topology if and only if X^n is normal and ***countably compact*** for every $n \in \mathbb{N}$. Sipacheva [18] has shown that for every countable space X with one nonisolated point x, $F(X)$ has the inductive limit topology if and only if for any collection $\{U_n\colon n \in \mathbb{N}\}$ of open neighbourhoods of x there exists a neighbourhood V of x such that $V \cap (U_n \setminus U_{n+1})$ is finite for all n. The same is valid for $A(X)$. Pestov and Yamada [17] proved that for a metrizable space X, $A(X)$ has the inductive limit topology if and only if X is ***locally compact*** and the set of all nonisolated points of X is ***separable***. In the non-Abelian case, for a metrizable space X, $F(X)$ has the inductive limit topology if and only if X is either locally compact, separable or discrete. As mentioned above, the natural map i_n is closed, and hence a quotient map if X is compact. Pestov [13] showed that i_n is not a quotient map even for $n = 2$ in general, that is, he proved that the map i_2 (for both $F_2(X)$ and $A_n(X)$) is a quotient map if and only if every neighbourhood of the diagonal $\Delta_X = \{(x, x)\colon x \in X\}$ belongs to the ***universal uniformity*** (see [E, 8.1.C]). Later, Yamada [26] improved the result as follows: the map i_n is closed if and only if every neighbourhood of the diagonal Δ_X is an element of the universal uniformity. This holds for both $F_2(X)$ and $A_2(X)$. Furthermore, for $n \geqslant 3$, he proved in the same paper, that for a metrizable space X the following are equivalent:

(1) i_n is a closed map for each $n \in \mathbb{N}$;
(2) i_3 is a closed map;
(3) X is compact or discrete.

In the Abelian case, he proved in [25] that for a metrizable space X the map i_3 is a quotient map if and only if X is locally compact or the set of all nonisolated points in X is compact, and also the following are equivalent:

(1) i_n is a quotient map for each $n \in \mathbb{N}$;
(2) i_4 is a quotient map;
(3) either X is locally compact and the set X' of all nonisolated points of X is separable, or X' is compact.

On the other hand, several mathematicians tried to construct neighbourhood bases of the unit elements of $F(X)$ and $A(X)$. For a ***metric*** space X, Graev [3] gave a procedure for extending a metric on X to a metric on $F(X)$ which is compatible with the group structure of $F(X)$ and whose topology is weaker than the topology for $F(X)$. Using the extended metrics on $F(X)$ and $A(X)$ on a metric space X, he simplified the Markov's proof of existence of the free topological group on a Tychonoff space. Furthermore, he proved that if compact metric spaces X and Y are M-equivalent, then $\dim X = \dim Y$, where $\dim X$ means the ***covering dimension*** of X. This result is generalized by Pestov [14] for Tychonoff spaces. Applying the Graev's method, every continuous pseudometric on a space X can be extended to a continuous pseudometric on $F(X)$ and $A(X)$. Tkačenko [20] constructed a neighbourhood base of the unit element of $A(X)$ applying the extended continuous pseudometrics. Then he proved in the same paper that $A(X)$ is **Weil complete** (i.e., it is complete relative to right and left uniformity of the group, see [E, Chapter 8]) if and only if X is Dieudonné complete (see [E, 8.5.13]). In the non-Abelian case, the same result was obtain by Pestov [15] and Sipacheva [19]. On the other hand, Tkačenko [21] introduced an intrinsic description of a neighbourhood base of the unit element of $F(X)$ using the universal uniformity on $(X \oplus X^{-1} \oplus \{e\})^n$ for each $n \in \mathbb{N}$. Then, he proved that for a pseudocompact space X the ***Souslin number*** of $F(X)$ is countable. Furthermore, Pestov [16] constructed a simpler neighbourhood base of the unit element of $A(X)$ using the universal uniformity on X.

Recently, there have been some interesting results obtained about the topological properties of $F(X)$ and $A(X)$ by applying the extended continuous pseudometrics on $F(X)$ and $A(X)$ and the above description of neighbourhood bases of the unit elements of $F(X)$ and $A(X)$. They are described in [2, 10] and [HvM, §4 in Chapter 2].

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h-4 Homogeneous Spaces

In general, geometry provides concrete and intuitive examples of groups, and group theory is used to clarify the geometry. In topology, for an arbitrary ***topological space*** X, the **autohomeomorphism group** $\mathrm{Aut}(X)$ consisting of all ***homeomorphism*** of X onto itself corresponds to such a group. It is well-known that every group is isomorphic to the autohomeomorphism group of some ***compact Hausdorff*** space. A topological space X is called **topologically homogeneous** (a **homogeneous space**) provided that for arbitrary points $x, y \in X$ there exists a homeomorphism $f: X \to X$ such that $f(x) = y$. From the setting of transformation group, it is said that homogeneous spaces are those spaces X on which $\mathrm{Aut}(X)$ **acts transitively**, i.e., for every x and y there is $\varphi \in \mathrm{Aut}(X)$ with $\varphi(x) = y$. There may be some extensive theory on the relation between the topological space X and its autohomeomorphism group $\mathrm{Aut}(X)$. Here we restrict our attention only to basic definitions and results in general topology. Hence, we will not discuss the group $\mathrm{Aut}(X)$ any more. It must be noticed as well that, in the context of ***Lie group***, the meaning of a homogeneous space is different from the definition above. That is, for a ***topological group*** G and its ***closed*** subgroup H, the **factor space** G/H is called a homogeneous space. Of course, topological groups and more generally factor spaces of topological groups are typical examples of homogeneous spaces in our sense of topological homogeneity. A much more complete survey of topological homogeneous spaces may be found in A.V. Arkhangel'skiĭ [1].

In a topologically homogeneous space X, a ***topological property*** which holds at one point of X also holds at all other points. For example, a topological space X with an ***isolated point*** is homogeneous if and only if X is a ***discrete space***. Hence, it follows that every topological space is the ***continuous*** image of some homogeneous space. The real line $\mathbb{R}$, which is one of the most important topological spaces in mathematics, is homogeneous. Since the ***product space*** of an arbitrary family of homogeneous spaces is homogeneous, the Euclidean n-space $\mathbb{R}^n$ is homogeneous, and more generally the ***Tychonoff product*** $\mathbb{R}^\tau$ is homogeneous for any τ. Consequently, it follows that every ***Tychonoff space*** is embedded in some homogeneous Tychonoff space. As an example of some non-trivial and fundamental results on homogeneous spaces, it may be appropriate to cite the following two results: (1) Though the unit interval $I = [0, 1]$ is not homogeneous, O.H. Keller [7] showed in 1931 that the countably infinite power I^ω of I is homogeneous. This space I^ω is called the ***Hilbert cube***, and important in the theory of ***infinite-dimensional*** topology; (2) Let $\beta\mathbb{N}$ be the ***Stone–Čech compactification*** of the discrete space $\mathbb{N}$ of natural numbers. The remainder $\beta\mathbb{N} - \mathbb{N}$ is denoted by $\mathbb{N}^*$. It seems that $\mathbb{N}^*$ is homogeneous since there is no topological difference among all infinite subsets of $\mathbb{N}$. In 1956, W. Rudin [11] showed that $\mathbb{N}^*$ is not homogeneous in the presence of the ***Continuum Hypothesis***. This result was generalized by Z. Frolík [6]. He showed without any additional set-theoretic assumption that $\beta X - X$ is not homogeneous if the space X is not ***pseudocompact***. Several interesting problems concerning homogeneous spaces still remain open. Especially, the following problem of E.K. van Douwen [5] (see [1], [vMR, Chapter 16]) which has been stimulating the research of homogeneous spaces: Is there a compact homogeneous space with Souslin number greater than $2^{\aleph_0}$? The **Souslin number** of a topological space is the supremum of all sizes of families of disjoint ***open*** subsets in the space. Considering the ***π-weight*** $\pi(X)$ of X in stead of the Souslin number of X, E.K. van Douwen showed that the inequality

$$|X| \leqslant 2^{\pi(X)}$$

is satisfied for any homogeneous Hausdorff space X. Further, it is known [3] that for every cardinal number κ there are homogeneous ***countably compact*** spaces and homogeneous ***σ-compact*** spaces, whose Souslin number exceeds κ.

V.V. Uspenskii [12] pointed out that for every topological space X there exists a topological space Y such that $X \times Y$ is homogeneous. The space Y can be constructed as a ***subspace*** of the product X^τ for some τ. Hence, if a topological property $\mathcal{P}$ is ***productive*** and ***hereditary***, then for a topological space X with the property $\mathcal{P}$ we can take a topological space Y such that $X \times Y$ is a homogeneous space with the same property $\mathcal{P}$. However, we cannot apply this argument to the following classes: compact Hausdorff spaces, ***normal*** spaces, ***paracompact*** spaces and so on. In fact, D.B. Motorov proved that there exists a compact ***metrizable*** space X such that $X \times Y$ is not homogeneous for any compact Hausdorff space Y. This compact metrizable space X has the stronger property that X is not the ***retract*** of any homogeneous compact Hausdorff space. However, the following problem of A.V. Arkhangel'skiĭ has not been solved yet: Is every compact Hausdorff space the continuous image of a homogeneous compact Hausdorff space? Obviously, if this problem were solved positively, then we would obtain a positive answer to E.K. van Douwen's problem. In an attempt to investigate whether or not a given compact Hausdorff space is represented as the retract of some homogeneous compact Hausdorff space, A.V. Arkhangel'skiĭ introduced and discussed the following concept: A **cellularity** of a topological space X is defined to be any map F of X into the set of all closed subsets of X such that

(1) $x \in F(x)$,
(2) if $y \in F(x)$ then $F(y) \subset F(x)$, and

(3) if $f : X \to X$ is a homeomorphism and $f(x) = y$, then $f(F(x)) = F(y)$.

The sets $F(x)$ are called the terms of the cellularity F. If for each $x, y \in X$ either $F(x) = F(y)$ or $F(x) \cap F(y) = \emptyset$, then the cellularity is called **disjoint**. There is a characterization of homogeneity: A Tychonoff space X is homogeneous if and only if every cellularity on X having a compact term is disjoint. One more general problem was raised by A.V. Arkhangel'skiĭ: Let $\mathcal{F}$ be a class of maps and $\mathcal{P}$ a topological property. Is it true that for any topological space X with the property $\mathcal{P}$ there exists a homogeneous topological space H with the property $\mathcal{P}$ such that $X = f(H)$ for some $f \in \mathcal{F}$. V.K. Bel'nov showed that for any Tychonoff space X there is the homogeneous Tychonoff space $H(X)$ which is called the **free homogeneous space** of X. For example, let us take the class of closed ***retractions*** as $\mathcal{F}$. Using free homogeneous spaces, N.G. Okromeshko gave affirmative answers for this problem of A.V. Arkhangel'skiĭ concerning each of the following topological properties; normality, ***hereditary normality***, paracompactness, ***hereditary paracompactness*** and ***Lindelöfness***. More precise statement of these are contained in [1].

The basic idea of showing that certain topological spaces are not homogeneous is found in the proof of the nonhomogeneity of $\mathbb{N}^*$ (see [13]): We can consider every point $p \in \mathbb{N}^*$ as a ***free ultrafilter*** of subsets of $\mathbb{N}$. Let us call free ultrafilters p and q **equivalent** if there exists an autohomeomorphism $\beta(f) : \beta\mathbb{N} \to \beta\mathbb{N}$ such that $\beta(f)(p) = q$. The equivalence classes of free ultrafilters are called **types** of ultrafilters. Since any countably infinite discrete subspace of $\mathbb{N}^*$ is ***C^*-embedded*** in $\mathbb{N}^*$, if X is such a subspace, then for any point $p \in \operatorname{cl} X - X$ we can define the **type** $\tau(p, X)$ of p relative to X. Further, define

$$\tau[p, \mathbb{N}^*] = \{\tau(p, X) : X \subset \mathbb{N}^*,\ X \approx \mathbb{N}\},$$

that is the set of relative types of p which occur relative to countable discrete subspaces of $\mathbb{N}^*$. Then the nonhomogeneity of $\mathbb{N}^*$ is obtained by showing that there must exist points r and s in $\mathbb{N}^*$ such that

$$\tau[r, \mathbb{N}^*] \neq \tau[s, \mathbb{N}^*].$$

Expanding this idea, E.K. van Douwen [5] showed that if the space X admits a continuous map f onto a Hausdorff space Y with $|Y| > 2^{\pi(Y)}$, then no power of X is homogeneous in each of the following cases:

(a) f is open or a retract and $d(X) \leqslant \pi(Y)$;
(b) f is ***perfect***, X is ***regular*** and $d(X) \leqslant \pi(Y)$; or
(c) X is compact Hausdorff and $w(X) \leqslant 2^{\pi(Y)}$;

where $d(X)$ is the ***density*** of X and $w(X)$ is the ***weight*** of X. In particular, no power of $\mathbb{N}^*$, $\beta\mathbb{Q} - \mathbb{Q}$, or $\beta\mathbb{R} - \mathbb{R}$ is homogeneous, where $\mathbb{R}$ is the real line with the usual topology and $\mathbb{Q}$ is the subspace of rational numbers.

Various other results on homogeneous spaces are known. In particular, J. van Mill has given some interesting examples and theorems: For example, he showed that there is a rigid space X such that $X \times X$ is homogeneous [8]. Here a topological space X is said to be **rigid** if there is no autohomeomorphism of X beyond the identity. A topological space X is called **uniquely homogeneous** provided that for all $x, y \in X$ there is a unique homeomorphism of X taking x onto y. For ***separable metric space*** X, if X is uniquely homogeneous, then a natural group structure is defined on X for which all left translations are homeomorphisms. J. van Mill showed that there is a unique homogeneous space X such that X does not admit the structure of a topological group [9]. The following result of A. Dow and E. Pearl [4] is also very interesting: For every ***zero-dimensional first-countable*** space X the ω-power X^ω of X is homogeneous. The possibility of this result was first pointed out by D.B. Motorov [10] who showed such a result for a zero-dimensional first-countable topological space with a ***dense*** set of isolated points.

Different kinds of homogeneities of topological spaces have been studied by many researchers. A ***Boolean algebra*** B is a **homogeneous Boolean algebra** if for any nonzero element a of B the relative algebra $B \mid a$ $(= \{x \in B : x \leqslant a\})$ is isomorphic to B. For a topological space X, the set $Clop(X)$ of all ***clopen*** subsets of X is an Boolean algebra. Hence, it is natural to consider the topological space X such that the corresponding Boolean algebra $Clop(X)$ is homogeneous. In particular, a zero-dimensional ***T_1-space*** X is called **h-homogeneous** if every non-empty clopen subsets of X is homeomorphic to the entire space X. It is known that every first-countable h-homogeneous space is homogeneous. Further, some important zero-dimensional topological spaces are h-homogeneous. For example, the ***Cantor discontinuum*** $\mathbb{C}$, the space $\mathbb{Q}$ of rational numbers, the space $\mathbb{P}$ of irrational numbers, the ***Sorgenfrey line***, $\beta\mathbb{N}$ and its remainder $\mathbb{N}^*$ are all h-homogeneous. For a first-countable or non-pseudocompact zero-dimensional T_1-space X, if X has a ***π-base*** consisting of clopen subsets which are ***homeomorphic*** to the entire space X, then X is h-homogeneous. A topological space X is called **n-homogeneous** if for any n points $x_1, x_2, \ldots, x_n$ of X and any n points $y_1, y_2, \ldots, y_n$ of X there is a homeomorphism of X onto itself that carries the set $\{x_1, x_2, \ldots, x_n\}$ onto $\{y_1, y_2, \ldots, y_n\}$. If there is such a homeomorphism which carries x_i to y_i for each $i = 1, 2, \ldots, n$, then X is said to be **strongly n-homogeneous**. Hence a homogeneous space is the same one as a (strongly) 1-homogeneous space. A topological space X is called **bihomogeneous** if for any pair of points p and q in X there is a homeomorphism $h : X \to Y$ such that $h(p) = q$ and $h(q) = p$. These types of homogeneities has been studied for a long time, especially in the theory of ***continuum***. For example, it is known that there is a 1-dimensional homogeneous subspace in the plane which is not bihomogeneous. This is one of results obtained by K. Kuratowski. A review article on homogeneous continua is found in [HvM, Chapter 15].

A separable topological space X is called **countable dense homogeneous** if for any pair A, B of countable dense subsets there exists an autohomeomorphism f of X such that $f(A) = B$. It is known that the Euclidean n-space $\mathbb{R}^n$ has this property. R.B. Bennett [2] proved that ***connected***, first-countable, countable dense homogeneous spaces are homogeneous. Assume that a topological space X has a ***σ-discrete*** dense subset. Then the concept of densely homogeneity is defined as follows: For any σ-discrete dense subsets A, B of a topological space X if there is a homeomorphism f from X onto itself such that $f(A) = B$, then X is said to be **densely homogeneous**. For example, the following results are known:

(1) Densely homogeneous connected T_1-spaces are homogeneous.
(2) There exists a countable dense homogeneous Hausdorff space which is not densely homogeneous.

There is a survey article of open problems concerning countable dense homogeneous spaces and densely homogeneous spaces [vMR, Chapter 15].

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h-5 Transformation Groups and Semigroups

Let G be a ***topological group*** and X be a ***topological space***. A **left action** of G on X is a ***continuous*** map $\varphi: G \times X \to X$ such that:

(1) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ for all $g, h \in G, x \in X$, and
(2) $\varphi(e, x) = x$ for all $x \in X$, where e denotes the identity element of G.

A **left G-space** is a pair (X, φ) consisting of a space X together with a left action φ of G on X. We usually denote the left G-space (X, φ) just by its underlying topological space X, and it is convenient to denote $\varphi(g, x)$ by gx. Then the above two rules are described by a familiar form $g(hx) = (gh)x$ and $ex = x$. A **right action** of G on X is a map $X \times G \to X$; $(x, g) \mapsto xg$ satisfying $(xh)g = x(hg)$ and $xe = x$. If $(x, g) \mapsto xg$ is a right action, then $(g, x) \mapsto xg^{-1}$ is a left action. Usually, a **G-space** means a left G-space.

Let X and Y be G-spaces. A map $f: X \to Y$ is a **G-map** or **equivariant** if $f(gx) = gf(x)$ for all $g \in G, x \in X$. For a subgroup H of G, a ***subspace*** A of X is called **H-invariant** if $g \in H$ and $x \in A$ implies $gx \in A$. A ***homeomorphism*** (respectively an ***embedding***) $X \to Y$ is a **G-homeomorphism** (respectively **G-embedding**) if it is a G-map. For a point $x \in X$, $G_x = \{g \in G\colon gx = x\}$ is called the **isotropy subgroup** of G at x, and $G(x) = \{gx\colon g \in G\}$ is called the **orbit** of x. We say that X is a **homogeneous space** if there exists a point x in X with $X = G(x)$, and that G acts **trivially** on X if $gx = x$ for all $g \in G$ and $x \in X$.

For a G-space X, we consider a relation $\sim$ on X defined by $x \sim y$ iff there exists $g \in G$ with $y = gx$. Then $\sim$ is an ***equivalence relation*** and the ***quotient space*** X/G with the ***quotient topology*** is called the **orbit space**.

Let G be a topological group and $f: G \to \mathbb{R}$ a real-valued function. For $h \in G$, define functions $R_h f: G \to \mathbb{R}$ and $L_h f: G \to \mathbb{R}$ by $(R_h f)(g) = f(gh)$ and $(L_h f)(g) = f(h^{-1}g)$, respectively. The existence of the **Haar integral** is known: Let G be a ***compact*** topological group. Then there exists a unique real-valued function A, called the Haar integral, defined for continuous real-valued functions on G, such that:

(a) $A(f_1 + f_2) = A(f_1) + A(f_2)$,
(b) $A(cf) = cA(f)$, where $c \in \mathbb{R}$,
(c) if $f(g) \geqslant 0$ for all $g \in G$, then $A(f) \geqslant 0$,
(d) $A(1) = 1$, and
(e) $A(R_h f) = A(f) = A(L_h f)$ for all $h \in G$.

A simple elementary proof of the existence of the Haar integral was given by J.V. Neumann and its proof can be found in [14].

Let V be a finite-dimensional real vector space. The set of invertible linear transformations from V to V forms a group $GL(V)$ which is identified with the general linear group $GL_n(\mathbb{R})$, where $n = \dim V$. A **representation** of G means a finite-dimensional real vector space V together with some inner product on V and a linear action of G on V preserving the inner product, i.e., a continuous group homomorphism $G \to GL(V)$ preserving the inner product.

Let X be a topological space and $\mathbb{R}^n$ the n-dimensional ***Euclidean space***. A pair (U, φ) consisting of an ***open set*** U of X and a homeomorphism φ from U onto an open subset of $\mathbb{R}^n$ is called a **chart** (or a **coordinate neighbourhood**) of X. When we write $\varphi(x)$ as $\varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))$ for $x \in U$, φ_i is called **ith coordinate function**. If a point x of X lies in U, then we say that (U, φ) is a **chart** at x.

A collection $\{(U_\alpha, \varphi_\alpha)\}$ of charts is an **atlas** on X (or a **system of coordinate neighbourhoods** on X) if $\{U_\alpha\}$ forms an ***open cover*** of X. The atlas is of **class C^∞** if for each pair of indices α, β with $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a C^∞ map. We say that two C^∞ atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ are **equivalent** if for each pair of indices α, β with $U_\alpha \cap V_\beta \neq \emptyset$, the map

$$\psi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap V_\beta) \to \psi_\beta(U_\alpha \cap V_\beta)$$

is a C^∞ map. The ***equivalence class*** of C^∞ atlases on X is called the C^∞ **structure** on X and the C^∞ structure represented by a C^∞ atlas $\{(U_\alpha, \varphi_\alpha)\}$ is denoted by $[(U_\alpha, \varphi_\alpha)]$. A pair $(X, [(U_\alpha, \varphi_\alpha)])$ consisting of a ***Hausdorff space*** X satisfying the ***second axiom of countability*** and a C^∞ structure $[(U_\alpha, \varphi_\alpha)]$ on X is called an **n-dimensional C^∞ manifold** (or simply a **C^∞ manifold**). To simplify notations, we simply write X to mean a C^∞ manifold.

Let $1 \leqslant r < \infty$. A continuous map $f: X \to Y$ between C^∞ manifolds $(X, [(U_\alpha, \varphi_\alpha)])$ and $(Y, [(V_\beta, \psi_\beta)])$ is said to be a **C^∞ map** (respectively a **C^r map**) if for each pair of indices α, β with $U_\alpha \cap f^{-1}(V_\beta) \neq \emptyset$, the map

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1}: \varphi_\alpha\bigl(U_\alpha \cap f^{-1}(V_\beta)\bigr) \to \psi_\beta(V_\beta)$$

is of class C^∞ (respectively C^r). A C^∞ map $f: X \to Y$ is called a C^∞ **diffeomorphism** if there exists a C^∞ map $h: Y \to X$ such that $f \circ h = \mathrm{id}_Y$ and $h \circ f = \mathrm{id}_X$. A subset K of an n-dimensional C^∞ manifold X is called an m-dimensional C^∞ submanifold of X if for each $x \in K$, there exists a chart (U, φ) of X at x such that

$$K \cap U = \bigl\{x \in U\colon \varphi_{m+1}(x) = \cdots = \varphi_n(x) = 0\bigr\},$$

where $\varphi_i(x)$ denotes the ith coordinate function of $\varphi(x)$.

Clearly $(\mathbb{R}^n, [(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})])$ is a C^∞ manifold. Some simple examples of C^∞ manifolds are the n-dimensional standard spheres

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}\colon x_1^2 + \dots + x_{n+1}^2 = 1\}$$

and the n-dimensional real projective spaces $P_n(\mathbb{R})$. Let $U_1 = S^n \setminus \{(0, \dots, 0, 1)\}$ and $U_2 = S^n \setminus \{(0, \dots, 0, -1)\}$. Then U_1 and U_2 are open subsets of S^n and $S^n = U_1 \cup U_2$. Clearly S^n is a Hausdorff space satisfying the second countability axiom. The map $\varphi_1 : U_1 \to \mathbb{R}^n$ defined by

$$\varphi_1(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\right)$$

is a homeomorphism and the inverse φ_1^{-1} maps the point $(y_1, \dots, y_n) \in \mathbb{R}^n$ to the point

$$\left(\frac{2y_1}{1 + \sum_{k=1}^n y_k^2}, \dots, \frac{2y_n}{1 + \sum_{k=1}^n y_k^2}, \frac{\sum_{k=1}^n y_k^2 - 1}{1 + \sum_{k=1}^n y_k^2}\right).$$

Similarly, the map $\varphi_2 : U_2 \to \mathbb{R}^n$ defined by

$$\varphi_2(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}}\right)$$

is a homeomorphism and the inverse φ_2^{-1} maps the point $(y_1, \dots, y_n) \in \mathbb{R}^n$ to the point

$$\left(\frac{2y_1}{1 + \sum_{k=1}^n y_k^2}, \dots, \frac{2y_n}{1 + \sum_{k=1}^n y_k^2}, \frac{1 - \sum_{k=1}^n y_k^2}{1 + \sum_{k=1}^n y_k^2}\right).$$

Thus S^n is a C^∞ manifold because

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2),$$
$$(x_1, \dots, x_n) \mapsto \left(\frac{x_1}{\sum_{k=1}^n x_k^2}, \dots, \frac{x_n}{\sum_{k=1}^n x_k^2}\right)$$

is a C^∞ map.

The n-dimensional real projective space $P_n(\mathbb{R})$ is the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation: $[x] = [y]$ iff there exists a non-zero real number t such that $y = tx$. Then $P_n(\mathbb{R})$ is a Hausdorff space satisfying the second countability axiom. Let

$$U_i = \{[x_1, \dots, x_{n+1}] \in P_n(\mathbb{R})\colon x_i \neq 0\},$$

for each $i = 1, \dots, n+1$. Then $\{U_i\colon 1 \leqslant i \leqslant n+1\}$ is an open cover of $P_n(\mathbb{R})$. Define a map $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i([x_1, \dots, x_{n+1}]) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right)$$

for each $i = 1, \dots, n+1$. Then φ_i is well defined and it is a homeomorphism with the inverse $\varphi_i^{-1}(y_1, \dots, y_n) = [y_1, \dots, y_{i-1}, 1, y_{i+1}, \dots, y_n]$. If $i < j$, then the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

which carries a point $(x_1, \dots, x_n)$ to the point

$$\left(\frac{x_1}{x_{j-1}}, \dots, \frac{x_{i-1}}{x_{j-1}}, \frac{1}{x_{j-1}}, \frac{x_i}{x_{j-1}}, \dots, \frac{\hat{x_{j-1}}}{x_{j-1}}, \dots, \frac{x_n}{x_{j-1}}\right),$$

is a C^∞ map, where ˆ means elimination of x_{j-1}/x_{j-1}. Similarly, $\varphi_j \circ \varphi_i^{-1}$ is a C^∞ map if $i \geqslant j$. Hence $P_n(\mathbb{R})$ is a C^∞ manifold.

Let $(X, [(U_\alpha, \varphi_\alpha)_{\alpha \in \Lambda}])$ be an n-dimensional C^∞ manifold. A **tangent vector** to X is an equivalence class $[x, \alpha, t]$ of triples $(x, \alpha, t) \in X \times \Lambda \times \mathbb{R}^n$ under the equivalence relation: $[x, \alpha, t] = [y, \beta, s]$ if and only if $x = y$ and $d(\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(x))t = s$. Here $d(\varphi_\beta \circ \varphi_\alpha^{-1})$ means the n-dimensional square matrix consisting of all first partial derivatives of components of $\varphi_\beta \circ \varphi_\alpha^{-1}$ (see, e.g., [11]). The set TX of all tangent vectors to X is called the **tangent space** of X, and the map $p_X : TX \to X$; $[x, \alpha, t] \mapsto x$ is well defined. For any subset $A \subset X$, we put $T_A X = p_X^{-1}(A)$. If $U \subset X$ is open, then $(U, [(U \cap U_\alpha, \varphi_\alpha | U \cap U_\alpha)_{\alpha \in \Lambda}])$ is also a C^∞ manifold, and we identify $T_U X$ with TU.

For any chart $(U_\alpha, \varphi_\alpha)$, there exists a well-defined bijective map

$$T\varphi_\alpha : TU_\alpha \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n,$$
$$[x, \alpha, t] \mapsto (\varphi_\alpha(x), t).$$

For any pair of indices α, β with $U_\alpha \cap U_\beta \neq \emptyset$, the map $(T\varphi_\beta) \circ (T\varphi_\alpha)^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n$ is the homeomorphism defined by

$$(y, t) \mapsto (\varphi_\beta \circ \varphi_\alpha^{-1}(y), d(\varphi_\beta \circ \varphi_\alpha^{-1})(y)t).$$

It follows that TX has a unique topology making each $T\varphi_\alpha$ a homeomorphism. Moreover the set of charts $\{(TU_\alpha, T\varphi_\alpha)_{\alpha \in \Lambda}\}$ is a C^∞ atlas on TX because each $(T\varphi_\beta) \circ (T\varphi_\alpha^{-1})$ is a C^∞ map. In this way, TX is a C^∞ manifold, and the charts $\{(TU_\alpha, T\varphi_\alpha)_{\alpha \in \Lambda}\}$ are called **natural charts** on TX. The map $p_X : TX \to X$ is a C^∞ map, and for each $x \in X$, $TX_x = p_X^{-1}(x)$ is called the **tangent space** of X at x.

A trivial example of tangent spaces is $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$.

Let $f : X \to Y$ be a C^∞ map. A C^∞ map $df : TX \to TY$ is defined as follows. Let $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, $\psi_\beta : V_\beta \to \mathbb{R}^m$ be charts for X, Y with $f(U_\alpha) \subset V_\beta$. The chain rule shows that the C^∞ map $(df)_{\alpha\beta} : TU_\alpha \to TV_\beta$ defined by

$$[x, \alpha, t] \to [f(x), \beta, d(\psi_\beta \circ f \circ \varphi_\alpha^{-1})(\varphi_\alpha(x))t]$$

is independent of α, β. Thus there exists a C^∞ map $df : TX \to TY$ which coincides with $(df)_{\alpha\beta}$ on TU_α and the restriction df_x of df on TX_x is a linear map $df_x : TX_x \to TY_{f(x)}$.

We say that a C^∞ map $f : X \to Y$ is a C^∞ **embedding** if for any $x \in X$, $df_x : TX_x \to TY_{f(x)}$ is injective and f is a homeomorphism onto $f(X)$ with the ***relative topology***. For example, the inclusion $S^n \to \mathbb{R}^{n+1}$ is a C^∞ embedding and the C^∞ map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (\cos x, \sin x)$ is not a C^∞ embedding because f is not a homeomorphism from $\mathbb{R}$ onto $f(\mathbb{R})$.

Let X, Y be C^∞ manifolds, $C^r(X, Y)$ denote the set of C^r maps from X to Y and $0 \leqslant r < \infty$, where a C^0 map means a continuous map. The C^r **topology** (the **weak** C^r **topology** or the **compact-open** C^r **topology**) is generated by the sets defined as follows: Let $f \in C^r(X, Y)$. Let (U, φ) and (V, ψ) be charts on X and Y, respectively, $K \subset U$ a compact set with $f(K) \subset V$ and $\varepsilon > 0$. A subbasic neighbourhood of f is defined to be the set of C^r maps $h : X \to Y$ such that $h(K) \subset V$ and for any k with $0 \leqslant k \leqslant r$, the square root of the sum of all kth partial derivatives of all components of $\psi \circ f \circ \varphi^{-1} - \psi \circ h \circ \varphi^{-1}$ is smaller than ε on $\varphi(K)$.

A topological group G is a **Lie group** if G is a C^∞ manifold and the group operations

$$G \times G \to G, \quad (g, h) \mapsto gh \quad \text{and}$$
$$G \to G, \quad g \mapsto g^{-1}$$

are C^∞ maps. The general linear group $GL_n(\mathbb{R})$ and the orthogonal group $O_n(\mathbb{R})$ are Lie groups.

Let G be a compact Lie group, Ω, Ξ representations of G, and $f : \Omega \to \Xi$ a continuous map. Fix $x \in \Omega$, and let $\dim \Xi = n$ and $g^{-1} f(gx) = (h_1, \ldots, h_n)$. Then each h_i is a continuous real-valued function on G. Hence $A(f)(x) = (A(h_1)(x), \ldots, A(h_n)(x))$ is a continuous map from Ω to Ξ. The following are some of the useful properties of the above map A (see [4]):

(a) $A(f)$ is equivariant, and $A(f) = f$ if f is equivariant,
(b) if f is a polynomial map, then so is $A(f)$, and
(c) the selfmap of $C^r(\Omega, \Xi)$ induced by A is continuous with respect to the C^r topology.

The following theorem concerning extension of G-maps is known (see, e.g., [3]): Let G be a compact topological group acting on a ***normal*** space X and $A \subset X$ a ***closed*** G-invariant subspace. Let $\rho : G \to GL(\mathbb{R}^n)$ be a representation map of G and $\varphi : A \to \mathbb{R}^n$ be equivariant, that is $\varphi(ga) = \rho(g)\varphi(a)$ for all $g \in G$, $a \in A$. Then there exists an equivariant extension $\psi : X \to \mathbb{R}^n$ of φ. This is an equivariant version of the ***Tietze–Urysohn theorem*** [E, 2.1.8].

Let G be a topological group, X a right G-space and Y a left G-space. Then a left action of G on $X \times Y$ is given by letting $g \in G$ take (x, y) to (xg^{-1}, gy). The **twisted product** $X \times_G Y$ is the orbit space of this action. If X is a left G'-space for another topological group G', then $X \times_G Y$ is also a left G'-space by letting $g' \in G'$ take $[x, y] \to [g'x, y]$, where $[x, y]$ denotes the equivalence class of (x, y).

Let X be a G-space and $x \in X$. A G_x-invariant subspace S containing x is called a **slice at** x if the map

$$G \times_{G_x} S \to X, \qquad [g, s] \mapsto gx$$

is a G-embedding onto a G-invariant ***open neighbourhood*** of $G(x)$ in X. G.D. Mostow [13] proved the existence of slices: If G is a compact Lie group, then any point x in a ***completely regular*** G-space admits a slice at x.

Let G be a compact topological group and $\mathcal{F}$ the class of Hausdorff homogeneous G-spaces. For $X, Y \in \mathcal{F}$, a relation $\sim$ on $\mathcal{F}$ defined by $X \sim Y \iff$ there exists a G-homeomorphism $f : X \to Y$ is an equivalence relation. Equivalence classes under this equivalence relation are called ***G*-orbit types**, and *type*(X) denotes the equivalence class of X. The set $\mathcal{F}/\!\sim$ of equivalence classes is called the **set of *G*-orbit types**. The ordering of $\mathcal{F}/\!\sim$ is given by *type*$(X) \geqslant$ *type*$(Y) \iff$ there exists a G-map $f : X \to Y$.

Two subgroups T_1 and T_2 of G are **conjugate subgroups** if there exists $g \in G$ with $T_2 = gT_1g^{-1}$. This relation is an equivalence relation, and these equivalence classes are called **conjugacy classes**.

For a subgroup G' of a compact topological group G, let (G') denote the conjugacy class of G'. For closed subgroups H, K of G, an ordering $\leqslant$ is given by $(H) \leqslant (K) \iff H$ is conjugate to a subgroup of K. This gives an ordering of the set of conjugacy classes $\mathcal{C}$ of closed subgroups of G. Then the following theorem is known (see, e.g., [10]): Let G be a compact topological group. Then for any orbit type *type*(X), there exists a closed subgroup H of G with *type*$(X) =$ *type*(G/H), and

$$\varphi : \mathcal{F}/\!\sim\, \to \mathcal{C}, \qquad \mathit{type}(G/H) \mapsto (H)$$

is an order-reversing isomorphism.

A **bundle** (E, p, X) is a triple consisting of topological spaces E and X, and a surjective continuous map $p : E \to X$. The space X is called the **base space**, the space E is called the **total space**, and the map p is called the **projection**. For each $x \in X$, $p^{-1}(x)$ is called the **fiber** of the bundle over $x \in X$. A k-dimensional ($\mathbb{R}$-) **vector bundle** over X is a bundle (E, p, X) together with the structure of a k-dimensional vector space over $\mathbb{R}$ on each fiber $p^{-1}(x)$ such that the following local trivial condition is satisfied: There exist an open cover $\{U_\alpha\}_{\alpha \in A}$ of X and homeomorphisms

$$h_\alpha : U_\alpha \times \mathbb{R}^k \to p^{-1}(U_\alpha)$$

such that $p \circ h_\alpha = p_{U_\alpha}$ and the restriction $\{x\} \times \mathbb{R}^k \to p^{-1}(x)$ is a linear isomorphism for each $x \in U_\alpha$, where $p_{U_\alpha} : U_\alpha \times \mathbb{R}^k \to U_\alpha$ denotes the ***projection*** map onto the first factor. We call $(U_\alpha, h_\alpha : U_\alpha \times \mathbb{R}^k \to p^{-1}(U_\alpha))_{\alpha \in A}$ a **family of local trivializations**. This family of local trivializations gives a family $\{l_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k(\mathbb{R})\}$ of continuous maps such that for any $x \in U_\alpha \cap U_\beta$ and $v \in \mathbb{R}^k$, $h_\beta^{-1} \circ h_\alpha(x, v) = (x, l_{\alpha\beta}(x)v)$. These maps verify that $l_{\alpha\alpha}(x)$ is the identity matrix for all $x \in U_\alpha$ and $l_{\alpha\beta}(x)l_{\beta\gamma}(x) = l_{\alpha\gamma}(x)$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$. We call this family of continuous maps a **family of transition functions**. Conversely, a family $\{l'_{ij} : V_i \cap V_j \to GL_k(\mathbb{R})\}$ of continuous maps such that $\{V_i\}_{i \in I}$ is an open cover of X, $l'_{ii}(x)$ is the identity matrix for all $x \in V_i$ and $l'_{ij}(x)l'_{js}(x) = l'_{is}(x)$ for all $x \in V_i \cap V_j \cap V_s$ defines a k-dimensional vector bundle over X.

The simplest example of vector bundles over a topological space X is a product vector bundle $(X \times \mathbb{R}^n, p, X)$, where $p: X \times \mathbb{R}^n \to X$ denotes the projection onto the first factor. Another simple example is the open Möbius band. Let

$$U_1 = S^1 \setminus \{(0,1)\} \quad \text{and} \quad U_2 = S^1 \setminus \{(0,-1)\}.$$

Then $U_1 \cap U_2$ has two connected components V_1 and V_2. Let E be the quotient space $U_1 \times \mathbb{R} \amalg U_2 \times \mathbb{R}/\sim$ of $U_1 \times \mathbb{R} \amalg U_2 \times \mathbb{R}$ with respect to the identification $(x,t) \sim (x,t)$ if $x \in V_1$ and $(x,t) \sim (x,-t)$ if $x \in V_2$ and $p: E \to S^1$, $p([x,t]) = x$. Then (E, p, S^1) becomes a 1-dimensional vector bundle over S^1 and E is homeomorphic to the open Möbius band, namely the interior of the Möbius band.

Let $\xi = (E, p, X)$ and $\xi' = (E', p', X)$ be two vector bundles. A continuous map $f: E \to E'$ is a **vector bundle morphism** if $p = p' \circ f$ and the restriction $p^{-1}(x) \to p'^{-1}(x)$ is a linear map for each $x \in X$. A vector bundle morphism $f: E \to E'$ is called a **vector bundle isomorphism** if there exists a vector bundle morphism $h: E' \to E$ such that $f \circ h = \mathrm{id}$ and $h \circ f = \mathrm{id}$.

Let $\xi = (E, p, X)$ be a vector bundle and Y a topological space. Given any continuous map $f: Y \to X$, one can construct the **pullback bundle** $f^*(\xi) = (E', p', Y)$ of ξ as follows: The total space E' of $f^*(\xi)$ is the subset $E' \subset Y \times E$ consisting of pairs (y, e) with $f(y) = p(e)$ and the projection $p'(y, e) = y$.

The following theorem is known (see, e.g., [5]): Let $f, h: X \to X'$ be two ***homotopic*** continuous maps, where X is a ***paracompact*** space, and let ξ be a vector bundle over X'. Then $f^*(\xi)$ and $h^*(\xi)$ are vector bundle isomorphic.

The **Grassmann manifold** $G_k(\mathbb{R}^n)$ is the set of all k-dimensional planes through the origin of the n-dimensional Euclidean space $\mathbb{R}^n$. This is to be topologized as a quotient space, as follows. A **k-frame** in $\mathbb{R}^n$ is a k-tuple of linearly independent vectors of $\mathbb{R}^n$. The collections of all k-frames in $\mathbb{R}^n$ forms an open subset of the k-fold Cartesian product $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$, called the **Stiefel manifold** $V_k(\mathbb{R}^n)$. There is a canonical map $q: V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ which takes each k-frame to the k-plane which it spans, and the topology of $G_k(\mathbb{R}^n)$ is given by the quotient topology. The canonical vector bundle

$$\gamma_k(\mathbb{R}^n) = (E_k(\mathbb{R}^n), u, G_k(\mathbb{R}^n))$$

over $G_k(\mathbb{R}^n)$ is provided so that the total space $E_k(\mathbb{R}^n)$ is the set of all pairs (k-dimensional linear subspace A of $\mathbb{R}^n$, vector in A) topologized as a subset of $G_k(\mathbb{R}^n) \times \mathbb{R}^n$ and the projection $u: E_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ is defined by $u(A, x) = A$. The **infinite Grassmann manifold** $G_k(\mathbb{R}^\infty)$ is the set of k-dimensional linear subspace of $\mathbb{R}^\infty$, topologized as the ***direct limit*** of the sequence $G_k(\mathbb{R}^k) \subset G_k(\mathbb{R}^{k+1}) \subset \cdots$. The **infinite Stiefel manifold** $V_k(\mathbb{R}^\infty)$ and the canonical vector bundle

$$\gamma_k(\mathbb{R}^\infty) = (E_k(\mathbb{R}^\infty), u, G_k(\mathbb{R}^\infty))$$

are defined similarly.

The following theorem is known (see, e.g., [5]): Let X be a paracompact space. For any k-dimensional vector bundle ξ over X, there exists a continuous map $f: X \to G_k(\mathbb{R}^\infty)$ such that ξ and $f^*(\gamma_k(\mathbb{R}^\infty))$ are vector bundle isomorphic.

Let G be a topological group. A vector bundle $\xi = (E, p, X)$ is a **G-vector bundle** if E and X are G-spaces, p is a G-map, and for each $g \in G$, $x \in X$, the map $p^{-1}(x) \to p^{-1}(gx)$; $y \mapsto gy$ is a linear isomorphism. A vector bundle morphism (respectively a vector bundle isomorphism) between G-vector bundles is called a **G-vector bundle morphism** (respectively a **G-vector bundle isomorphism**) if it is a G-map.

Let G be a compact topological group and Ω a representation of G. Then the **Grassmann G-manifold** $G_k(\Omega)$, the **Stiefel G-manifold** $V_k(\Omega)$ and $\gamma_k(\Omega)$ are defined in a way similar to the above. Two continuous G-maps $f, h: X \to Y$ between G-spaces are called **G-homotopic** if there exists a continuous G-map $H: X \times [0,1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$, where the action on $[0,1]$ of G is trivial.

The following results are equivariant versions of the previous two theorems: (1) (See [2] and [12].) Let G be a compact Lie group, X a paracompact G-space, and ξ a G-vector bundle over a G-space Y. If $f, h: X \to Y$ are G-homotopic continuous G-maps, then $f^*(\xi)$ and $h^*(\xi)$ are G-vector bundle isomorphic. (2) (See [1] and [15].) Let G be a compact topological group and X a compact G-space. If ξ is a k-dimensional G-vector bundle over X, then there exist a representation Ω of G and a continuous G-map $\varphi: X \to G_k(\Omega)$ such that ξ is G-vector bundle isomorphic to $\varphi^*(\gamma_k(\Omega))$.

A **semigroup** is a pair (S, ψ) consisting of a set S and an operation $\psi: S \times S \to S$ such that $\psi(\psi(a,b),c) = \psi(a, \psi(b,c))$ for all $a, b, c \in S$. We usually denote a semigroup by S instead of (S, ψ), and write ab instead of $\psi(a,b)$. A semigroup G is an **Abelian semigroup** or **commutative semigroup** if $ab = ba$ for any $a, b \in G$. A map f from a semigroup G to a semigroup G' is called a **semigroup homomorphism** if $f(ab) = f(a)f(b)$ for all $a, b \in G$. A semigroup homomorphism $f: G \to G'$ is a **semigroup isomorphism** if there exists a semigroup homomorphism $h: G' \to G$ such that $f \circ h = \mathrm{id}$ and $h \circ f = \mathrm{id}$. A **semi-ring** is a triple (S, α, μ) consisting of a set S, $\alpha: S \times S \to S$ is the additive function usually denoted by $\alpha(a,b) = a + b$, and $\mu: S \times S \to S$ is the multiplicative function usually denoted by $\mu(a,b) = ab$ such that they satisfy all the axioms of a commutative ring except the existence of an additive inverse. A map f from a semi-ring S to a semi-ring S' is called a **semi-ring homomorphism** if $f(ab) = f(a)f(b)$ and $f(a+b) = f(a) + f(b)$ for all $a, b \in S$, $f(0) = 0$ and $f(1) = 1$. A semi-ring homomorphism $f: S \to S'$ is a **semi-ring isomorphism** if there exists a semi-ring homomorphism $h: G' \to G$ such that $f \circ h = \mathrm{id}$ and $h \circ f = \mathrm{id}$. The **ring completion** of a semi-ring S is a pair (S^*, θ) consisting of a ring S^* and a semi-ring homomorphism $\theta: S \to S^*$ such that for any semi-ring homomorphism $f: S \to R$ into a ring R, there exists a unique ring homomorphism $f': S^* \to R$ with $f' \circ \theta = f$. Every semiring admits the ring completion (see, e.g., [5]). The **group completion** of a semigroup is similarly defined, and does exist (see, e.g., [5]).

Let ξ, ξ' be vector bundles over a topological space X whose families of local trivializations are

$$\{U_\alpha, h_\alpha: U_\alpha \to U'_\alpha \times \mathbb{R}^n\}_{\alpha\in A} \quad \text{and}$$
$$\{V_\beta, k_\beta: V_\beta \to V_\beta \times \mathbb{R}^m\}_{\beta\in B},$$

respectively, where the dimensions of ξ and ξ' are n and m, respectively. Considering a refinement $\{U_\alpha \cap V_\beta\}_{\alpha\in A,\beta\in B}$ of two open covers $\{U_\alpha\}_{\alpha\in A}$ and $\{V_\beta\}_{\beta\in B}$, we may assume that the above families are

$$\{U_\alpha, h_\alpha: U_\alpha \to U_\alpha \times \mathbb{R}^n\}_{\alpha\in A} \quad \text{and}$$
$$\{U_\alpha, k_\alpha: U_\alpha \to U_\alpha \times \mathbb{R}^m\}_{\alpha\in A}.$$

These families give two families of transition functions

$$\{\varphi_{\alpha\beta}: U_\alpha \cap U_\beta \to GL_n(\mathbb{R})\}_{\alpha\beta} \quad \text{and}$$
$$\{\psi_{\alpha\beta}: U_\alpha \cap U_\beta \to GL_m(\mathbb{R})\}_{\alpha\beta}.$$

The two maps $\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta}: U_\alpha \cap U_\beta \to GL_{n+m}(\mathbb{R})$ and $\varphi_{\alpha\beta} \otimes \psi_{\alpha\beta}: U_\alpha \cap U_\beta \to GL_{nm}(\mathbb{R})$ are defined by

$$(\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta})(x) = \begin{pmatrix} \varphi_{\alpha\beta}(x) & 0 \\ 0 & \psi_{\alpha\beta}(x) \end{pmatrix}$$

and

$$(\varphi_{\alpha\beta} \otimes \psi_{\alpha\beta})(x) = \begin{pmatrix} a_{11}(x)\psi_{\alpha\beta}(x) & \dots & a_{1n}(x)\psi_{\alpha\beta}(x) \\ \vdots & & \vdots \\ a_{n1}(x)\psi_{\alpha\beta}(x) & \dots & a_{nn}(x)\psi_{\alpha\beta}(x) \end{pmatrix},$$

respectively, where $\varphi_{\alpha\beta}(x) = (a_{ij}(x))$. Then $\{\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta}\}_{\alpha\beta}$ and $\{\varphi_{\alpha\beta} \otimes \psi_{\alpha\beta}\}_{\alpha\beta}$ are families of transition functions. The vector bundle over X defined by $\{\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta}\}_{\alpha\beta}$ (respectively $\{\varphi_{\alpha\beta} \otimes \psi_{\alpha\beta}\}_{\alpha\beta}$) is called the **Whitney sum** $\xi \oplus \xi'$ (respectively the **tensor product** $\xi \otimes \xi'$) of ξ and ξ'.

Let F denote the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$, or the quaternion field $\mathbb{H}$. Then F-vector bundles over a topological space are defined similarly. Let $\mathrm{Vect}_F(X)$ denote the set of isomorphism classes of F-vector bundles over a topological space X. For $F = \mathbb{R}$ or $\mathbb{C}$, $\mathrm{Vect}_F(X)$ admits a natural commutative semi-ring structure, where $(\xi, \xi') \mapsto \xi \oplus \xi'$ is the additive function and $(\xi, \xi') \mapsto \xi \otimes \xi'$ is the multiplicative function. For $F = \mathbb{H}$, $\mathrm{Vect}_F(X)$ admits a natural commutative semigroup structure, where $(\xi, \xi') \mapsto \xi \oplus \xi'$ is the additive function. The ring $K_F(X)$ (or group for $F = \mathbb{H}$) is the ring (or group) completion of $\mathrm{Vect}_F(X)$.

The following periodicity theorem is known (see, e.g., [5]): Let X be a compact space. Then $K_\mathbb{C}(X) \otimes K_\mathbb{C}(S^2)$ is isomorphic to $K_\mathbb{C}(X \times S^2)$ and $K_\mathbb{R}(X) \otimes K_\mathbb{R}(S^8)$ is isomorphic to $K_\mathbb{R}(X \times S^8)$.

One can consider C^∞ versions of G-spaces and G-vector bundles.

Let G be a Lie group. A **$C^\infty G$-manifold** X is a pair (X, φ) consisting of a C^∞ manifold and an action $\varphi: G \times X \to X$ of G on X, which is a C^∞ map. Let X and Y be $C^\infty G$-manifolds. A C^∞ submanifold of X is a **$C^\infty G$-submanifold** of X if it is G-invariant. A C^∞ map $f: X \to Y$ is a **$C^\infty G$-map** if f is a G-map. A C^∞ diffeomorphism (respectively a C^∞ embedding) $X \to Y$ is a **$C^\infty G$-diffeomorphism** (respectively a **$C^\infty G$-embedding**) if it is a G-map.

The following embedding theorem holds (see, e.g., [3]): Let G be a compact Lie group. Then every compact $C^\infty G$-manifold is $C^\infty G$-embeddable into some representation of G.

Let G be a Lie group and X a $C^\infty G$-manifold. A k-dimensional **$C^\infty G$-vector bundle** over X is a G-vector bundle (E, p, X) over X consisting of $C^\infty G$-manifolds E and X, and a $C^\infty G$-map $p: E \to X$ with a family of local trivializations $(U_\alpha, \varphi_\alpha: U_\alpha \times \mathbb{R}^k \to p^{-1}(U_\alpha))_{\alpha\in A}$ such that for any $\alpha, \beta \in A$,

$$\varphi_\beta^{-1} \circ \varphi_\alpha \mid (U_\alpha \cap U_\beta) \times \mathbb{R}^k : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

is a C^∞ map. Let $\xi = (E, p, X)$ and $\xi' = (E', p', X)$ be two $C^\infty G$-vector bundles. A G-vector bundle morphism $f: E \to E'$ is a **$C^\infty G$-vector bundle morphism** if it is a C^∞ map. A $C^\infty G$-vector bundle morphism $f: E \to E'$ is called a **$C^\infty G$-vector bundle isomorphism** if there exists a $C^\infty G$-vector bundle morphism $h: E' \to E$ such that $f \circ h = \mathrm{id}$ and $h \circ f = \mathrm{id}$.

Let $\xi = (E, p, X)$ be a $C^\infty G$-vector bundle and Y a $C^\infty G$-manifold. Given any $C^\infty G$-map $f: Y \to X$, the pullback bundle $f^*(\xi)$ becomes a $C^\infty G$-vector bundle over Y. The map $x \mapsto$ the origin of $p^{-1}(x)$ becomes a $C^\infty G$-map from X to E (see, e.g., [10]), and its image is called the **zero section** of ξ which is identified with X. The Whitney sum of two $C^\infty G$-vector bundles is defined similarly.

Two $C^\infty G$-maps $f, h: X \to Y$ between $C^\infty G$-mani folds are called **$C^\infty G$-homotopic** if there exists a $C^\infty G$-map H: $X \times \mathbb{R}$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$, where G acts trivially on $\mathbb{R}$.

A.G. Wasserman [16] proved the following theorem: Let G be a compact Lie group, X and Y $C^\infty G$-manifolds and ξ a $C^\infty G$-vector bundle over Y. If $f, h: X \to Y$ are $C^\infty G$-homotopic $C^\infty G$-maps, then $f^*(\xi)$ and $h^*(\xi)$ are $C^\infty G$-vector bundle isomorphic.

Let G be a compact Lie group and X a $C^\infty G$-manifold. Then the tangent space TX of X is a $C^\infty G$-manifold, p_X is a $C^\infty G$-map, and $T(X) = (TX, p_X, X)$ becomes a $C^\infty G$-vector bundle over X. This bundle is called the **tangent**

$C^\infty G$**-bundle of** X. Let A be a $C^\infty G$-submanifold of X. Then the restriction $(p_X^{-1}(A), p_X|p_X^{-1}(A), A)$ of $T(X)$ over A is a $C^\infty G$-vector bundle, and it is denoted by $T(X)|A$. The fiber of $T(A)$ over $x \in A$ is a vector subspace of $T(X)|A$ over x. The quotient space of $p_X^{-1}(A)$ under the equivalence relation:

$$z \sim z' \iff p_X(z) = p_X(z') \text{ and } z - z' \text{ lies in the fiber of } T(A) \text{ over } p_X(z)$$

is denoted by $p_X^{-1}(A)/TA$. Then there exists a continuous map $p' : p_X^{-1}(A)/TA \to A$ such that $p_X|p_X^{-1}(A) = p' \circ \pi$, where $\pi : p_X^{-1}(A) \to p_X^{-1}(A)/TA$ denotes the projection. It is known that $\nu = (p_X^{-1}(A)/TA, p', A)$ becomes a $C^\infty G$-vector bundle (see, e.g., [10]), and ν is called the **normal** $C^\infty G$**-vector bundle of** A **in** X.

Let $\xi = (E, p, X)$ be a $C^\infty G$-vector bundle and let $\langle \cdot, \cdot \rangle_x$ denote an inner product on the fiber $p^{-1}(x)$. We say that ξ has a G**-invariant** C^∞ **metric** if the map

$$E \to \mathbb{R}, \qquad t \mapsto \langle t, t \rangle_{p(t)}$$

is a C^∞ map and $\langle gt, gs \rangle_{gx} = \langle t, s \rangle_x$ for all $g \in G$, $t, s \in p^{-1}(x)$ and $x \in X$. A $C^\infty G$-manifold X has a G-invariant C^∞ metric if $T(X)$ admits a G-invariant C^∞ metric.

If X has a G-invariant C^∞ metric and A is a $C^\infty G$-submanifold of X, then there exists a $C^\infty G$-vector bundle η over A such that $T(X)|A$ is $C^\infty G$-vector bundle isomorphic to $T(A) \oplus \eta$ and such that for any $x \in A$, the fiber of η over x is the orthogonal complement of that of $T(A)$ over x in the fiber of $T(X)|A$ over x (see, e.g., [10]). The bundle η is called the **orthogonal complement** of $T(A)$ in $T(X)|A$ and denoted by $T(A)^\perp$. Moreover $T(A)^\perp$ is $C^\infty G$-vector bundle isomorphic to ν (see, e.g., [10]). A $C^\infty G$**-tubular neighbourhood** of A in X is a $C^\infty G$-embedding $i : \nu \to X$ such that the restriction of i to the zero section A of ν is the inclusion of A in X.

The following theorem (see, e.g., [10]) guarantees the existence of $C^\infty G$-tubular neighbourhoods: Let G be a compact Lie group and X a $C^\infty G$-manifold. Then every $C^\infty G$-submanifold A of X admits a $C^\infty G$-tubular neighbourhood of A in X.

On the other hand, one can consider G-spaces and G-vector bundles in various categories, for example, the algebraic category, the Nash category, the definable category. Results on these categories can be found in [6–9].

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h-6 Topological Discrete Dynamical Systems

Recently, terms such as chaos theory and nonlinear analysis have become familiar among many people without common precise definition. The dynamical systems' point of view is becoming increasingly important in a broad area of science and technology. Taken in its broad sense, dynamical systems involves many branches of mathematics; differential equations, differential topology, general topology, ergodic theory, complex analysis, etc. In this section of the encyclopedia we shall take up the subjects in a narrow sense, omitting differential equations and complex analysis from the above list and minimizing differential topology and ergodic theory, to introduce basic concepts in dynamical systems.

1. Phase spaces

The phase space of a dynamical system is usually taken to be a differentiable manifold.

A ***continuous*** surjection $f: M \to N$ between ***metric spaces*** is a **homeomorphism** if it is injective and the inverse map $f^{-1}: N \to M$ is also continuous. A metric space M is called an n-dimensional **topological manifold** if there exist ***open*** subsets U_i of M and homeomorphisms α_i of U_i onto open subsets of the n-dimensional Euclidean space $\mathbb{R}^n$, such that $\{U_i\}$ covers M, i.e., the union of all U_i's coincides with M. It follows that the composite map $\alpha_i \circ \alpha_j^{-1}: \alpha_j(U_i \cap U_j) \to \alpha_i(U_i \cap U_j)$ is a homeomorphism between nonempty open subsets of $\mathbb{R}^n$ whenever U_i and U_j overlap. If, in addition, all the composite maps $\alpha_i \circ \alpha_j^{-1}$ are r-times continuously differentiable, then M is said to be an n-dimensional **differentiable manifold** of class C^r ($1 \leqslant r \leqslant \infty$). The collection $\{(U_i, \alpha_i)\}$ is called an **atlas** of class C^r for M, while each (U_i, α_i) is called a **local chart** of M. A differentiable manifold of class C^∞ is, in particular, said to be **smooth**. Let M and N be differentiable manifolds, and let $f: M \to N$ be a continuous map. Then we can choose atlases $\{(U_i, \alpha_i)\}$ for M and $\{(V_j, \beta_j)\}$ for N, such that for each i the ***closure*** of $f(U_i)$ in N is a subset of some V_{j_i}. We say that f is a **differentiable map** of class C^r if $\beta_{j_i} \circ f \circ \alpha_i^{-1}: \alpha_i(U_i) \to \beta_{j_i}(V_{j_i})$ is r-times continuously differentiable for all i; if $r = \infty$ then we say f is a **smooth map**. This definition does not depend on the choice of atlases which are consistent with the differentiable structures of M and N, respectively. Suppose that M is ***compact***, and denote by $C^r(M, N)$ the set of all differentiable maps of class C^r from M to N. Let $f \in C^r(M, N)$, and let $\{(U_i, \alpha_i)\}$ and $\{(V_j, \beta_j)\}$ be as above. Since M is compact, we can choose $\{U_i\}$ to be a finite ***open cover*** of M, and take a ***subcover*** $\{U_i'\}$ of M such that the closure of each U_i' is contained in U_i. Then there exists a ***neighbourhood*** $N(f)$ of f in $C^r(M, N)$ with respect to the ***compact-open topology*** such that if $g \in N(f)$, then the closure of $g(U_i')$ is also a subset of V_{j_i} for each i. Hence, if $1 \leqslant r < \infty$, the distance d^r between f and g is defined by

$$\begin{aligned} d^r(f, g) &= \max\Big\{\max_{x \in M} d\big(f(x), g(x)\big), \\ &\qquad \max_{1 \leqslant k \leqslant r} \max_i \sup_{x \in \alpha_i(U_i')} \big\| D^k\big(\beta_{j_i} \circ f \circ \alpha_i^{-1}\big)(x) \\ &\qquad\qquad - D^k\big(\beta_{j_i} \circ g \circ \alpha_i^{-1}\big)(x) \big\| \Big\} \end{aligned}$$

where d is the ***metric*** for N, $D^k(\cdot)$ is the kth derivative of $\cdot$; and $\|\cdot\|$ denotes the Euclidean norm. By the d^r a ***neighbourhood base*** of f in $C^r(M, N)$ is induced, which defines a ***topology*** of $C^r(M, N)$, called the C^r **topology**. The C^0 **topology** for the set $C^0(M, N)$ of all continuous maps from M to N is defined by the metric;

$$d^0(f, g) = \max_{x \in M} d\big(f(x), g(x)\big)$$

for $f, g \in C^0(M, N)$, which is consistent with the compact-open topology. We omit the definition of the C^∞ topology because it is not relative to this article. The space $C^r(M, N)$ endowed with the C^r topology is ***separable*** and ***metrizable*** whenever $0 \leqslant r < \infty$. If, in addition, N is ***complete***, then so is the space $C^r(M, N)$, and hence $C^r(M, N)$ is, in particular, a ***Baire space***. The space $\mathrm{Hom}(M, N)$ of all homeomorphisms from M onto N with the C^0 topology is a complete metric space if the metric is given by

$$d^{0'}(f, g) = \max\Big\{d^0(f, g),\ \max_{x \in N} d'\big(f^{-1}(x), g^{-1}(x)\big)\Big\}$$

where d' is the metric for M. If $f: M \to N$ is a homeomorphism and if both f and f^{-1} are differentiable maps of class C^r ($r \geqslant 1$), then the homeomorphism $f: M \to N$ is called a **diffeomorphism** of class C^r. It is known that the set $\mathrm{Diff}^r(M, N)$ of all diffeomorphisms of class C^r from M onto N is open in $C^r(M, N)$ with respect to the C^r topology.

By a C^r **curve** in a differentiable manifold M we mean a C^r map into M of an open interval of the real line. If $\gamma: I \to M$ is a C^r curve, then the **tangent vector** to γ at $x \in \gamma(I)$ can be defined. For $x \in M$ all such tangent vectors form a real n-dimensional vector space $T_x M$, called the **tangent space** to M at x. The set of all pairs (x, v), where $x \in M$ and $v \in T_x M$, can be vested with the structure of a $2n$-dimensional differentiable manifold TM, called the **tangent bundle** of M. If M is of class C^r, then TM is of class C^{r-1}. An inner product $\langle\ ,\ \rangle_x$ on $T_x M$ is a **Riemannian**

metric of class C^{r-1} for M if the map $(u, v, x) \mapsto \langle u, v\rangle_x$ is of class C^{r-1}. By using the Riemannian metric a norm $\| \ \|$ on each tangent space $T_x M$ is defined by $\|v\|^2 = \langle v, v\rangle_x$. Let $f : M \to N$ be a differentiable map of class C^r between differentiable manifolds M and N. If γ is a C^r curve in M, then the composite map $f \circ \gamma$ is also a C^r curve in N, and hence the tangent vector $v \in T_x M$ defined by γ is sent to some tangent vector, say $D_x f(v)$, of $T_{f(x)}N$ defined by $f \circ \gamma$, which induces a differentiable map $Df : TM \to TN$ of class C^{r-1}, called the **derivative** of f.

2. Dynamical systems

We are now ready to define a dynamical system. A **dynamical system with continuous time**, or **flow** on a metric space X is a family $\{\varphi_t\colon t \in \mathbb{R}\}$ of homeomorphisms of X, such that the map $(t, x) \mapsto \varphi_t(x)$ is continuous and $\varphi_{t+s}(x) = \varphi_t \circ \varphi_s(x)$ for all $x \in X$ and all $t, s \in \mathbb{R}$. It follows that φ_0 is the identity map on X and φ_{-t} is the inverse map of φ_t. A family $\{\varphi_t\colon t \geqslant 0\}$ of continuous maps of X is called a **semi-flow** if φ_0 is the identity map and if $\varphi_{t+s}(x) = \varphi_t \circ \varphi_s(x)$ for all $x \in X$ and all $t, s \geqslant 0$, and $(t, x) \mapsto \varphi_t(x)$ is continuous. The **orbit** of a point $x \in X$ under the flow or semi-flow is the set of points $\{\varphi_t(x)\colon t \in \mathbb{R}\}$ or $\{\varphi_t(x)\colon t \geqslant 0\}$. The point x is said to be a **fixed point** if $\varphi_t(x) = x$ for all t, and to be a **periodic point** if $\varphi_p(x) = x$ for some $p > 0$. This implies $\varphi_{t+p}(x) = \varphi_t(x)$ for all t, and the orbit $\{\varphi_t(x)\colon t \in \mathbb{R}\} = \{\varphi_t(x)\colon 0 \leqslant t \leqslant p\}$ is a ***simple closed curve*** in X, which is called the **periodic orbit**. The smallest number among the p's satisfying $\{\varphi_t(x)\colon t \in \mathbb{R}\} = \{\varphi_t(x)\colon 0 \leqslant t \leqslant p\}$ is called the **period** of the point x. A dynamical system with discrete time on a metric space X is a family $\{f^n\colon n \in \mathbb{Z}\}$ of homeomorphisms of X, or a family $\{f^n\colon n \geqslant 0\}$ of continuous maps of X with f^0 the identity map, such that $f^{n+m}(x) = f^n \circ f^m(x)$ for all $x \in X$ and all n, m. It follows that $f^n = f \circ \cdots \circ f$ (n-times) for $n > 0$, where $f = f^1$, and if f is a homeomorphism then $f^{-n} = (f^n)^{-1}$ for $n > 0$. Conversely, the iterations of a homeomorphism or a continuous map $f : X \to X$ form a discrete dynamical system. The **orbit** of a point $x \in X$ under the homeomorphism or continuous map is the set of points $\{f^n(x)\colon n \in \mathbb{Z}\}$ or $\{f^n(x)\colon n \geqslant 0\}$. A point x is said to be a **fixed point** if $f(x) = x$, and be a **periodic point** if $f^p(x) = x$ for some $p > 0$. In this case the orbit through x is a finite set, called the **periodic orbit**. The cardinality of the periodic orbit is the **period** of x.

A dynamical system with continuous time is sometimes obtained as solutions of a differential equation of form

$$\frac{d}{dt}(x) = F(x)$$

where x belongs to a ***domain*** D of $\mathbb{R}^n$ and $F : D \to \mathbb{R}^n$ is a map on the phase space D. In the theory of differential equations a fundamental theorem says that if F is continuously differentiable then, for any given point x_0, the equation has a unique solution $\varphi_t(x_0)$ which is defined for $|t|$ sufficiently small and satisfies the initial condition $\varphi_0(x_0) = x_0$. The time-dependence together with uniqueness ensures that $\varphi_{t+s}(x_0) = \varphi_t(\varphi_s(x_0))$. The restriction to small value of $|t|$ is essential. In many problems the phase space is a domain of the real n-dimensional space $\mathbb{R}^n$. However, other spaces also arise quite naturally. For example, if the differential equation can be restricted on a compact m-dimensional manifold M contained in the domain D (i.e., for every $x \in M$ the value $F(x)$ is a tangent vector of M at x), for any given point $x_0 \in M$ the equation has then a solution $\varphi_t(x_0) \in M$ for all $t \in \mathbb{R}$. Therefore we have a flow $\{\varphi_t\}$ on the phase space M. There are two ways to relate such an equation $\frac{d}{dt}x = F(x)$ to a dynamical system with discrete time. The first method is to discretize time, i.e., for any choice of small $t_0 > 0$ this leads to the equation

$$x_{n+1} = \varphi_{t_0}(x_n) = f(x_n).$$

Then the iterations $\{f^n\}$ of the map f becomes a dynamical system with discrete time on the space. The other method can be applied if the equation has a periodic solution, i.e., $\varphi_T(x_0) = x_0$ for some x_0 and some $T > 0$. Then we consider a hypersurface S, called the **cross section**, transverse to the curve $t \mapsto \varphi_t(x_0)$, and in this hypersurface a neighbourhood U of x_0. Then a map $g : U \to S$ is induced by associating to $y \in U$ the next intersection with S of the (forward) orbit $\{\varphi_t(y)\colon t > 0\}$. If the first such intersection occurs at y', then we define $g(y) = y'$, and g is a map $U \to S$, reduced from the flow, which is called the **Poincaré map**. Thus we have a dynamical system with discrete time modeled on the differential equation. We can discuss mostly dynamical systems with discrete time on a compact metric space X. As a special case, we regard X as a differentiable manifold and f as a differentiable map of X. The general case may be called a **topological dynamical system** and the special case a **differentiable dynamical system**. We are primarily interested in differentiable maps or diffeomorphisms on a differentiable manifold. However, such a map may have invariant sets which are not manifolds. The theory of topological dynamical systems is useful to investigate the behavior of a differentiable map on invariant sets.

We give an important example of a discrete dynamical system with a topological structure. Let $k \geqslant 1$ be an integer and let $Y_k^{\mathbb{Z}}$ denote the set of all two-sided infinite sequences $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$ where $x_n \in Y_k = \{0, 1, \ldots, k-1\}$. The set $Y_k^{\mathbb{Z}}$ becomes a compact metric space if we define the distance between two points x, y by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 2^{-m}$ if $x \neq y$ where m is the largest integer such that $x_n = y_n$ for all n with $|n| < m$. The set $Y_k^{\mathbb{Z}}$ is homeomorphic to the ***Cantor set***. The **shift map** σ defined by $\sigma(x) = y$ where $x = (x_n)$, $y = (y_n)$ and $y_n = x_{n+1}, n \in \mathbb{Z}$, is a homeomorphism of $Y_k^{\mathbb{Z}}$. Thus the family of homeomorphisms σ^n, $n \in \mathbb{Z}$, is a discrete dynamical system, called a **symbolic dynamics**. The periodic points of this system are ***dense*** in $Y_k^{\mathbb{Z}}$ and there exists a point $x \in Y_k^{\mathbb{Z}}$ whose orbit $O_\sigma(x) = \{\sigma^n(x)\colon n \in \mathbb{Z}\}$ is dense in $Y_k^{\mathbb{Z}}$. If S is a ***closed*** subset of $Y_k^{\mathbb{Z}}$ and σ-invariant (i.e., $\sigma(S) = S$), then the restriction of σ to S is called a **subshift**, denoted by $\sigma_{|S}$. The subshift $\sigma_{|S}$ defines a discrete dynamical systems on S. As an important class of subshifts we give the following one. Let $A = (a_{ij})$ be a $k \times k$ matrix of 0s and 1s and let S_A denote the set of all $x = (x_n) \in Y_k^{\mathbb{Z}}$ with $a_{x_n x_{n+1}} = 1$ for all $n \in \mathbb{Z}$. Then S_A is closed in $Y_k^{\mathbb{Z}}$ and $\sigma(S_A) = S_A$. A subshift of this type is called a **Markov subshift**.

A continuous surjection $f : X \to X$ of a metric space is said to be **topologically transitive** if for any nonempty open sets U, V there is $n > 0$ such that $f^n(U) \cap V$ is nonempty. It is said to be **topologically mixing** if for any nonempty open sets U, V there is $N > 0$ such that $f^n(U) \cap V$ is nonempty for all $n \geqslant N$. When X is compact, a continuous surjection $f : X \to X$ is topologically transitive if and only if there is $x_0 \in X$ such that the orbit $O_f^+(x_0) = \{x_0, f(x_0), \dots\}$ is dense in X. A homeomorphism $f : X \to X$ of a compact metric space is topologically transitive if and only if there exists $x_0 \in X$ such that the orbit $O_f(x_0) = \{f^n(x_0)\colon n \in \mathbb{Z}\}$ is dense in X.

A Markov subshift $\sigma_{|S_A} : S_A \to S_A$ is topologically transitive if and only if the matrix A is irreducible, and it is topologically mixing if and only if every element of A^m is positive for some $m > 0$. It may seem that Markov subshifts have no relevance to physical systems. But the fact is that they are realized as basic components of some physical systems. For example, Smale's theorem says that in a neighbourhood of the orbit through a transverse homoclinic point, associated to a hyperbolic fixed point of a diffeomorphism, there is an invariant set such that some iterations of the diffeomorphism on the invariant set act as the shift map of $Y_k^{\mathbb{Z}}$ for some $k > 0$. This theorem applies to the restricted three-body problem. For example, see J. Moser [36].

3. 1-dimensional dynamics

The iterations of a map of an interval into itself certainly present an example of nonlinear dynamical systems. The dynamics on an interval has many applications to physical systems. Let $f : [0, 1] \to [0, 1]$ be a continuous map of the interval. The iterations of the form $x_n \mapsto x_{n+1} = f(x_n)$ can be viewed as a discrete time version of a topological dynamical system. Here n plays the role of time variable. In natural sciences, the dynamical system often depends on quantities, called **parameters**. Therefore it is important to study a one-parameter family of maps which is the most basic system. For example, we consider a continuous map $f_r(x) = rx(1 - x)$ from $[0, 1]$ to itself, which is called the **logistic map**. Fix r with $0 \leqslant r \leqslant 4$ and define a ***sequence*** $\{x_n\}$ by $x_{n+1} = rx_n(1 - x_n)$ for $n \geqslant 0$. Then the behaviors of these maps, with the parameter r, where $0 \leqslant r \leqslant 4$, has the following properties:

(1) If $0 \leqslant r < 1$ then the sequence $\{x_n\}$ is decreasing and $x_n \to 0$ as $n \to \infty$.
(2) If $1 \leqslant r \leqslant 2$ then the sequence $\{x_n\}$ is increasing and $x_n \to 1 - 1/r$ as $n \to \infty$.
(3) If $2 < r \leqslant 3$ then the sequence $\{x_n\}$ ***converges*** to $1 - 1/r$.
(4) If $3 < r \leqslant 1 + \sqrt{6}$ then the sequence $\{x_n\}$ converges to some periodic orbit with period 2.
(5) If $1 + \sqrt{6} < r \leqslant 4$ then the behavior of $\{x_n\}$ is very complicated, and when r varies increasingly in the interval $[1 + \sqrt{6}, 4]$ there exists a sequence of parameters r_n in $[1 + \sqrt{6}, 4]$ such that f_{r_n} has periodic points with period 2^n.

On the set of natural numbers we define an ordering, which is called the **Sharkovskii ordering**, as follows.

$$
\begin{aligned}
&3 > 5 > 7 > 9 > \cdots \\
&\quad > 2 \cdot 3 > 2 \cdot 5 > 2 \cdot 7 > 2 \cdot 9 > \cdots \\
&\quad \dots \\
&\quad > 2^n \cdot 3 > 2^n \cdot 5 > 2^n \cdot 7 > 2^n \cdot 9 > \cdots \\
&\quad \dots \\
&\quad > \cdots > 2^m > \cdots > 8 > 4 > 2 > 1
\end{aligned}
$$

The Šharkovskii's theorem says that if $f : [0, 1] \to [0, 1]$ is continuous and has a periodic point with period p, then it has a periodic point with period q for every $q < p$ with respect to the above ordering. From this result it follows that if a continuous map has a periodic point with period 3 then it has periodic points of all periods. Concerning the question as to whether the behavior of the continuous map is chaotic, T. Li and J. Yorke [28] have shown that a 3-period implies chaos, i.e., if $f : [0, 1] \to [0, 1]$ is a continuous map with a periodic point with period 3, then there is an uncountable set S, called the **scrambled set**, of points and $\varepsilon > 0$ such that for every $x, y \in S$ with $x \neq y$

$$
\limsup_{n\to\infty} \left| f^n(x) - f^n(y) \right| \geqslant \varepsilon,
$$

$$
\liminf_{n\to\infty} \left| f^n(x) - f^n(y) \right| = 0.
$$

It can be shown that the logistic map f_r has a periodic point with period 3 if $r > 3.84$. Thus the Li–Yorke's theorem holds for such a map and its behavior is chaotic.

Let X and Y be metric spaces. If continuous maps $f : X \to X$ and $g : Y \to Y$ satisfy the relation $f \circ h = h \circ g$ for some homeomorphism $h : Y \to X$, then we say that f is **topologically conjugate** to g. This relation is an equivalence relation in the set of all continuous maps of metric spaces. When the relation $f \circ h = h \circ g$ holds for some continuous surjection $h : Y \to X$, we say that f is **topologically semi-conjugate** to g.

For the case $r = 4$ in particular, the logistic map f_4 is topologically conjugate to a continuous map $g : [0, 1] \to [0, 1]$, called the **tent map**, i.e., $g(x) = 2x$ for $0 \leqslant x \leqslant 1/2$ and $g(x) = 2(1 - x)$ for $1/2 \leqslant x \leqslant 1$. It can be shown that the map g is topologically semi-conjugate to a symbolic dynamics.

4. Higher dimensional dynamics

The 2-**dimensional torus** $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is obtained from the 2-dimensional Euclidean space $\mathbb{R}^2$ by identifying the points (x_1, y_1) and (x_2, y_2) when $x_2 - x_1$ and $y_2 - y_1$ are integers, i.e., $(x_2 - x_1, y_2 - y_1) \in \mathbb{Z}^2$. The torus is a compact smooth manifold. A linear map of the plane with matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

induces a map $f = f_A$ of the torus to itself if a, b, c, d are integers. The map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a differentiable map. If the determinant $ad - bc$ of A is nonzero, then f is called a **toral endomorphism**. If $ad - bc = \pm 1$ then f is bijective, and we call it a **toral automorphism**. A toral automorphism (endomorphism) f is said to be a **hyperbolic automorphism** if A has no eigenvalues on the unit circle in the complex plane. Then the behavior of f is quite complicated. It follows that a point p is periodic if and only if any coordinate of p is a rational number, and that the periodic points are dense in the torus. Moreover, there exists a point x whose orbit $O_f(x)$ is dense on the torus. It may be asked whether the behavior of f is chaotic. However, the question is somewhat vague because there is no widely accepted definition of chaotic behavior of higher dimension. Nevertheless, the hyperbolic toral automorphism (endomorphism) f is chaotic according to reasonable definitions which are not equivalent in general.

5. Anosov systems

In a similar way we can define an n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ from the n-dimensional Euclidean space $\mathbb{R}^n$. An automorphism (endomorphism) $f = f_A$ of $\mathbb{T}^n$ is induced by an $n \times n$ matrix A of integers. If the determinant of A is nonzero and A has no eigenvalues on the unit circle, then $\mathbb{R}^n$ is the direct sum of two subspaces E^s and E^u, invariant under A, such that every eigenvalue of the restriction of A to E^s (respectively E^u) has an absolute value less than one (respectively greater than one). Thus the behavior of f is similar to that in the 2-dimensional case.

To generalize the situation further, we consider a compact smooth manifold M. A diffeomorphism $f : M \to M$ is said to be an **Anosov diffeomorphism** if there are constants $C > 0$ and $0 < \lambda < 1$, and a continuous splitting $TM = E^s \oplus E^u$ of the tangent bundle, which is left invariant by the derivative Df, such that for all $n \geqslant 0$

$$\begin{aligned} \|Df^n(v)\| &\leqslant C\lambda^n \|v\| \quad \text{if } v \in E^s, \\ \|Df^{-n}(v)\| &\leqslant C\lambda^n \|v\| \quad \text{if } v \in E^u \end{aligned}$$

where $\|\cdot\|$ is a Riemannian metric. For $x \in M$ we define the **stable** and **unstable manifolds**

$$\begin{aligned} W^s(x) &= \{y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}, \\ W^u(x) &= \{y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty\} \end{aligned}$$

where d is the metric for M. A hyperbolic toral automorphism is an example of Anosov diffeomorphisms. If f is an Anosov diffeomorphism, by the stable manifold theorem it follows that each $W^\sigma(x)$ $(\sigma = s, u)$ is actually a submanifold of M, which is an injective immersion of the vector space E^σ_x that is the fiber of x for the bundle E^σ, and $T_x W^\sigma(x) = E^\sigma_x$. Since the splitting $TM = E^s \oplus E^u$ is continuous, we have that the **stable** and **unstable foliations**

$$\mathcal{F}^s = \{W^s(x) : x \in M\}, \qquad \mathcal{F}^u = \{W^u(x) : x \in M\}$$

are of class C^0, and they are **transverse**, that is, for each point $x \in M$ we can take a neighbourhood N of x in M, called a **product neighbourhood**, on which $\mathcal{F}^s$ and $\mathcal{F}^u$ has a **product structure**, i.e., there is a homeomorphism $\varphi : D^s \times D^u \to N$, where D^σ is a disc with the same dimension as $\mathcal{F}^\sigma$, such that each of $\varphi(\{y\} \times D^u)$ is a ***component*** of the intersection of N and an unstable manifold, and each of $\varphi(D^s \times \{z\})$ is a component of the intersection of N and a stable manifold. By using this structure it can be shown that every Anosov diffeomorphism $f : M \to M$ has the expansivity and the shadowing property which are defined in the following paragraph.

A homeomorphism $f : X \to X$ of a metric space is said to be **expansive** if there is a constant $e > 0$, called an **expansive constant**, such that if $x, y \in M$ and $x \neq y$ then $d(f^n(x), f^n(y)) > e$ for some $n \in \mathbb{Z}$, where d is the metric for X. A dynamics $f : X \to X$ with the expansivity could be considered to have a chaotic behavior, because it depends on initial conditions sensitively. A sequence (x_i) of points in X is called a δ-**pseudo orbit** of f if $d(f(x_i), x_{i+1}) < \delta$ for all i. This is quite a natural notion since on account of rounding errors a computer actually calculate a pseudo orbit, rather than an orbit. For given $\varepsilon > 0$ a δ-pseudo orbit (x_i) is said to be ε-**traced** by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ for all i. We say that f has the **shadowing property** (in other words, the **pseudo orbit tracing property**) if for $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of f is ε-traced by some point of X. It is easy to see that the expansivity and the shadowing property are independent of the choice of ***compatible metrics*** for X when X is compact, but not so for noncompact spaces. Every subshift $\sigma_{|S}$ of a symbolic dynamics $\sigma : Y_k^{\mathbb{Z}} \to Y_k^{\mathbb{Z}}$ is expansive, and a subshift $\sigma_{|S}$ has the shadowing property if and only if $\sigma_{|S}$ is a Markov subshift (P. Walters [65]). A toral automorphism $f_A : \mathbb{T}^n \to \mathbb{T}^n$ is expansive if and only if it is hyperbolic, and f_A has the shadowing property if and only if it is also hyperbolic.

By D. Anosov [2] it has been obtained that the set of all Anosov diffeomorphisms $f : M \to M$ of class C^1 is an open subset of the space $\mathrm{Diff}^1(M, M)$ of all diffeomorphisms with the C^1 topology, and that each Anosov diffeomorphism $f : M \to M$ is **structurally stable**, i.e., there is a neighbourhood $N(f)$ of f in $\mathrm{Diff}^1(M, M)$ such that if $g \in N(f)$ then g is topologically conjugate to f. S. Smale has posed in [62] the problem of finding all Anosov diffeomorphisms up to topological conjugacy. As known so far there are only algebraic examples such as hyperbolic toral automorphisms (J. Franks [17], S. Newhouse [37], A. Manning [33], S. Smale [63], M. Brin and A. Manning [14], and K. Hiraide [22]).

A differentiable map $f: M \to M$ is said to be **expanding** if there are constants $C > 0$, $0 < \lambda < 1$ such that for all $n \geqslant 0$ and all $v \in TM$

$$\|Df^n(v)\| \geqslant C\lambda^{-n}\|v\|.$$

It is known that an expanding map $f: M \to M$ has stronger properties than ones of Anosov diffeomorphisms. For instance, every expanding differentiable map has at least one fixed point, while it is still open the problem of whether every Anosov diffeomorphism has a fixed point or not. For the properties of expanding maps, the readers may refer to M. Shub [59], J. Franks [17], M. Gromov [18], and M. Shub and D. Sullivan [61].

For a differentiable map $f: M \to M$ of a compact smooth manifold a point $x \in M$ is called a **regular point** if $D_x f: T_x M \to T_{f(x)} M$ is surjective, and a **critical point** otherwise. If x is a regular point, by the inverse function theorem some open neighbourhood of x is mapped diffeomorphically onto an open set of M by f. Hence the set $S(f)$ of all critical points of f is closed in M. If $S(f)$ is empty, then f is called a **regular map**. By definition it follows that any expanding map is a regular map. Denote by $R^r(M, M)$ the space of all regular C^r maps of M. It follows that $R^r(M, M)$, $r \geqslant 1$, is open in $C^r(M, M)$ with respect to the C^r topology.

A regular differentiable map $f: M \to M$ is said to be an **Anosov endomorphism** if there are constants $C > 0$ and $0 < \lambda < 1$ such that for any orbit (x_i) of f (consisting of both of backward orbit and forward orbit), i.e., $f(x_i) = x_{i+1}$, $\forall i \in \mathbb{Z}$, there is a splitting

$$\bigcup_{i\in\mathbb{Z}} T_{x_i} M = E^s \oplus E^u = \bigcup_{i\in\mathbb{Z}} E^s_{x_i} \oplus E^u_{x_i},$$

which is left invariant by the derivative Df, such that for all $n \geqslant 0$

$$\|Df^n(v)\| \leqslant C\lambda^n\|v\| \quad \text{if } v \in E^s,$$
$$\|Df^n(v)\| \geqslant C^{-1}\lambda^{-n}\|v\| \quad \text{if } v \in E^u$$

where $\|\cdot\|$ is a Riemannian metric. When $(x_i) \neq (y_i)$ and $x_0 = y_0$, we have $E^u_{x_0} \neq E^u_{y_0}$ in general. Hence, we sometimes write $E^u_{x_0} = E^u_{x_0}((x_i))$. On the other hand, even if $(x_i) \neq (y_i)$, it follows that $E^s_{x_0} = E^s_{y_0}$ whenever $x_0 = y_0$, from which we have the **stable bundle**

$$E^s = \bigcup_{x\in M} E^s_x$$

which is a continuous subbundle of the tangent bundle TM. We say that f is **special** if for orbits (x_i), (y_i) with $x_0 = y_0$, $E^u_{x_0} = E^u_{y_0}$. In this case we have the **unstable bundle**

$$E^u = \bigcup_{x\in M} E^u_x$$

which is also a continuous subbundle of TM. It follows that if an Anosov endomorphism $f: M \to M$ is injective then f is special and it is an Anosov diffeomorphism, and that if $E^s = 0$, i.e., $E^u = TM$ then f is an expanding map, all of which form another class of special Anosov endomorphisms. F. Przytycki [46] has shown that the set of all Anosov endomorphisms $f: M \to M$ of class C^1 is an open subset of the space $R^1(M, M)$ with the C^1 topology, as well as Anosov diffeomorphisms and expanding maps. However, the behavior of an Anosov endomorphism is more complicated than the ones of an Anosov diffeomorphism and an expanding map, because any nonempty open subset consisting of Anosov endomorphisms $f: M \to M$, which are neither an expanding map nor an Anosov diffeomorphism, in the space $R^r(M, M)$ contains an uncountable set whose two distinct elements are not semi-conjugate by surjective semi-conjugacy (F. Przytycki [46, 47]), which implies that if an Anosov endomorphism $f: M \to M$ is structurally stable in $R^r(M, M)$ then f must be an expanding map or an Anosov diffeomorphism.

Let $f: X \to X$ be a continuous map of a compact metric space, and denote the set of all orbits of f by

$$\varprojlim(X, f) = \left\{(x_i) \in \prod_{-\infty}^{\infty} M\colon f(x_i) = x_{i+1},\ \forall i \in \mathbb{Z}\right\},$$

which is called the **inverse limit** for f. Let d be the metric for X, and define a metric $\tilde{d}$ for $\prod_{-\infty}^{\infty} X$ by

$$\tilde{d}\big((x_i), (y_i)\big) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} d(x_i, y_i)$$

and the **shift** $\sigma: \prod_{-\infty}^{\infty} X \to \prod_{-\infty}^{\infty} X$ by $\sigma((x_i)) = (x_{i+1})$. Then $\varprojlim(X, f)$ is a closed σ-invariant subset. The homeomorphism $\sigma: \varprojlim(X, f) \to \varprojlim(X, f)$ is called the **inverse limit system** for f.

It can be shown that if $f: M \to M$ is an Anosov endomorphism, then the correspondence $\varprojlim(M, f) \ni (x_i) \mapsto E^u_{x_0} = E^u_{x_0}((x_i))$ is continuous, and furthermore the inverse limit system $\sigma: \varprojlim(M, f) \to \varprojlim(M, f)$ is expansive and has the shadowing property. N. Aoki and K. Hiraide [5] have shown that the inverse limit system for an Anosov endomorphism of the n-dimensional torus $\mathbb{T}^n$ is topologically conjugate to the inverse limit system for some hyperbolic toral endomorphism.

6. Invariant sets of dynamical systems

Let $f: X \to X$ be a homeomorphism of a compact metric space with metric d. A point x belonging to M is said to be a **nonwandering point** if for any neighbourhood U of x there is an integer $n \neq 0$ such that $U \cap f^n(U)$ is nonempty. The set $\Omega(f)$ of all nonwandering points is called the **nonwandering set**. Clearly $\Omega(f)$ is closed in X and invariant under f. If f has the shadowing property, then so does $f_{|\Omega(f)}$ (N. Aoki [3]). If, in addition, f is expansive, then the set of all periodic points, $\mathrm{Per}(f)$, is dense in $\Omega(f)$. If $f_{|\Omega(f)}: \Omega(f) \to \Omega(f)$ is expansive and has the shadowing property, then the following properties hold:

(1) (the spectral decomposition theorem due to S. Smale) $\Omega(f)$ contains a finite sequence B_i ($1 \leqslant i \leqslant \ell$) of f-invariant closed subsets such that (i) $\Omega(f) = \bigcup_{i=1}^{\ell} B_i$ (disjoint union), (ii) $f_{|B_i}: B_i \to B_i$ is topologically transitive, and

(2) (the decomposition theorem due to R. Bowen) for each B_k there exist $a_k > 0$ and a finite sequence C_i ($0 \leqslant i \leqslant a_k - 1$) of closed subsets such that (i) $C_i \cap C_j = \emptyset$ ($i \neq j$), $f(C_i) = C_{i+1}$ and $f^{a_k}(C_i) = C_i$, (ii) $B_k = \bigcup_{i=0}^{a_k-1} C_i$, and (iii) $f^{a_k}_{|C_i}: C_i \to C_i$ is topologically mixing.

The sets B_i and C_j are called **basic set**s and **elementary sets**, respectively. For a subset A of X define

$$W^s(A) = \bigcup_{x \in A} W^s(x) \quad \text{and} \quad W^u(A) = \bigcup_{x \in A} W^u(x),$$

where $W^s(x)$ and $W^u(x)$ are the **stable** and **unstable set** of x defined by

$$W^s(x) = \{y \in X : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\},$$
$$W^u(x) = \{y \in X : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty\},$$

respectively. Then, if $f_{|\Omega(f)}: \Omega(f) \to \Omega(f)$ is expansive and has the shadowing property, and if $B_1, \ldots, B_\ell$ are all basic sets, we have

$$X = \bigcup_{i=1}^{\ell} W^s(B_i) = \bigcup_{i=1}^{\ell} W^u(B_i).$$

For $x, y \in X$ and $\alpha > 0$, x is **α-related** to y if there are α-pseudo orbits of f such that $x_0 = x, x_1, \ldots, x_k = y$ and $y_0 = y, y_1, \ldots, y_\ell = x$. If x is α-related to y for any $\alpha > 0$, then x is **related** to y (written $x \sim y$). The set $CR(f) = \{x \in X: x \sim x\}$ is said to be the **chain recurrent set** of f. The relation $\sim$ is an equivalence relation in $CR(f)$. It is clear that $CR(f) = f(CR(f))$ and $\Omega(f) \subset CR(f)$. Moreover, $CR(f)$ is closed in X. If $f_{|CR(f)}: CR(f) \to CR(f)$ is expansive and has the shadowing property, then we have that $\Omega(f) = CR(f)$. Also, if $f: X \to X$ has the shadowing property, then $\Omega(f) = CR(f)$. The chain recurrent set of $f|_{CR(f)}$ coincides with $CR(f)$ (C. Robinson [52]), however it does not hold in general for nonwandering sets. Let 2^X be the family of all nonempty closed subsets of X. The metric ρ defined by

$$\rho(A, B) = \max\Big\{\sup_{b \in B} d(A, b),\ \sup_{a \in A} d(a, B)\Big\},$$
$$A, B \in 2^X$$

where $d(A, b) = \inf\{d(a, b)\colon a \in A\}$, is called the **Hausdorff metric** for 2^X. Let $\mathrm{Hom}(X, X)$ be the space of all homeomorphisms of X with the C^0 topology. Then the map from $\mathrm{Hom}(X, X)$ to 2^X defined by $f \mapsto CR(f)$ is ***upper semi-continuous***.

For a point x in X, the **α-limit set** of x, denoted by $\alpha(x)$, is defined as the set consisting of the points $y \in X$ such that $y = \lim_{j \to \infty} f^{n_j}(x)$ for some strictly decreasing sequence of integers n_j. The **ω-limit set** of x, denoted by $\omega(x)$, is similarly defined for strictly increasing sequence. A point $x \in X$ is said to be **recurrent** for a homeomorphism f if, for any neighbourhood U of x, there exist infinitely many n with $f^n(x) \in U$, which is equivalent to $x \in \alpha(x) \cap \omega(x)$. Every recurrent point belongs to $\Omega(f)$. The set of all recurrent points is called the **recurrent set**, and is denoted by $R(f)$. The notion of a chain recurrent point is a generalization of the notion of a recurrent point. The closure of $R(f)$, $c(f)$, is called the **Birkhoff center**. Let $\Omega_1 = \Omega(f)$ and for $i \geqslant 1$ let Ω_{i+1} be the nonwandering set of $f_{|\Omega_i}: \Omega_i \to \Omega_i$. Then $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_i \supset \cdots$ and $\bigcap_{i \geqslant 1} \Omega_i \supset c(f)$.

A closed f-invariant subset E of X is an **isolated invariant set** if there is a neighbourhood U of E such that $\bigcap_{-\infty}^{\infty} f^n(U) = E$. If $f: \Omega(f) \to \Omega(f)$ is expansive and has the shadowing property, then $\Omega(f)$ is isolated and so are all the B_i's in the spectral decomposition theorem stated before. An isolated set, Λ, is called a **basic set** for f if $f_{|\Lambda}: \Lambda \to \Lambda$ is topologically transitive.

Let Λ be a basic set for f and let U be a compact neighbourhood of Λ such that $\bigcap_{-\infty}^{\infty} f^n(U) = \Lambda$. Define

$$W^s_{\mathrm{loc}}(\Lambda, U) = \bigcap_{n \geqslant 0} f^{-n}(U)$$

and

$$W^u_{\mathrm{loc}}(\Lambda, U) = \bigcap_{n \geqslant 0} f^{n}(U).$$

Then we have

$$W^s(\Lambda)' = \bigcup_{n \geqslant 0} f^{-n}\big(W^s_{\mathrm{loc}}(\Lambda, U)\big)$$

and

$$W^u(\Lambda)' = \bigcup_{n \geqslant 0} f^{n}\big(W^u_{\mathrm{loc}}(\Lambda, U)\big),$$

where

$$W^s(\Lambda)' = \{x \in X : d(f^n(x), \Lambda) \to 0 \text{ as } n \to \infty\},$$
$$W^u(\Lambda)' = \{x \in X : d(f^{-n}(x), \Lambda) \to 0 \text{ as } n \to \infty\}.$$

If $f:\Omega(f)\to\Omega(f)$ is expansive and has the shadowing property and the B_i's are the basic sets, it follows that $W^\sigma(B_i)'=W^\sigma(B_i)$ for $\sigma=s,u$.

Let $\{\Lambda_1,\dots,\Lambda_\ell\}$ be a finite family of basic sets. We say that the family has no cycles if there is no sequence $\Lambda_{i_1},\dots,\Lambda_{i_k}$ of distinct members such that $\Lambda_{i_1}=\Lambda_{i_k}$ and

$$\big(W^u(\Lambda_{i_j})'\setminus\Lambda_{i_j}\big)\cap\big(W^s(\Lambda_{i_{j+1}})'\setminus\Lambda_{i_{j+1}}\big)\neq\emptyset$$

for $1\leqslant j<k$. If $\{\Lambda_1,\dots,\Lambda_\ell\}$ is a family of disjoint isolated sets for f having no cycles such that $\alpha(x)\subset\bigcup_{i=1}^\ell\Lambda_i$ for all $x\in X$, then there exist compact sets

$$\emptyset=X_0\subset X_1\subset\cdots\subset X_\ell=X$$

such that $f(X_i)$ is contained in the ***interior*** of X_i and

$$\bigcap_{n=-\infty}^{\infty}f^n(X_i\setminus X_{i-1})=\Lambda_i,$$

and $\Omega(f)\subset\bigcup_{i=1}^\ell\Lambda_i$. The above sequence $\emptyset=X_0\subset X_1\subset\cdots\subset X_\ell=X$ is called the **filtration**. I. Malta [29] has shown that if the Birkhoff center $c(f)$ is an isolated set for f and admits a decomposition $c(f)=\Lambda_1\cup\cdots\cup\Lambda_\ell$ into basic sets having no cycles, then $c(f)=\Omega(f)$.

We say that f is a **minimal homeomorphism** if, for all $x\in X$, the orbit $O_f(x)=\{f^n(x),\ n\in\mathbb{Z}\}$ is dense in X. Obviously every minimal homeomorphism is topologically transitive. A homeomorphism f is minimal if and only if $E=\emptyset$ or X whenever $f(E)=E$ and E is closed. A nonempty f-invariant closed subset Λ is a **minimal set** if $f_{|\Lambda}:\Lambda\to\Lambda$ is minimal. Since X is compact, by Zorn's Lemma it follows that there is at least one minimal set for any homeomorphism $f:X\to X$.

For a continuous surjection $f:X\to X$, a point $x\in X$ is said to be a **nonwandering point** if for any open neighbourhood U of x there is $n>0$ such that $f^n(U)\cap U\neq\emptyset$. Denote by $\Omega(f)$ the set of all nonwandering points of f ($\Omega(f)$ is called the **nonwandering set** of f). Then $\Omega(f)$ is a nonempty closed subset and $f(\Omega(f))\subset\Omega(f)$. If f has the shadowing property, then $f(\Omega(f))=\Omega(f)$, and moreover $f_{|\Omega(f)}:\Omega(f)\to\Omega(f)$ also has the shadowing property. The **chain recurrent set** of a continuous surjection f, $CR(f)$, is defined in the same way as for a homeomorphism. Then $f(CR(f))\subset CR(f)$ and $CR(f)$ is closed. If f has the shadowing property, then $\Omega(f)=CR(f)$. Define the **ω-limit set** $\omega(x)$ of a point x for f as in the case of homeomorphisms, and denote by $c(f)$ the closure of $R(f)=\{x\in X: x\in\omega(x)\}$. Then $c(f)$ is called the **Birkhoff center** for the continuous surjection. The set $R(f)$ is the **recurrent set**. The **minimal set** of the continuous surjection f is also defined in the same way as given for a homeomorphism. Let $\sigma:\varprojlim(X,f)\to\varprojlim(X,f)$ be the inverse limit system for f. Then the following properties hold:

(1) if $n>0$ is an integer and F_n is the set of periodic points of f with period n, then $\varprojlim(F_n,f)$ is the set of periodic points of σ with period n and the natural projection $p_0:\varprojlim(F_n,f)\to F_n$ defined by $(x_i)\mapsto x_0$ is bijective,
(2) if $x\in X$ and $\mathbf{x}=(\dots,x,x_1,x_2,\dots)\in\varprojlim(X,f)$, then $\omega(\mathbf{x},\sigma)=\varprojlim(\omega(x),f)$ is the ω-limit set for σ,
(3) $c(\sigma)=\varprojlim(c(f),f)$ is the Birkhoff center for σ,
(4) $\Omega(\sigma)=\varprojlim(\Omega(f),f)$ is the nonwandering set for σ,
(5) $CR(\sigma)=\varprojlim(CR(f),f)$ is the chain recurrent set for σ,
(6) if A is a minimal subset of X, then the inverse limit system $\sigma:\varprojlim(A,f)\to\varprojlim(A,f)$ is minimal, and if $\mathbf{A}$ is a minimal subset of $\sigma:\varprojlim(X,f)\to\varprojlim(X,f)$, then $p_0(\mathbf{A})$ is a minimal subset of $f:X\to X$, where $p_0:\varprojlim(X,f)\to X$ is the natural projection to the 0th coordinate,
(7) $f:X\to X$ is topologically transitive if and only if $\sigma:\varprojlim(X,f)\to\varprojlim(X,f)$ is topologically transitive, and
(8) $f:X\to X$ is topologically mixing if and only if $\sigma:\varprojlim(X,f)\to\varprojlim(X,f)$ is topologically mixing.

7. Expansive behaviors on compact metric spaces

Let $f:X\to X$ be a homeomorphism of a compact metric space with metric d. A finite open cover α of X is a **generator** (respectively **weak generator**) for f if for every two-sided sequence (A_n) of members of α, $\bigcap_{-\infty}^{\infty}f^{-n}(\mathrm{cl}(A_n))$ is at most one point (respectively $\bigcap_{-\infty}^{\infty}f^{-n}(A_n)$ is at most one point). Here $\mathrm{cl}(A)$ is the closure of A. The following are equivalent; (1) f is expansive, (2) f has a generator, (3) f has a weak generator. It can be shown in general that if $f:X\to X$ is an expansive homeomorphism, then there exist an integer $k>0$ and a closed subset Ω of $Y_k^{\mathbb{Z}}$ such that $\sigma(\Omega)=\Omega$ (σ is the shift map of $Y_k^{\mathbb{Z}}$) and a continuous surjection $\pi:\Omega\to X$ such that $f\circ\pi=\pi\circ\sigma$.

It is known that every compact metric space does not necessarily admit an expansive homeomorphism. For instance, a closed arc and also the unit circle do not admit an expansive homeomorphism (J. Jacobsen and W. Utz [23]). K. Hiraide [20] and J. Lewowicz [27] have shown independently that there exit no expansive homeomorphisms on the 2-dimensional sphere S^2, the projective plane P^2 or the Klein bottle K^2. On the other hand, every compact orientable surface except S^2 admits expansive homeomorphisms (T. O'Brien and W. Reddy [39]). The **topological dimension** of a space X is said to be less than n if for all $\gamma>0$ there exists a ***cover*** α of X by open sets with diameter $<\gamma$ such that each point belongs to at most $n+1$ of the sets from α. R. Mañé [31] has shown that if $f:X\to X$ is expansive, then the topological dimension of X is finite, and moreover that if, in addition, $f:X\to X$ is minimal, then the dimension of X must be zero.

Let $x \in X$ and $\varepsilon > 0$. We define the **local stable set** and **local unstable set** of x by

$$W_\varepsilon^s(x,d) = \{y \in X: d(f^n(x), f^n(y)) \leqslant \varepsilon,\ \forall n \geqslant 0\},$$
$$W_\varepsilon^u(x,d) = \{y \in X: d(f^{-n}(x), f^{-n}(y)) \leqslant \varepsilon,\ \forall n \geqslant 0\}.$$

It is easily checked that f is an expansive homeomorphism with expansive constant c if and only if there is some $c > 0$ such that for any $\varepsilon > 0$ there exists $N > 0$ such that for all $x \in X$ and $n \geqslant N$

$$f^n(W_c^s(x,d)) \subset W_\varepsilon^s(f^n(x), d) \quad \text{and}$$
$$f^{-n}(W_c^n(x,d)) \subset W_\varepsilon^u(f^{-n}(x), d).$$

This implies that if f is an expansive homeomorphism with expansive constant c, then for $0 < \varepsilon < c$

$$W^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(W_\varepsilon^s(f^n(x), d)),$$
$$W^u(x) = \bigcup_{n=0}^{\infty} f^{n}(W_\varepsilon^u(f^{-n}(x), d)).$$

W. Reddy [49] has shown that if f is expansive then there exists a compatible metric D for X and constants $r > 0$, $C > 0$, $0 < \lambda < 1$ such that

(1) if $y \in W_r^s(x, D)$ then for all $n \geqslant 0$

$$D(f^n(x), f^n(y)) \leqslant C\lambda^n D(x,y),$$

(2) if $y \in W_r^u(x, D)$ then for all $n \geqslant 0$

$$D(f^{-n}(x), f^{-n}(y)) \leqslant C\lambda^n D(x,y).$$

Such a metric D is called the **hyperbolic metric** for f and λ is called the **shewness** of f.

A continuous surjection $f: X \to X$ is called **positively expansive** if there is a constant $e > 0$ such that if $x \neq y$ then $d(f^n(x), f^n(y)) > e$ for some nonnegative integer n, where e is called an **expansive constant** for f. This notion does not depend on the choice of the metric compatible with the topology of X. A toral endomorphism of an n-torus $\mathbb{T}^n$ is positively expansive if and only if it has only eigenvalues whose absolute values are greater than one. An expanding differentiable map of a compact smooth manifold is positively expansive. If $f: X \to X$ is positively expansive, then the inverse limit system $\sigma: \varprojlim(X, f) \to \varprojlim(X, f)$ is expansive. W. Reddy [48] has shown that if $f: X \to X$ is a positively expansive map, then there exist a compatible metric D and constants $\delta > 0$, $\lambda > 1$ such that for $x, y \in X$

$$D(x,y) \leqslant \delta \implies D(f(x), f(y)) \geqslant \lambda D(x,y).$$

Such a metric D is also called a hyperbolic metric for f. Using this fact we have that (1) if a homeomorphism of a compact metric space is positively expansive, then the space is a set consisting of finite points, (2) the closed interval $I = [0, 1]$ does not admit positively expansive maps, and (3) if $f: X \to X$ is positively expansive and open and if X is connected, then f has at least one fixed point in X. For the problem of whether a positively expansive map is open, we can construct an example. Consider the subset X of the plane defined by

$$X = \{z: |z| = 1\} \cup \{z: |z - 3/2| = 1/2\} \cup \{z: |z + 3/2| = 1/2\}.$$

Give X the arc length metric, and define a map f as follows; stretch each of the small circles onto the big circle, stretch each of upper and lower semicircles of the big circle first around a small circle, and then across the other semicircle and finally around the other small circle. More precisely we describe f by

$$f(z) = \begin{cases} 2(z - 1/2) & \text{if } Re(z) > 1, \\ 2(z + 1/2) & \text{if } Re(z) \leqslant -1, \\ (1/2)z^6 - 3/2 & \text{if } 1/2 \leqslant Re(z) \leqslant 1, \\ z^3 & \text{if } -1/2 \leqslant Re(z) \leqslant 1/2, \\ (-1/2)z^6 + 3/2 & \text{if } -1 \leqslant Re(z) \leqslant -1/2. \end{cases}$$

Then f is positively expansive, but not open.

8. Markov partitions

Let $f: X \to X$ be a homeomorphism of a compact metric space with metric d. Let $\Delta(\varepsilon) = \{(x,y): d(x,y) \leqslant \varepsilon\}$ for $\varepsilon > 0$. Then we say that f has the **local product structure** if the following conditions (A) and (B) are satisfied:

(A) There is a $\delta_0 > 0$ and a continuous map $[\ ,\ ]: \Delta(\delta_0) \to X$ such that for $x, y, z \in X$

$$[x,x] = x, \qquad [[x,y],z] = [x,z],$$
$$[x,[y,z]] = [x,z],$$
$$f([x,y]) = [f(x), f(y)]$$

when the two sides of these relations are defined.

(B) There exist $0 < \delta_1 < \delta_0/2$ and $0 < \rho < \delta_1$ such that for each $x \in X$, the sets

$$V_{\delta_1}^u(x) = \{y \in W_{\delta_0}^u(x,d): d(x,y) < \delta_1\},$$
$$V_{\delta_1}^s(x) = \{y \in W_{\delta_0}^s(x,d): d(x,y) < \delta_1\},$$
$$N_x = [V_{\delta_1}^u(x), V_{\delta_1}^s(x)],$$

satisfy the following conditions; (a) N_x is an open set of X and $\operatorname{diam}(N_x) < \delta_0$, (b) $[\ ,\ ]: V_{\delta_1}^u(x) \times V_{\delta_1}^s(x) \to N_x$ is a homeomorphism, and (c) $N_x \supset B_\rho(x)$ where $B_\rho(x) = \{y \in X: d(x,y) \leqslant \rho\}$.

If $f : X \to X$ has the shadowing property, for $\delta > 0$ small we have $W^s_\delta(x, d) \cap W^u_\delta(y, d) \neq \emptyset$ whenever x is very near to y. If, in addition, f is expansive, then we see easily that $W^s_\delta(x, d) \cap W^u_\delta(y, d)$ is a set consisting of a single point and it is denoted by $[x, y]$. For such a homeomorphism the existence of the local product structure is ensured.

Suppose that a homeomorphism $f : X \to X$ has the local product structure. A subset R of X is called a **rectangle** if $\operatorname{diam}(R) \leqslant \rho$ and $[x, y] \in R$ for $x, y \in R$. A rectangle R is said to be proper if $R = \operatorname{cl}(\operatorname{int}(R))$, where $\operatorname{int}(R)$ is the interior of R in X. It can be shown that the homeomorphism has a **Markov partition** which is defined as a finite cover $\{R_1, \ldots, R_m\}$ of X such that (a) each R_i is a proper rectangle, (b) $\operatorname{int}(R_i) \cap \operatorname{int}(R_j) = \emptyset$ for $i \neq j$, and (c) if $x \in \operatorname{int}(R_i) \cap f^{-1} \operatorname{int}(R_j)$, then

$$f\bigl(V^s(x, R_i)\bigr) \subset V^s\bigl(f(x), R_j\bigr),$$
$$f\bigl(V^u(x, R_i)\bigr) \supset V^u\bigl(f(x), R_j\bigr)$$

where $V^s(x, R_i) = V^s_{\delta_1}(x) \cap R_i$ and $V^u(x, R_i) = V^u_{\delta_1}(x) \cap R_i$ (Y. Sinai [58], R. Bowen [9, 11], D. Ruelle [55], K. Hiraide [19], and M. Dateyama [16]).

Let $f : X \to X$ be a homeomorphism such that $f_{|\Omega(f)} : \Omega(f) \to \Omega(f)$ is expansive and has the shadowing property. By the spectral decomposition theorem stated before, $\Omega(f)$ can be decomposed into the union $\Omega(f) = \bigcup_{s=1}^{\ell} B_s$ of basic sets. Let B_s be one of the basic sets. Then, there exists a Markov partition $\mathcal{R} = \{R_1, \ldots, R_m\}$ of B_s. Define the transition matrix $A = A(\mathcal{R})$ by

$$A_{ij} = \begin{cases} 1 & \text{if } \operatorname{int}(R_i) \cap f^{-1}\bigl(\operatorname{int}(R_j)\bigr) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let Σ_A be the compact subset of $Y_m^{\mathbb{Z}}$ defined by

$$\Sigma_A = \bigl\{x = (x_i) \colon A_{x_i x_{i+1}} = 1 \text{ for } i \in \mathbb{Z}\bigr\}$$

and $\sigma : \Sigma_A \to \Sigma_A$ the shift map defined by $\sigma((x_i)) = (x_{i+1})$. For each $a = (a_i) \in \Sigma_A$, the set $\bigcap\{f^{-j}(R_{a_j}) \colon j \in \mathbb{Z}\}$ consists of a single point which is denoted by $\pi(a)$. The map $\pi : \Sigma_A \to B_s$ is a continuous surjection such that the diagram

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\ \sigma\ } & \Sigma_A \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B_s & \xrightarrow[\ f\ ]{} & B_s \end{array} \quad \text{commutes,}$$

and π is injective on a Baire set $Y = B_s \setminus \bigcup\{f^j(\partial\mathcal{R}) \colon j \in \mathbb{Z}\}$, where $\partial\mathcal{R}$ is the union of the ***boundaries*** of the R_i's. Thus, the behavior of the homeomorphism f on B_s is represented by the symbolic dynamics $\sigma : \Sigma_A \to \Sigma_A$. The shift map $\sigma : \Sigma_A \to \Sigma_A$ is topologically transitive. If $f_{|B_s}$ is topologically mixing, so is $\sigma : \Sigma_A \to \Sigma_A$. R. Bowen [10] has shown that there exists an integer d such that $\pi : \Sigma_A \to B_s$ is at most a d-to-one map, i.e., $\operatorname{card}(\pi^{-1}(x)) \leqslant d$ for all $x \in B_s$.

9. Stability

Let M be a compact smooth manifold and $f : M \to M$ a differentiable map. The set $A(f) = \bigcap_{n \geqslant 0} f^n(M)$ is the maximal closed f-invariant set, i.e., $f(A(f)) = A(f)$. If Λ is a closed invariant set ($f(\Lambda) = \Lambda$) then Λ is a subset of $A(f)$, and $f_{|\Lambda} : \Lambda \to \Lambda$ is a local diffeomorphism if $\Lambda \cap S(f) = \emptyset$, where $S(f)$ is the set of all critical points. As before let

$$\varprojlim(M, f) = \bigl\{(x_i) \colon x_i \in A(f),\ f(x_i) = x_{i+1},\ i \in \mathbb{Z}\bigr\}$$

be the inverse limit. A closed invariant set Λ is said to be a **hyperbolic set** of f if there exist constant $C > 0$ and $0 < \lambda < 1$ such that for every $\mathbf{x} = (x_i) \in \varprojlim(\Lambda, f) = \{(x_i) \colon x_i \in \Lambda \text{ and } f(x_i) = x_{i+1}, i \in \mathbb{Z}\}$ there is a splitting

$$T_{\mathbf{x}} M = \bigcup_{i \in \mathbb{Z}} T_{x_i} M = \bigcup_{i \in \mathbb{Z}} E^s_{x_i} \oplus E^u_{x_i},$$

which is left invariant by the derivative Df (the phenomenon that $Df(E^s_{x_i}) \subset E^s_{x_{i+1}}$ but $Df(E^s_{x_i}) \neq E^s_{x_{i+1}}$ is here permitted), so that for all $n \geqslant 0$

$$\bigl\|D_{x_i} f^n(v)\bigr\| \leqslant C\lambda^n \|v\| \quad \text{if } v \in E^s_{x_i},$$
$$\bigl\|D_{x_i} f^n(v)\bigr\| \geqslant C^{-1}\lambda^{-n} \|v\| \quad \text{if } v \in E^u_{x_i}.$$

If, in particular, $T_{\mathbf{x}} M = \bigcup_i E^u_{x_i}$ for all $\mathbf{x} = (x_i) \in \varprojlim(\Lambda, f)$, then $f_{|\Lambda} : \Lambda \to \Lambda$ is said to be **expanding**. In this case, by definition, it follows that $\Lambda \cap S(f) = \emptyset$. If the entire space M is a hyperbolic set of f and if $S(f) = \emptyset$, then $f : M \to M$ is an Anosov endomorphism. It can be shown that if Λ is a hyperbolic set of f then the inverse limit system $\sigma : \varprojlim(\Lambda, f) \to \varprojlim(\Lambda, f)$ is expansive, and has the shadowing property provided that $\varprojlim(\Lambda, f)$ is isolated. When a hyperbolic invariant set Λ is a periodic orbit of some point p, the point p is called a **hyperbolic periodic point**.

Here we restrict ourselves to the case of diffeomorphisms. A diffeomorphism $f : M \to M$ is said to satisfy **Axiom A** if the set $\operatorname{Per}(f)$ of all periodic points is dense in the nonwandering set $\Omega(f)$ and if $\Omega(f)$ is a hyperbolic set of f. In this case $f_{|\Omega(f)} : \Omega(f) \to \Omega(f)$ is expansive and has the shadowing property. The spectral decomposition theorem says that if f satisfies Axiom A then $\Omega(f)$ can be written as the finite disjoint union $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_s$ of closed invariant sets Λ_i such that each $f_{|\Lambda_i}$ is topologically transitive and each Λ_i is a basic set. As before, for $x \in M$ the stable and unstable manifolds are given by

$$W^s(x) = \bigl\{y \in M \colon d\bigl(f^n(x), f^n(y)\bigr) \to 0 \text{ as } n \to \infty\bigr\},$$
$$W^u(x) = \bigl\{y \in M \colon d\bigl(f^{-n}(x), f^{-n}(y)\bigr) \to 0 \text{ as } n \to \infty\bigr\}.$$

The stable manifold theorem ensures that if f satisfies Axiom A then both $W^s(x)$ and $W^u(x)$ are actually manifolds each of which is an injective immersion of some Euclidean space, and $M = \bigcup_{x \in \Omega(f)} W^\sigma(x)$ for $\sigma = s, u$. An Axiom A diffeomorphism $f : M \to M$ is said to satisfy the

strong transversality if, for $x \in M$, the stable manifold $W^s(x)$ and the unstable manifold $W^u(x)$ are **transverse**, i.e., $T_yM = T_yW^s(x) + T_yW^u(x)$ if $y \in W^s(x) \cap W^u(x)$. From Smale's theorem it follows that f has no cycles with respect to $\Omega(f)$ if it satisfies the strong transversality. In this case $CR(f) = \Omega(f)$. By the works of J. Robbin [50] and C. Robinson [51] it has been shown that if a diffeomorphism $f : M \to M$ of class C^1 satisfies Axiom A and satisfies the strong transversality then f is structurally stable, i.e., there is a neighbourhood $N(f)$ of f in $\mathrm{Diff}^1(M, M)$ such that if a diffeomorphism $g : M \to M$ is in $N(f)$ then g is topologically conjugate to f. In 1988 R. Mañé [32] has proved that the converse is true, that is, if a diffeomorphism $f : M \to M$ is structurally stable in $\mathrm{Diff}^1(M, M)$, then f must satisfy Axiom A and satisfy the strong transversality. It is still open the problem of whether the C^r structural stability, $r \geqslant 2$, implies Axiom A or not.

The interior of the set of all diffeomorphisms having the shadowing property in $\mathrm{Diff}^1(M, M)$ with respect to the C^1 topology is equal to the set of Axiom A diffeomorphisms satisfying the strong transversality (K. Moriyasu [34], K. Sakai [57]), while a diffeomorphism belonging to the interior of the set of all expansive diffeomorphisms in $\mathrm{Diff}^1(M, M)$ satisfies Axiom A (R. Mañé [30]).

It is known that the set of diffeomorphisms $f : M \to M$ satisfying the structural stability is not dense in $\mathrm{Diff}^1(M, M)$ if the dimension of M is greater than 2. However, a diffeomorphism satisfying the following conditions;

(1) all periodic points of f are hyperbolic, and
(2) given all the pairs (p, q) of periodic points, the stable manifold $W^s(p)$ and the unstable manifold $W^u(q)$ are transverse,

is called a **Kupka–Smale diffeomorphism**, and the set of all Kupka–Smale diffeomorphisms of class C^r is a **residual subset** of $\mathrm{Diff}^r(M, M)$ in any dimension for all $r \geqslant 1$, that is, a subset which can be written as the intersection of countable-many open dense subsets of $\mathrm{Diff}^r(M, M)$. Since $\mathrm{Diff}^r(M, M)$ is a Baire space, we remark that a residual subset of $\mathrm{Diff}^r(M, M)$ is dense in $\mathrm{Diff}^r(M, M)$. For a hyperbolic periodic point p of a diffeomorphism f, a point in the intersection of the stable manifold $W^s(p)$ and the unstable manifold $W^u(p)$, $x \in W^s(p) \cap W^u(p) \setminus \{p\}$, is called a **homoclinic point**. Smale's theorem says that if x is a transverse homoclinic point then in a neighbourhood of x there is a closed subset Λ invariant under f^N for some $N > 0$ such that $f^N : \Lambda \to \Lambda$ is topologically conjugate to the shift map of $Y_k^{\mathbb{Z}}$ for some $k > 0$.

For $f \in \mathrm{Diff}^r(M, M)$, we denote the set of isolated periodic points of period $\leqslant n$ (isolated fixed points of f^n) by

$$P_n(f) = \{x \in M : x = f^n(x) \text{ and } x \text{ is isolated}\}.$$

M. Artin and B. Mazur [7] have shown that there exists a dense set $\mathcal{D}$ in $\mathrm{Diff}^r(M, M)$ $(r \geqslant 1)$ such that for any $f \in \mathcal{D}$ the number $\sharp P_n(f)$ grows at most exponentially with n, that is, for some $C > 0$

$$\sharp P_n(f) \leqslant \exp(Cn) \quad \text{for } n \geqslant 0.$$

We can give the notion of Axiom A for a differentiable map in a similar fashion. Then every differentiable map satisfying Axiom A yields the spectral decomposition theorem. Thus there are no cycle conditions for differentiable maps. However we cannot find in the literature any definition of strong transversality for differentiable map. This seems likely due to the obstruction caused by critical points. In particular let S^1 be the unit circle and $R^1(S^1, S^1)$ the set of regular maps of S^1 endowed with the C^1 topology. Then the set of maps in $R^1(S^1, S^1)$ satisfying structural stability coincides with the set of maps in $R^1(S^1, S^1)$ satisfying Axiom A and the no cycle condition. Recently, N. Aoki, K. Moriyasu and N. Sumi [6] have shown that the interior of the set of maps f satisfying the following conditions (1) and (2) in $C^1(M, M)$ with respect to the C^1 topology; (1) all periodic points are hyperbolic and (2) each critical point p belonging to the nonwandering set is a **sink**, i.e., there exists a neighbourhoods U of p such that $\bigcap_{n \geqslant 0} f^n(U) = \{p\}$, coincides with the set of Axiom A maps having no cycles.

Let X be a compact metric space with metric d. A homeomorphism $f : X \to X$ is said to be **topologically stable in the class of homeomorphisms** if for $\varepsilon > 0$ there is $\delta > 0$ such that for a homeomorphism $g : X \to X$ with $d(f(x), g(x)) < \delta$ for all x there is a continuous map $h : X \to X$ so that $h \circ g = f \circ h$ and $d(h(x), x) < \varepsilon$ for all x. P. Walters [65] has shown that if a homeomorphism $f : X \to X$ is expansive and has the shadowing property, then f is topologically stable in the class of homeomorphisms. Conversely, if a homeomorphism $f : M \to M$ of a compact topological manifold M is topologically stable in the class of homeomorphisms, then the set of all periodic points of f, $\mathrm{Per}(f)$, is dense in $\Omega(f)$. Moreover, if in addition the dimension of M is greater than one, then the homeomorphism f has the shadowing property (P. Walters [65]). This is also true for the one-dimensional case and there exists a homeomorphism of the circle which has shadowing property, but is not topologically stable (K. Yano [68, 69]). It remains a problem, whether or not, a homeomorphism $f : M \to M$ of a compact topological manifold which is topologically stable in the class of homeomorphisms has the property that the restriction $f_{|\Omega(f)} : \Omega(f) \to \Omega(f)$ is expansive.

10. Topological entropy

We first define the topological entropy by open covers. All logarithms will be to the base "e". Let X be a compact ***topological space***. Open covers of X are denoted by α, $\beta, \ldots$. The **join of two covers** $\alpha \vee \beta$ is given by $\alpha \vee \beta = \{A \cap B : A \in \alpha,\ B \in \beta\}$. β is a **refinement** of α (written $\alpha \leqslant \beta$) if every member of β is a subset of some member of α. Thus $\alpha \leqslant \alpha \vee \beta$ and $\beta \leqslant \alpha \vee \beta$. If $f: X \to X$ is a continuous map, then $f^{-1}(\alpha) = \{f^{-1}(A)\colon A \in \alpha\}$ is an open cover of X. It is clear that $f^{-1}(\alpha \vee \beta) = f^{-1}(\alpha) \vee f^{-1}(\beta)$ and $f^{-1}(\alpha) \leqslant f^{-1}(\beta)$ if $\alpha \leqslant \beta$. Let $N(\alpha)$ be the number of members in a finite subcover of α with the smallest cardinality. We define the **entropy** of α by $H(\alpha) = \log N(\alpha)$. If $\{a_n\}_{n \geqslant 1}$ satisfies $a_n \geqslant 0$ and $a_{n+m} \leqslant a_n + a_m$ for $n, m \geqslant 1$, then $\lim a_n/n = \inf a_n/n$ exists. Thus, for every continuous map f, $\lim (1/n) H(\bigvee_0^{n-1} f^{-i}(\alpha))$ exists. The **topological entropy** of f is given by

$$h(f) = \sup_\alpha \lim_n \frac{1}{n} H\left(\bigvee_0^{n-1} f^{-i}(\alpha)\right)$$

where the supremum is taken over all open covers α of X.

Let X_1 and X_2 be compact spaces. If $f_i: X_i \to X_i$ are continuous maps and topologically conjugate, then $h(f_1) = h(f_2)$. If, in particular, $f: X \to X$ is a homeomorphism, then $h(f) = h(f^{-1})$.

We give another definition of topological entropy by R. Bowen [11]. Let (X, d) be a metric space and let $x \in X$. Let $f: X \to X$ be a ***uniformly continuous*** map, and let $n > 0$, $\varepsilon > 0$. If $K \subset X$, then a subset F of X is said to (n, ε)-**span** K with respect to f if for $x \in K$ there is $y \in F$ such that

$$\max\{d(f^i(x), f^i(y))\colon 0 \leqslant i \leqslant n-1\} \leqslant \varepsilon.$$

For K compact let $r_n(\varepsilon, K)$ be the smallest cardinality of (n, ε)-spanning sets for K with respect to f. Then $r_n(\varepsilon, K) < \infty$ and thus we write

$$\bar{r}_f(\varepsilon, K) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon, K).$$

A set $E \subset X$ is (n, ε)-**separated** with respect to f if $x, y \in E$ $(x \neq y)$ then

$$\max\{d(f^i(x), f^i(y))\colon 0 \leqslant i \leqslant n-1\} > \varepsilon.$$

For K compact let $s_n(\varepsilon, K)$ denote the largest cardinality of (n, ε)-separated subsets of K with respect to f. Then $s_n(\varepsilon, K) < \infty$. We set

$$\bar{s}_f(\varepsilon, K) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, K).$$

If

$$h(f, K) = \lim_{\varepsilon\to 0} \bar{r}_f(\varepsilon, K) = \lim_{\varepsilon\to 0} \bar{s}_f(\varepsilon, K),$$

then we define

$$h_d(f) = \sup\{h(f, K)\colon K \text{ is compact}\}.$$

Let d and d' be ***uniformly equivalent***, and f be uniformly continuous. Then $h_d(f) = h_{d'}(f)$. If X is compact and $h(f)$ is the topological entropy defined by open covers, then $h_d(f) = h(f)$. Also we have

$$h(f) = \lim \frac{1}{n} \log r_n(\delta, X) = \lim \frac{1}{n} \log s_n(\delta, X)$$

when X is compact and f is expansive.

We say that a homeomorphism f of a compact metric space X satisfies the **specification** if for $\varepsilon > 0$ there is an $M = M(\varepsilon) > 0$ such that for a finite sequence $x_1, x_2, \ldots, x_n \in X$, integers $a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_n \leqslant b_n$ with $a_j - b_{j-1} \geqslant M$ $(2 \leqslant j \leqslant n)$ and $p > M(b_n - a_1)$, there is a periodic point $x \in X$ with period p such that $d(f^i(x), f^{i-a_j}(x_j)) < \varepsilon$ $(a_j \leqslant i \leqslant b_j,\ 1 \leqslant j \leqslant n)$. If $f: X \to X$ has the specification, then it is topologically mixing, and when f is topologically mixing, expansive and has shadowing property, f satisfies the specification. If f has the specification, then $h(f) > 0$.

Denote by $P_n(f)$ the set of all periodic points with period $\leqslant n$, and by $\sharp E$ the cardinality of E. If f is expansive and satisfies the specification, then

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log \sharp P_n(f).$$

It was asked by Bowen in 1978 whether the property that

$$h(f) = \limsup_{n\to\infty} \frac{1}{n} \sharp P_n(f) \qquad (*)$$

is C^r**-generic** in the space $\mathrm{Diff}^r(M, M)$ of all diffeomorphisms with the C^r topology (that is, the set of all diffeomorphisms having $(*)$ is a residual set in $\mathrm{Diff}^r(M, M)$). Recently, it was answered in V. Kaloshin [17] that the property of having $(*)$ is not C^r-generic $(2 \leqslant r < \infty)$.

11. Topological pressure

Let X be a compact metric space, and denote by $C(X, \mathbb{R})$ the set of all real-valued continuous functions of X. A norm for $C(X, \mathbb{R})$ is given by $\|\varphi\| = \max\{|\varphi(x)|\colon x \in X\}$ for $\varphi \in C(X, \mathbb{R})$. Such a norm is called the **uniform norm**. Let $f: X \to X$ be a continuous map. For $\varphi \in C(X, \mathbb{R})$ define

$$S_n\varphi(x) = \sum_{i=0}^{n-1} \varphi(f^i x) \quad (\text{for } n \geqslant 1),$$

and for $\varepsilon > 0$

$$Q_n(f, \varphi, \varepsilon) = \inf\left\{\sum_{x\in F} e^{S_n\varphi(x)}\colon F \text{ is an } (n, \varepsilon)\text{-spanning set}\right\}.$$

We set

$$Q(f, \varphi, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log Q_n(f, \varphi, \varepsilon).$$

Then

$$P(f,\varphi)=\lim_{\varepsilon\to 0} Q(f,\varphi,\varepsilon)$$

is called the **topological pressure** with respect to f on $C(X,\mathbb{R})$ (P. Walters [64]). $P(f,0)$ is consistent with the topological entropy, i.e., $P(f,0)=h(f)$. $P(f,\cdot)$ has the following properties; for $\varphi,\psi\in C(X,\mathbb{R})$, $\varepsilon>0$ and $c\in\mathbb{R}$

(1) if $\varphi\leqslant\psi$ then $P(f,\varphi)\leqslant P(f,\psi)$,
(2) $0\leqslant P(f,0)\leqslant\infty$,
(3) if $P(f,0)<\infty$, then $P(f,p\varphi+(1-p)\psi)\leqslant pP(f,\varphi)+(1-p)P(f,\psi)$ for $0<p<1$ (that is, $P(f,\cdot)$ is a convex function),
(4) $P(f,\varphi+c)=P(f,\varphi)+c$,
(5) $P(f,\varphi+\psi)\leqslant P(f,\varphi)+P(f,\psi)$,
(6) $P(f,c\varphi)\leqslant cP(f,\varphi)$ $(c\geqslant 1)$, $P(f,c\varphi)\geqslant cP(f,\varphi)$ $(c\leqslant 1)$,
(7) $|P(f,\varphi)|\leqslant P(f,|\varphi|)$,
(8) $|P(f,\varphi)-P(f,\psi)|\leqslant\|\varphi-\psi\|$,
(9) $P(f_{|Y},\varphi_{|Y})\leqslant P(f,\varphi)$ when Y is a closed f-invariant subset.

Let Z be a subset of X. Then we can define for $\varphi\in C(X,\mathbb{R})$

$$\begin{aligned}&Q_{n,Z}(f,\varphi,\varepsilon)\\&\quad=\inf\left\{\sum_{x\in F} e^{S_n\varphi(x)}\colon F \text{ is an } (n,\varepsilon)\text{-spanning set of } Z\right\}\end{aligned}$$

and put

$$Q_Z(f,\varphi,\varepsilon)=\limsup_{n\to\infty}\frac{1}{n}\log Q_{n,Z}(f,\varphi,\varepsilon).$$

We say that

$$P_Z(f,\varphi)=\lim_{\varepsilon\to 0} Q_Z(f,\varphi,\varepsilon)$$

is the **topological pressure** with respect to $f_{|Z}$ on $C(X,\mathbb{R})$. It is clear that $P_Z(f,0)=h(f,Z)$. If $Z_1\subset Z_2$, then $P_{Z_1}(f,\varphi)\leqslant P_{Z_2}(f,\varphi)$ for $\varphi\in C(X,\mathbb{R})$. We have $P_Z(f,\varphi)=P_{f(Z)}(f,\varphi)$ when f is bijective.

12. Ergodic theory

Measure-theoretical aspects (ergodic theory) of continuous surjections of a compact metric space are sometimes applied to topological dynamics. The origins of ergodic theory were problems in statistical mechanics, related to Hamiltonian flows, one-parameter groups of diffeomorphisms preserving Liouville measure, geodesic flows, etc. However, many theorems in ergodic theory are formulated in a much simpler setting, i.e., the main objects of the theory are measure preserving transformations of measure spaces.

Let X be a set. A **σ-algebra** of subsets of X is a collection $\mathcal{B}$ of subsets of X such that

(1) $\mathcal{B}$ contains the entire space X,
(2) $X\setminus B\in\mathcal{B}$ when $B\in\mathcal{B}$,
(3) $\bigcup_{i\geqslant 1} B_i\in\mathcal{B}$ when each B_i is in $\mathcal{B}$.

A **measure space** is a triple $(X,\mathcal{B},m)$ where m is a function $m:\mathcal{B}\to\mathbb{R}^+$ satisfying $m(\bigcup_{n=1}^{\infty} B_n)=\sum_{n=1}^{\infty} m(B_n)$ if $\{B_n\}$ is a pairwise disjoint sequence of elements of $\mathcal{B}$. If $m(X)=1$ then $(X,\mathcal{B},m)$ is called a **probability space**. We say that a map $f:X\to X$ is a **measure preserving transformation** if $E\in\mathcal{B}$ implies $f^{-1}(E)\in\mathcal{B}$ and $m(f^{-1}(E))=m(E)$. Here m is said to be **f-invariant**. Let $f:X\to X$ be a measure preserving transformation of a probability space $(X,\mathcal{B},m)$. For a function $\xi:X\to\mathbb{R}$ such that $\int_X|\xi|\,dm<\infty$ we define the **time mean** of ξ to be

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\xi\bigl(f^i(x)\bigr)$$

and the **space mean** of ξ to be $\int_X\xi(x)\,dm$. The first major result in ergodic theory is the ergodic theorem that was obtained in 1931 by G.D. Birkhoff [8]. The ergodic theorem says that these means are almost everywhere equal if and only if $f:X\to X$ is **ergodic**, i.e., if $f(E)=E$ and $E\in\mathcal{B}$ then $m(E)=0$ or $m(E)=1$. Such a measure m is called an **ergodic measure** and $(X,\mathcal{B},m,f)$ is said to be an **ergodic system**.

Let X be a compact metric space and $\mathcal{B}$ the family of **Borel set**s, i.e., the smallest σ-algebra containing all open subsets of X. If $C(X)$ is the ***Banach space*** of continuous real-valued functions of X with the sup-norm, then every probability measure μ on X induces a nonnegative linear functional on $C(X)$ by the map $\xi\mapsto\int\xi\,d\mu$. Conversely, the Riesz representation theorem says that to any nonnegative linear functional J on $C(X)$ with $J(1)=1$, there corresponds a unique probability measure μ on X such that $J(\xi)=\int_X\xi\,d\mu$ for $\xi\in C(X)$. The space $\mathcal{M}(X)$ of all probability measures on X is obviously a ***convex set***. We may define a topology in $\mathcal{M}(X)$ by taking as a ***neighbourhood base*** at $\mu\in\mathcal{M}(X)$ the sets

$$\begin{aligned}&V_\mu(\xi_1,\ldots,\xi_k;\varepsilon_1,\ldots,\varepsilon_k)\\&\quad=\left\{\nu\in\mathcal{M}(X)\colon\left|\int\xi_j\,d\mu-\int\xi_j\,d\nu\right|<\varepsilon_j,\ 1\leqslant j\leqslant k\right\}\end{aligned}$$

with $\varepsilon_j>0$ and $\xi_j\in C(X)$. This topology is called the **weak topology** of $\mathcal{M}(X)$. With this topology $\mathcal{M}(X)$ becomes a compact metrizable space. A sequence $\mu_n\in\mathcal{M}(X)$ converges to μ if and only if one of the following equivalent conditions holds:

(1) $\lim_{n\to\infty}\int\xi\,d\mu_n=\int\xi\,d\mu$ for all $\xi\in C(X)$,
(2) $\liminf_{n\to\infty}\mu_n(U)\geqslant\mu(U)$ for all open $U\subset X$,
(3) $\lim_{n\to\infty}\mu_n(A)=\mu(A)$ for all $A\in\mathcal{B}$ with $\mu(\partial A)=0$.

Here ∂A denotes the boundary of the set A. The **support** $\mathrm{Supp}(\mu)$ of a probability measure μ is the smallest closed set C with $\mu(C) = 1$. Equivalently, $\mathrm{Supp}(\mu)$ is the set of all $x \in X$ with the property that $\mu(U) > 0$ for any open U containing x. Let $f : X \to X$ be a continuous surjection of a compact metric space. It can be shown that the set $\mathcal{M}(f)$ of all f-invariant probability measures on X is nonempty and a compact convex subset of $\mathcal{M}(X)$. A probability measure μ is said to be **absolutely continuous** with respect to ν, written $\mu \ll \nu$, if every set of ν measure zero has also μ measure zero. In this case, the Radon–Nikodým theorem says that there exists a ν-integrable function ξ such that $\mu(B) = \int_B \xi\, d\nu$ for all $B \in \mathcal{B}$. If $\mu \ll \nu$ and $\nu \ll \mu$ then μ and ν are said to be **equivalent**. It can be shown that if μ and ν are ergodic measures in $\mathcal{M}(f)$ and $\mu \ll \nu$, then $\mu = \nu$, and that ergodic measures in $\mathcal{M}(f)$ are exactly the **extremal points** of $\mathcal{M}(f)$. Here we mean by an extremal point one that cannot be written in the form $\alpha\mu_1 + (1-\alpha)\mu_2$ with $\mu_1, \mu_2 \in \mathcal{M}(f)$, $\mu_1 \neq \mu_2$ and $0 < \alpha < 1$. The Krein–Milman theorem says that every invariant probability measure is a limit of convex combinations $\mu = c_1\mu_1 + \cdots + c_k\mu_k$, $\sum_{j=1}^k c_j = 1$, $c_j \geqslant 0$ $(1 \leqslant j \leqslant k)$, of ergodic measures $\mu_1, \ldots, \mu_k$.

If f is a continuous (surjective) map of a compact metric space and if $\mu \in \mathcal{M}(f)$, then there exists a family of probability measures $\{\mu_y\} \subset \mathcal{M}(f)$ satisfying $\mu(E) = \int \mu_y(E)\, d\mu(y)$ for every Borel set E such that for $\xi \in C(X, \mathbb{R})$,

(1) ξ is μ_y-integrable,
(2) $y \mapsto \int \xi\, d\mu_y$ is measurable,
(3) $\int \xi\, d\mu = \int (\int \xi\, d\mu_y)\, d\mu(y)$,
(4) $\lim_{n\to\infty} \frac{1}{n}\sum_0^{n-1} \xi(f^j(y)) = \int \xi\, d\mu_y$ (μ-a.e.y).

This result is called the **ergodic decomposition theorem** for f.

A measure $\mu \in \mathcal{M}(f)$ is said to be **strongly mixing** if

$$\lim_{j\to\infty} \{\mu(A \cap f^{-j}(B)) - \mu(A)\mu(B)\} = 0$$

and **weakly mixing** if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \left|\mu(A \cap f^{-j}(B)) - \mu(A)\mu(B)\right| = 0$$

for all $A, B \in \mathcal{B}$. By definition a strongly mixing measure is weakly mixing, and a weakly mixing measure is ergodic. It can be shown that a factor of an ergodic system $(X, \mathcal{B}, \mu, f)$ is ergodic, and that the product $(X_1 \times X_2, \mathcal{B}_1 \times \mathcal{B}_2, \mu_1 \times \mu_2, f_1 \times f_2)$ of two ergodic systems $(X_1, \mathcal{B}_1, \mu_1, f_1)$, $(X_2, \mathcal{B}_2, \mu_2, f_2)$ need not be ergodic, and moreover that the product of an ergodic system with a weakly mixing system is ergodic. Here $\mathcal{B}_1 \times \mathcal{B}_2$ is the smallest σ-algebra containing $\{B_1 \times B_2 \colon B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\}$ and $\mu_1 \times \mu_2$ is the measure on $X_1 \times X_2$ induced by $\mu_1 \times \mu_2(B_1 \times B_2) = \mu_1(B_1)\mu(B_2)$ $(B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2)$. A system is weakly mixing if and only if the product with itself is ergodic.

The set R of all recurrent points is a Borel set and $\mu(R) = 1$ for all $\mu \in \mathcal{M}(f)$. This is, in essence, Poincaré's recurrence theorem which is one of the oldest results in ergodic theory. Let $\mu \in \mathcal{M}(f)$ be positive on all nonempty open sets of X. It can be shown that f is topologically transitive if μ is ergodic, and topologically mixing if μ is strongly mixing. However, there exits an example of topologically transitive homeomorphism such that some measures in $\mathcal{M}(f)$ are positive on all nonempty open sets but none of them are ergodic.

As before, let $\sigma : Y_k^{\mathbb{Z}} \to Y_k^{\mathbb{Z}}$ be a symbolic dynamics for a fixed $k \geqslant 1$. Let $\ell < m$ be integers and $a_\ell, \ldots, a_m$ be a finite sequence of points in Y_k. An open subset $[a_\ell, \ldots, a_m] = \{(x_i) \in Y_k^{\mathbb{Z}} \colon x_i = a_i \text{ for } \ell \leqslant i \leqslant m\}$ is called a **cylinder**. A probability measure μ on $Y_k^{\mathbb{Z}}$ is σ-invariant if for all cylinders $[a_\ell, \ldots, a_m]$, μ satisfies $\mu([a_\ell, \ldots, a_m]) = \mu([a_{\ell+1}, \ldots, a_{m+1}])$. In the set of σ-invariant probability measures, Bernoulli measures are important. The k-tuple $\pi = (p_1, \ldots, p_k)$ with $p_j \geqslant 0$ and $\sum_{j=1}^k p_j = 1$ defines a **Bernoulli measure** μ by

$$\mu([a_\ell, \ldots, a_m]) = p_{a_\ell} p_{a_{\ell+1}} \cdots p_{a_m}.$$

This is the simplest type of probability measures on $Y_k^{\mathbb{Z}}$. The system $(Y_k^{\mathbb{Z}}, \mu, \sigma)$ is called a **Bernoulli shift** if μ is a Bernoulli measure.

Probability spaces $(X, \mathcal{B}, \mu)$ and $(X', \mathcal{B}', \mu')$ are said to be **isomorphic** if there are $M \in \mathcal{B}$ with $\mu(M) = 1$, $M' \in \mathcal{B}'$ with $\mu'(M') = 1$ and a bijection $h : M \to M'$ such that h and h^{-1} are measurable and $\mu(h^{-1}(A')) = \mu'(A')$ for all A' in the restriction of $\mathcal{B}'$ to M'. If a measurable map $h : M \to M'$ is surjective and satisfies $\mu(h^{-1}(A')) = \mu(A')$ for all A' in the restriction of $\mathcal{B}'$ to M', then we say that $(X', \mathcal{B}', \mu')$ is a **factor** of $(X, \mathcal{B}, \mu)$.

Let $f : X \to X$ and $f' : X' \to X'$ be measure preserving continuous maps of compact metric spaces. Then $(X, \mathcal{B}, \mu, f)$ and $(X', \mathcal{B}', \mu', f')$ are said to be **measure theoretically conjugate** if h is a measurable bijection and satisfies $h \circ f = f' \circ h$. When h is a measurable surjection and satisfies $h \circ f = f' \circ h$, $(X', \mathcal{B}', \mu', f')$ is called a **factor** of $(X, \mathcal{B}, \mu, f)$. A system $(X, \mathcal{B}, \mu, f)$ is called a **Bernoulli shift** if the system is measure theoretically conjugate to a Bernoulli shift $(Y_k^{\mathbb{Z}}, \mu, \sigma)$.

Let $(X, \mathcal{B}, \mu)$ be a probability space. A family α of subsets of X is said to be a **measurable partition** of X if $C \in \mathcal{B}$ for all $C \in \alpha$, $C \cap C' = \emptyset$ for distinct elements $C, C' \in \alpha$, and $X = \bigcup_{C\in\alpha} C$. A measurable partition α is called countable if α is a countable family, and finite if it is a finite family.

Let μ be a probability Borel measure on a compact metric space X and α be a measurable partition of X. Then there exists a Borel subset Y satisfying $\mu(Y) = 1$ and a family $\{\mu_x^\alpha \colon x \in Y\}$ of probability measures satisfying the following (1) and (2):

(1) $x \mapsto \mu_x^\alpha(B)$ is a measurable function and

$$\mu(E \cap B) = \int_E \mu_x^\alpha(B)\, d\mu(x) \quad (E \in \mathcal{B}),$$

(2) if, in particular, there are countable partitions α_n $(n \geqslant 1)$ such that $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \bigvee_{n\geqslant 1} \alpha_n = \alpha$, then $\mu_x^\alpha(\alpha(x)) = 1$ ($\alpha(x)$ is an element of α containing x) for every $x \in Y$, that is, μ_x^α is a probability measure.

The family $\{\mu_x^\alpha \colon x \in Y\}$ satisfying (1) is called a **family of conditional probability**, and the family satisfying (1) and (2) is called a **standard system of conditional probability**.

Since X is a compact metric space, there exists a sequence $\{\alpha_n\}$ of finite measurable partitions such that (i) $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \bigvee_1^\infty \alpha_n$ and (ii) $\bigvee_1^\infty \alpha_n$ is the pointwise partition of X.

13. Metric entropy

Let $(X, \mathcal{B}, \mu)$ be a probability space. Let α and α' be countable measurable partitions of X. If, for $A \in \alpha'$ there is $B \in \alpha$ such that $B \subset A$ (μ-a.e.) we write $\alpha \leqslant \alpha'$ (μ-a.e.), and α' is said to be a **refinement** to α. If $\alpha \leqslant \alpha'$ (μ-a.e.) and $\alpha' \leqslant \alpha$ (μ-a.e.), then write $\alpha = \alpha'$ (μ-a.e.). Define

$$H_\mu(\alpha) = \sum_{C\in\alpha} -\mu(C)\log\mu(C).$$

Then $0 \leqslant H_\mu(\alpha) \leqslant \infty$, and $H_\mu(\alpha) \leqslant \log k$ if α is finite and k is the number of elements in α.

The join of α and η is defined by $\alpha \vee \eta = \{A \cap B \colon A \in \alpha, B \in \eta\}$. Then, by definition $H_\mu(\alpha \vee \eta) \leqslant H_\mu(\alpha) + H_\mu(\eta)$. If $f : X \to X$ is a measure preserving transformation, obviously $H_\mu(f^{-1}(\alpha)) = H_\mu(\alpha)$. Let $n > 0$. Since $\alpha \vee f^{-1}(\alpha) \vee \cdots \vee f^{-(n-1)}(\alpha) = \bigvee_0^{n-1} f^{-i}(\alpha)$ is a countable measurable partition, we set $a_n = H_\mu(\bigvee_0^{n-1} f^{-i}(\alpha))$, and then $a_n \geqslant 0,\ a_{n+m} \leqslant a_n + a_m$. Thus the existence of

$$h_\mu(f,\alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_0^{n-1} f^{-i}(\alpha)\right)$$

is ensured. We define the **metric entropy** by

$$h_\mu(f) = \sup\bigl\{h_\mu(f,\alpha)\colon \alpha \text{ is a countable measurable partition}\bigr\}.$$

Let α be a finite measurable partition. If the smallest σ-algebra containing $\bigvee_0^\infty f^{-i}(\alpha)$ equals $\mathcal{B}$, then $h_\mu(f,\alpha) = h_\mu(f)$. If $f : X \to X$ and $g : X \to X$ are Bernoulli and $h_\mu(f) = h_\mu(g)$, then f is isomorphic to g (D. Ornstein [41]).

Let X be a compact metric space and $\mathcal{B}$ the set of all Borel sets in X. Let μ be a Borel probability measure on X. If α is a countable measurable partition of X such that

$$H_\mu(\alpha) = -\int \log\mu\bigl(\alpha(x)\bigr)\,d\mu < \infty,$$

then there exists an f-invariant integrable function $h_\mu(f, \alpha, \cdot)$ satisfying the following (1) and (2):

(1) $h_\mu(f,\alpha,x) = \lim_{n\to\infty} -\frac{1}{n}\log\mu\left(\bigcap_{i=0}^{n-1} f^{-i}\alpha\bigl(f^i x\bigr)\right)$ μ-a.e.,

(2) $\int h_\mu(f,\alpha,x)\,d\mu = h_\mu(f,\alpha)$.

(Shannon-McMillan–Breinman's theorem). We define

$$h_\mu(f,x) = \sup_Z h_\mu(f,\alpha,x)$$

where $Z = \{\alpha\colon H_\mu(\alpha) < \infty\}$. Then $h_\mu(f,x)$ is measurable and it is called the **information function**. If μ is ergodic, then $h_\mu(f,\alpha,x) = h_\mu(f,\alpha)$.

The following result called **local entropy** is derived from the theorem stated above. If a homeomorphism $f : X \to X$ is μ-measure preserving and $h_\mu(f) < \infty$, and μ is ergodic, then

$$\begin{aligned} h_\mu(f) &= \lim_{r\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log\mu\bigl(B_n(x,r)\bigr) \\ &= \lim_{r\to 0}\liminf_{n\to\infty} -\frac{1}{n}\log\mu\bigl(B_n(x,r)\bigr) \end{aligned}$$

where $B_n(x,r) = \{y \in X\colon d(f^i(x), f^i(y)) \leqslant r \text{ for } 0 \leqslant i \leqslant n-1\}$ (M. Brin and A. Katok [13]).

Let α be a measurable partition of X, and let $\{\mu_x^\alpha\}$ be a standard system of conditional probability measure with respect to a probability Borel measure μ. Define

$$\widehat{H}_\mu(\beta \mid \alpha) = \begin{cases} -\int \log\mu_x^\alpha\bigl(\beta(x)\bigr)\,d\mu & \bigl(\mu_x^\alpha\bigl(\beta(x)\bigr) > 0\bigr), \\ \infty & \bigl(\mu_x^\alpha\bigl(\beta(x)\bigr) = 0\bigr) \end{cases}$$

for a measurable partition β. $\widehat{H}_\alpha(\beta \mid \alpha)$ is called the **quasi entropy** of β with respect to α (V. Rohlin [54]).

Let α_j $(j \geqslant 1)$ be a sequence of countable measurable partitions and let $H_\mu(\alpha_j) < \infty$ $(j \geqslant 1)$ and $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \bigvee_1^\infty \alpha_j = \alpha$. Then $h_\mu(f) \geqslant \widehat{H}_\mu(\alpha \mid f(\alpha))$ if $\alpha \leqslant f^{-1}(\alpha)$. There exists a measurable partition α satisfying the following conditions; $\alpha \leqslant f(\alpha)$, $\bigvee_0^\infty f^i(\alpha)$ is a pointwise partition and $\widehat{H}_\mu(f(\alpha) \mid \alpha) = h_\mu(f)$.

14. Fractal dimensions

Let μ be a Borel probability measure on a compact metric space X. Then

$$HD(\mu) = \inf\bigl\{HD(Y)\colon \mu(Y) = 1\bigr\}$$

is said to be the Hausdorff dimension of X with respect to μ. In this definition $HD(Y)$ denotes the Hausdorff dimension of Y. The Hausdorff dimension is defined as follows. Let Y be any subset of X. For $\delta > 0$ and $\varepsilon > 0$ we put

$$H_\varepsilon^\delta(Y) = \inf\left\{\sum_{i=1}^\infty \bigl(\operatorname{diam}(U_i)\bigr)^\delta\colon \{U_i\} \text{ is a cover of } Y \text{ with } \operatorname{diam}(U_i) < \varepsilon\right\},$$

and write $H^\delta(Y) = \lim_{\varepsilon\to 0} H^\delta_\varepsilon(Y)$. Then

$$\delta_0 = \inf\{\delta\colon H^\delta(Y) = 0\} = \sup\{\delta\colon H^\delta(Y) = \infty\}$$

exists, and $\delta_0 = HD(Y)$ is the **Hausdorff dimension** of Y. We set

$$\underline{\delta}(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

$$\overline{\delta}(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

where $B(x,r) = \{y \in X\colon d(x,y) \leqslant r\}$. If $\underline{\delta}(x) = \overline{\delta}(x)$, we write $\delta(x)$ as the limit value, and call $\delta(x)$ the **local dimension** of μ.

If $\log \mu(B(x,r))/\log r$ converges for μ-a.e. x, i.e.,

$$\delta = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

then $\delta = HD(\mu)$ holds. Such a Borel probability measure μ is called a **completely dimensional measure**. Let Z be a subset of X. For $\varepsilon > 0$ let $N(Z,\varepsilon)$ denote the smallest cardinality of open covers with diameter less than or equal to ε which covers Z, and define

$$\underline{\dim}_B(Z) = \liminf_{\varepsilon\to 0} \frac{\log N(Z,\varepsilon)}{\log(1/\varepsilon)},$$

$$\overline{\dim}_B(Z) = \limsup_{\varepsilon\to 0} \frac{\log N(Z,\varepsilon)}{\log(1/\varepsilon)}.$$

If we have

$$\underline{\dim}_B(Z) = \overline{\dim}_B(Z) = \dim_B(Z),$$

then $\dim_B(Z)$ is called the **box dimension** of Z. Obviously, $HD(Z) \leqslant \dim_B(Z)$. We can define the box dimension with respect to μ as follows; write

$$\underline{\dim}_B(\mu) = \lim_{\delta\to 0}\{\inf\{\underline{\dim}_B(Z)\colon \mu(Z) > 1-\delta\}\},$$

$$\overline{\dim}_B(\mu) = \lim_{\delta\to 0}\{\inf\{\overline{\dim}_B(Z)\colon \mu(Z) > 1-\delta\}\}.$$

Then $\underline{\dim}_B(\mu) \leqslant \overline{\dim}_B(\mu)$. If $\underline{\dim}_B(\mu) = \overline{\dim}_B(\mu) = \dim_B(\mu)$, then we say that $\dim_B(\mu)$ is the **box dimension with respect to** μ. Let ξ be a countable partition with $H_\mu(\xi) = -\sum_{c\in\xi} \mu(c)\log\mu(c) < \infty$, and denote

$$H_\mu(\varepsilon) = \inf\{H_\mu(\xi)\colon \operatorname{diam}(\xi) \leqslant \varepsilon\}.$$

If $\underline{I}(\xi) = \bar{I}(\xi) = I(\xi)$ where

$$\underline{I}(\xi) = \liminf_{\varepsilon\to 0} \frac{H_\mu(\varepsilon)}{\log(1/\varepsilon)},$$

$$\bar{I}(\xi) = \limsup_{\varepsilon\to 0} \frac{H_\mu(\varepsilon)}{\log(1/\varepsilon)},$$

then $I(\xi)$ is called the **information dimension** with respect to μ.

If μ is a Borel probability measure on $\mathbb{R}^n$ (the n-dimensional Euclidean space) and is completely dimensional, then $HD(\mu) = \dim_B(\mu) = I(\mu)$ (L. Young [70]).

15. Variational Principle

Let X be a compact metric space and let $f\colon X \to X$ be a continuous map. For $\varphi \in C(X,\mathbb{R})$ we have the equation

$$P(f,\varphi) = \sup\left\{h_\mu(f) + \int \varphi\, d\mu\colon \mu \in \mathcal{M}_f(x)\right\}$$

which is called the **Variational Principle**. Obviously, $P(f,0) = h(f)$ and $h(f) = \sup\{h_\mu(f)\colon \mu \in \mathcal{M}_f(X)\}$. If $f\colon X \to X$ is expansive, for $\varphi \in C(X,\mathbb{R})$ there is a $\mu \in \mathcal{M}_f(X)$ such that

$$P(f,\varphi) = h_\mu(f) + \int \varphi\, d\mu.$$

Such a μ is called an **equilibrium measure** for φ. If $\mu \in \mathcal{M}_f(X)$ is ergodic, then we have $\mu(B(\mu)) = 1$ where

$$B(\mu) = \left\{x \in X\colon \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i x) = \int \varphi\, d\mu, \text{ for all } \varphi \in C(X,\mathbb{R})\right\},$$

and then

$$P_{B(\mu)}(f,\varphi) = h_\mu(f) + \int \varphi\, d\mu$$

for $\varphi \in C(X,\mathbb{R})$. For fixed $\varphi \in C(X,\mathbb{R})$ define $\Psi_{B(\mu)}(t) = P_{B(\mu)}(f, t\varphi)$ for $t \in \mathbb{R}$. Then $\Psi_{B(\mu)}(t)$ is called the **pressure function**, and $\Psi_{B(\mu)}(t) = 0$ is called the **Bowen's equation**.

If $h(f, B(\mu)) = P_{B(\mu)}(f,0) < \infty$ and $P_{B(\mu)}(f,\varphi) < 0$, then $\Psi_{B(\mu)}(t)$ is a ***Lipschitz continuous***, strictly decreasing convex function. Thus $\Psi_{B(\mu)}(t) = 0$ has a unique solution s, and then (1) if $h(f, B(\mu)) < \infty$ then $0 \leqslant s < \infty$ and (2) $h(f, B(\mu)) = 0$ when $s = 0$. Such a solution s equals the Hausdorff dimension with respect to μ, provided that X is a 2-dimensional smooth manifold and $f\colon X \to X$ is a diffeomorphism of class C^2, which means that μ is a completely dimensional measure.

We say that a diffeomorphism $f\colon M \to M$ of class C^2 is **u-conformal** (respectively **s-conformal**) on a hyperbolic basic set Λ if there exists a continuous function $a^u(x)$ (respectively $a^s(x)$) on Λ such that $Df_{|E^u_x} = a^u(x)\mathrm{Isom}_x$ for every $x \in \Lambda$ (respectively $Df_{|E^s_x} = a^s(x)\mathrm{Isom}_x$), where Isom_x denotes an isometry of E^u_x to $E^u_{f(x)}$ (respectively of E^s_x to $E^s_{f(x)}$). Since the subspaces E^u_x and E^s_x depend Hölder continuously on x, the functions $a^u(x)$ and $a^s(x)$ are also Hölder continuous. Note that $|a^u(x)| > 1$ and $|a^s(x)| < 1$

for every $x \in \Lambda$. A diffeomorphism is called **conformal** on Λ if it is both u-conformal and s-conformal on Λ.

Suppose that $f : M \to M$ is a diffeomorphism of class C^2 and Λ is a basic set on which f is conformal. Let δ^u and δ^s be the unique solutions of Bowen's equations

$$P_\Lambda\big(f, -t \log |a^u|\big) = 0, \qquad P_\Lambda\big(f, t \log |a^s|\big) = 0$$

respectively. Then

$$\dim_H(\Lambda) = \dim_B(\Lambda) = \delta^u + \delta^s$$

(cf. Y. Pesin [45]).

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h-7 Fixed Point Theorems

A **fixed point** of a self-map $f : X \to X$ of a ***topological space*** X is a point x of X such that $f(x) = x$. The set of all fixed points of f is denoted by $\mathrm{Fix}(f)$. A topological space X is said to have the **fixed-point property** if every ***continuous*** self-map of X has a fixed point. Finding a fixed point of a given map is often an important problem in mathematics and its applications, e.g., topology, ***dynamical systems***, functional analysis, theory of differential equations, economics, game theory, optimal control, etc.

The exposition of the present article will focus on various generalizations of the Brouwer Fixed Point Theorem and will stay at an elementary level.

There are several obvious omissions. For instance, results on fixed-point sets of group actions on topological spaces will not be discussed. In spite of the importance and the close connection with various areas of mathematics, the fixed point indices, Nielsen fixed point theory will not be discussed. Although fixed point theorems for multivalued maps play an important role in applications, they will not be treated in this article. No applications will be illustrated either.

Several English textbooks on fixed-point theory are available and the readers are referred to, for example, [25, 55, 90, 98] (for general references) [14, 58] (for the aspect of algebraic topology) [1, 37, 61, 88, 108] (for the aspect of functional analysis and metric fixed point theory) [41, 81] (for fixed-point theory for multivalued maps and selection theory) etc. The article [15] is an excellent survey on the topic of Section 1. Also proceedings [16, 27, 57, 67], etc. contain relevant articles.

The following theorem is the most basic and fundamental result and serves as a prototype for the theory of fixed points.

THEOREM 1 (**Brouwer Fixed Point Theorem**). *Every* ***n-cell*** ($n \geqslant 1$) *has the fixed-point property.*

Besides the standard proof using homology theory, several proofs are known. For a proof, from the view point of point-set topology, see [E].

1. The Lefschetz fixed point theorem and related topics

The Lefschetz fixed point theorem provides an elegant generalization of the Brouwer fixed point theorem to ***non-contractible*** spaces.

We follow [14] for the presentation below. For a topological space X and a ***field*** F, $\mathrm{H}^*(X; F)$ denotes the ***singular cohomology*** with the coefficient F. For a continuous map $f : X \to Y$, the ***induced homomorphism*** between ***cohomology group***s is denoted by $f^* : \mathrm{H}^*(Y; F) \to \mathrm{H}^*(X; F)$. Standard textbooks on algebraic topology (for example, [10], [23] etc.) contain expositions on the Lefschetz fixed point theorem as well.

DEFINITION 2. Let X be a ***compact*** ANR and F a field. For a continuous self-map $f : X \to X$, the **Lefschetz number** of f with respect to F is defined by the following formula:

$$L(f; F) = \sum_i (-1)^i \operatorname{tr}(f^*)^i,$$

where $(f^*)^i : \mathrm{H}^i(X; F) \to \mathrm{H}^i(X; F)$ is the induced homomorphism (= a linear map between vector spaces over F) between the ith ***cohomology*** $\mathrm{H}^i(X; F)$ and $\operatorname{tr}(f^*)^i$ denotes the trace of $(f^*)^i$.

Notice that, since X has the ***homotopy type*** of a compact ***polyhedron***, the above sum makes sense. Also it is clear that if $g : X \to X$ is ***homotopic*** to f, then $L(f; F) = L(g; F)$. Observe that $L(\mathrm{id}_X; F)$ is the Euler characteristic of X.

THEOREM 3 (**Lefschetz Fixed Point Theorem**). *Let X be a compact* ANR *and $f : X \to X$ be a self-map of X. If $L(f, F) \neq 0$, then f has a fixed point.*

COROLLARY 4. *Every compact* AR*, and in particular every compact contractible polyhedron, has the fixed-point property.*

The AR assumption in the above corollary cannot be dropped. Kinoshita constructed a 2-dimensional compact contractible metric space without the fixed-point property (see [6]).

The converse of the above theorem does not hold for an obvious reason: take a compact ANR with zero Euler characteristic, then $L(\mathrm{id}_X; F) = 0$, but (of course) each point of X is a fixed point of id_X. An appropriate formulation of the converse statement should be: if $L(f; F) = 0$, then there exists a fixed-point free map g which is homotopic to f. Actually this is the case when X is either a ***simply connected manifold*** of ***dimension*** at least 3 or an ***H-space*** (e.g., a ***topological group***), see [14, Chapter 8]. For a result on non-simply connected manifolds, see [28].

The Lefschetz numbers of iterations of a map satisfy a certain congruence relation. This was first proved by Zabréiko-Krasnosel'ski and Steinlein for some special cases and was later generalized by Dold [24]. See [31] for related results.

The fixed point index and the Nielsen number

In addition to the existence of fixed points, the estimate of the number of fixed points is an important problem. The

Nielsen number $N(f)$ of a self-map $f : X \to X$ of a compact ANR X provides a lower bound of $|\mathrm{Fix}(f)|$. Intuitively $N(f)$ is the number of fixed points of f which are "essential" in the sense that these points "remain" under homotopic perturbations of f. To make the above statement precise, we need to introduce the notion of "fixed point classes" and "fixed point indices". Textbooks like [14, 23, 58] and the articles in [67] are good references on this subject.

Variations, generalizations and related topics
Here we give a brief description on variations and generalizations of Lefschetz and Nielsen fixed-point theory. Only a few explanations will be given and the readers are referred to appropriate references which are quoted in each section.

The Lefschetz-type coincidence theorems. For two maps $f, g : X \to Y$, we may ask as to whether there exists a point $x \in X$ such that $f(x) = g(x)$. A theorem that guarantees the existence of such a point is called a **coincidence theorem**. A fixed point theorem is then termed as a coincidence theorem for which one of the above two maps is the identity. When the spaces under consideration are (often orientable) manifolds, one can define the "Lefschetz coincidence number" and the "Nielsen coincidence number" and obtain coincidence theorems. See, for example, [8, 10, 21, 38, 39, 83, 89, 104], etc.

Relative Nielsen numbers and Lefschetz numbers. For a given map $f : X \to X$, the Nielsen number serves as a lower bound of $|\mathrm{Fix}(f)|$, the number of the fixed points of f, and the bound is often the best possible, but not in general. Being motivated to improve the lower bound and to obtain information on the "location" of fixed points, H. Schirmer introduced the notion of the "relative Nielsen numbers" for a map $f : (X, A) \to (X, A)$. See the survey article of Schirmer in [67]. Also see [29, 56, 76], etc.

The Atiyah–Bott and the Atiyah–Singer fixed point theorems. For a ***smooth map*** on a compact ***smooth manifold***, the Lefschetz fixed point theorem is extensively generalized in the context of the "Atiyah–Singer Index Theorem", one of the great achievement of algebraic-differential topology. For example, the book [86] serves as a good introduction to the subject.

The Lefschetz fixed point theorem for nearly extendable map. Watanabe [102] developed *approximative shape theory* and proved the Lefschetz fixed point theorem for nearly extendable maps on general compact spaces. For other results in this direction, see [85] and [36].

Miscellaneous. A characterization of a finite polyhedron which admits fixed-point free deformations is given by U.K. Scholz. See [79, 103]. For related topics to this section, see [16, 17, 27, 35, 40, 57, 68, 91, 97], etc.

2. Fixed point theorems on Banach and Hilbert spaces

In addition to the references cited in the introduction, the following proceedings provide good sources on the recent advances [34, 37, 42, 74, 87, 94, 95], etc.

We start with the Banach Contraction Principle, the most well known and most useful result in the context of metric fixed-point theory.

The Banach contraction principle and some variations
THEOREM 5 (**Banach Contraction Principle**). *Let (X, d) be a **complete** metric space and $f : X \to X$ be a self-map of X satisfying the following condition: there exists a constant k, $0 < k < 1$, such that $d(f(x), f(y)) \leqslant k \cdot d(x, y)$ for each pair of points $x, y \in X$. Then f has the unique fixed point.*

The fixed point above is given by $\lim_{n\to\infty} f^n(x_0)$, where x_0 is an arbitrary point of X. Analogous conclusion can be obtained by "localizing" the above inequality [26, 82], etc. Also a topological analogue, replacing the above metric condition with the one on open covers, is proved for compact ***Hausdorff spaces*** in [93].

The Schauder fixed point theorem
The Brouwer Fixed Point Theorem relies on the fact that there is no ***retraction*** of n-cell onto its ***boundary***, while as was shown by Kakutani [59], this is not the case for ***infinite-dimensional Hilbert spaces***. Some compactness assumption must be imposed to obtain an infinite-dimensional analogue of the Brouwer Fixed Point Theorem. A **topological linear space** (a **topological vector space** or **linear topological space**) is a real vector space endowed with a topology so that the addition and the scalar multiplication are continuous. A topological linear space X is said to be **locally convex** if each point of X (or equivalently, the origin of X) has a ***neighbourhood base*** consisting of convex sets. Every ***normed linear space*** is a locally convex topological linear space.

THEOREM 6 [90, p. 32]. *Let X be a locally convex topological linear space and C be a convex subset of X. Let $f : C \to C$ be a map such that $\overline{f(C)}$ is compact. Then f has a fixed point.*

COROLLARY 7 (**Schauder–Tychonoff Fixed Point Theorem**). *Every compact convex subset of each locally convex topological linear space has the fixed-point property.*

The compactness hypothesis above is applied to obtain a "finite-dimensional approximation". The approximation basically reduces the proof of the above theorem to the Brouwer Fixed Point Theorem. The problem as to whether the local convexity in the hypothesis of the above theorem can be dropped is still open (Schauder conjecture [vMR, Problem 986]). This is closely related to the problem [vMR, Problem 985] for compact convex sets. A partial answer is given in [75].

The **measure of noncompactness** γ originated by K. Kuratowski is defined below to state a consequence of the Schauder–Tychonoff fixed-point theorem. For a bounded subset A of a ***Banach space*** B, let

$$\gamma(A) = \inf\left\{r\colon A = \bigcup_{i=1}^{m} A_i,\ m < \infty,\ \mathrm{diam}(A_i) < r\right\}.$$

Notice that $\gamma(A) = 0$ if and only if $\bar{A}$ is compact.

We state the theorem below, basically following [77], but in a weaker form for simplicity.

THEOREM 8 (Darbo). *Let B be a Banach space and C be a closed convex subset of B. If $f : C \to C$ is a map satisfying the following condition: there exists a constant k, $0 < k < 1$, such that $\gamma(f(A)) \leqslant k \cdot \gamma(A)$ for each bounded set A with $\gamma(A) < \infty$, then f has a fixed point.*

The Leray–Schauder degree and the fixed point index

Let B be a Banach space and W be an ***open bounded subset*** of B. For a map $f : \overline{W} \to B$ such that $\mathrm{Fix}(f) \cap \mathrm{Fr}\, W = \emptyset$ and $\overline{f(W)}$ is compact, the Leray–Schauder degree of $\mathrm{id} - f$ is defined and plays an important role in functional analysis. In a topological term, this is interpreted as the fixed point index of f, see [12, 43, 44, 74, 77], etc.

Nonexpansive maps

The hypothesis on the constant "$k < 1$" is well known to be essential in the Banach Contraction Principle. If we replace the above hypothesis "$k < 1$" with "$k \leqslant 1$", then the theorem does not hold anymore.

DEFINITION 9. A self-map f of a metric space (X, d) is said to be **nonexpansive** if $d(f(x), f(y)) \leqslant d(x, y)$ for each pair of points $x, y \in X$.

Nonexpansive maps behave differently on Banach spaces and on Hilbert spaces from the view point of fixed-point theory. For Banach spaces, one may only expect the existence of "approximate fixed points".

THEOREM 10 [90, p. 35]. *Let $(B, \|\cdot\|)$ be a Banach space and C be a bounded convex subset of B. For each nonexpansive map $f : C \to C$ and for each $\varepsilon > 0$, there exist a point $x_\varepsilon \in C$ such that $\|f(x_\varepsilon) - x_\varepsilon\| < \varepsilon$.*

On the other hand, a fixed point theorem on nonexpansive maps on Hilbert spaces is available as follows ("if" part is due to F.E. Browder and "only if" part is due to W.O. Ray).

THEOREM 11 [11, 80]. *Let H be a Hilbert space and C be a closed convex subset of H. Then each nonexpansive map $f : C \to C$ has a fixed point if and only if C is bounded.*

3. Fixed point theorems for planar or one-dimensional continua

Throughout this section, a **continuum** means a compact connected metric space.

Fixed point theorems for nonseparating plane continua and related results

The Brouwer Fixed Point Theorem implies that the 2-cell has the fixed point property. The following question is a natural attempt to generalize this result. It has not been answered yet.

OPEN PROBLEM 1 [6, Question 3]. Let X be a **nonseparating continuum** in the plane, that is, a subcontinuum of the plane $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus X$ is ***connected***. Does X have the fixed-point property?

Some partial answers are given below. See [3, 4, 48, 69] and the cited references in these papers for more general results. The method for some of the proofs may be traced back to [9].

THEOREM 12. *Let X be a nonseparating plane continuum.*

(1) *(Hagopian) If X is **arcwise connected**, then X has the fixed point property.*
(2) *(Minc) If X is a continuous image of arc-like continuum (see (b) below), then X has the fixed-point property.*
(3) *(Bell) Each **homeomorphism** $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(X) = X$ has a fixed point in X.*
(4) *(Bell) Each holomorphic map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\mathrm{Fr}\, X) \subset X$ has a fixed point in X.*

Another type of fixed point theorem for 2-cell, from the view point of dynamical systems, is proved in [50].

Orientation-reversing homeomorphisms of the plane may have many fixed points. A lower bound is given below [63].

THEOREM 13. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-reversing homeomorphism and let X be a subcontinuum of $\mathbb{R}^2$ such that $h(X) = X$. If $\mathbb{R}^2 \setminus X$ has at least 2^k components which are invariant under h, then h has at least $k + 2$ fixed points in X.*

Tree-like continua

A **tree** means a one-dimensional compact connected polyhedron which contains no circles. By the corollary of the Lefschetz Fixed Point Theorem, every tree has the fixed point property. A continuum is said to be **tree-like** (respectively **arc-like**, **circle-like**) if it is an ***inverse limit*** of trees (respectively arcs, circles).

THEOREM 14 [49]. *Every arc-like continuum has the fixed-point property.*

Not all tree-like continua have the fixed-point property. An example was first constructed by Bellamy [5]. Some variations are given by [30, 78, 70, 71], etc. On the other hand, each homeomorphism on each arcwise connected tree-like continuum has a fixed point [72]. Extensive research have been made on the fixed-point property of tree-like continua. See, for example, [47, 66, 107], etc.

Appendix: Bing asked twelve questions in [6] on the fixed-point property. Here we briefly discuss the current status of these questions. The reader is referred to [6] for the statement of each question. Question 2: No. [5]; Question 3: Open; Question 4: Yes [48]; Question 6: Yes [72]; Question 11: No [53].

The Poincaré–Birkhoff theorem and the Brouwer translation theorem
The following famous theorem is motivated by celestial mechanics. A homeomorphism $h : A \to A$ of an annulus A is called a **twist homeomorphism** if h is ***isotopic*** to id_A and maps the boundary circles in the opposite directions.

THEOREM 15 (**Poincaré–Birkhoff Theorem**). *If a twist homeomorphism $h : A \to A$ preserves the areas (i.e., preserves the Lebesgue measure on A), then h has at least two fixed points.*

The above theorem has some topological analogues, dropping the area-preserving hypothesis [7, 18, 20, 46], etc. Also some generalizations have been made [2, 32].

The following is the fundamental result on the fixed-point free homeomorphism of the plane.

THEOREM 16 (**Brouwer Translation Theorem**). *For each orientation-preserving fixed-point free homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$, there exists a homeomorphism $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

(1) $g \circ T = h \circ g$, *where* $T(x, y) = (x + 1, y)$, *for each* $(x, y) \in \mathbb{R}^2$.
(2) $g(\{x\} \times \mathbb{R})$ *is closed in* $\mathbb{R}^2$ *for each* $x \in \mathbb{R}$.

Modern proofs are available in, for example, [13, 33]. The theorem is generalized to certain homeomorphisms on surfaces [45]. Extensive research has been made on surface homeomorphisms in connection with surface dynamics (see [19] for an exposition on Nielsen–Thurston Theory), but it is out of the scope of this article.

4. Miscellaneous results

This section collects miscellaneous results on fixed-point theory that are scattered throughout the literature. Although they are not necessarily related to the Brouwer Fixed Point Theorem, it is worth putting these results in a single place for the convenience of the reference.

Fixed point theorems for partially ordered spaces
THEOREM 17 (Bourbaki–Kneser [108, p. 504], Tarski–Kantorovitch [25, p. 15]). *Let $(P, \leqslant)$ be a partially ordered set and $F : P \to P$ be a continuous map with a point $b \in P$ such that*

(1) $b \leqslant F(b)$, *and*
(2) *every countable chain in* $\{x \in P\colon x \geqslant b\}$ *has a supremum.*

Then F has a fixed point.

For Nielsen Theory of partially ordered sets, see [105].

Cones, products, unions and hyperspaces
The ***cone*** of a continuum X with the fixed-point property need not have the fixed-point property [62]. Some positive results are, for example, in [96] and [52]. The ***product*** of two manifolds with the fixed-point property need not have the fixed-point property [53]. The union $X \cup Y$ of two spaces X and Y with the fixed-point property need not have the fixed-point property even when the intersection $X \cap Y$ is "nice" [6, 64]. A positive result is in [92]. See also [65] for a related topic. For a survey on hyperspaces with the fixed-point property, see [73] and [54].

The essential components of fixed point sets
Besides the essentiality in terms of fixed point indices, there is another notion of "essential components" of a fixed-point set [60]. For a self-map $f : X \to X$ of a compact metric space (X, d), we say that a ***connected component*** C of $\mathrm{Fix}(f)$ is **essential** if, for each open ***neighbourhood*** U of C, there exists a $\delta > 0$ such that each map $g : X \to X$ with $d(g, f) < \delta$ has a fixed point in U. A compact metric space X with the fixed-point property is said to have the **f*. p. p.** if every self-map $f : X \to X$ of X has an essential component of $\mathrm{Fix}(f)$. Every compact AR has the f*. p. p. [60]. There exists a non-***locally connected*** continuum which has the fixed point property but does not have the f*. p. p. [106].

The complete invariance property
A space X is said to have the **complete invariance property** if each nonempty closed subset of X is the fixed-point set of a self-map of X. Every compact manifold, every convex subset of each Banach space, every ***locally compact metrizable topological group***, every 1-dimensional ***Peano continuum***, the ***Menger Universal Space*** μ^n $(n \geqslant 1)$ have the complete invariance property. A survey on the results before 1980 is given in [84]. Also [51] contains a good survey on this subject.

Colorings of maps
Let $f : X \to X$ be a fixed-point free map of a ***Tychonoff space***. Then the extension $\beta f : \beta X \to \beta X$ is fixed-point free if and only if there exists a finite ***functionally closed*** cover $\mathcal{F}$ such that $f(F) \cap F = \emptyset$ for each $F \in \mathcal{F}$. Such maps are said to be **colorable** and the above cover $\mathcal{F}$ is called a (functionally closed) **color** of f. Under this terminology, a theorem of E.K. van Douwen [99] can be stated as follows: each fixed-point free homeomorphism of every finite-dimensional ***paracompact*** Hausdorff space admits a closed coloring. The problem of estimating the number of closed color of a fixed-point free map via the ***dimension*** of the spaces has been investigated by J.M. Aarts, R. Fokkink, M.A. van Hartskamp, J. van Mill, J. Vermeer et al. For example, if a paracompact space X has $\dim X \leqslant n$, then each fixed-point free homeomorphism has a closed color consisting of at most $n + 3$ elements. See [100] for the map coloring theory (cf. [101], the volume of which is not specified in [100]).

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h-8 Topological Representations of Algebraic Systems

Algebraic systems can be represented by topological means in various ways. Here we present three of them that allow representations of rather large classes of algebraic systems.

1. Representations within dualities

The **clopen** (= ***closed***-and-***open***) subsets of any ***topological space*** X form a Boolean algebra. In 1937, M.H. Stone proved that every Boolean algebra can be represented in this way. Given a **Boolean algebra** B, with 0 and 1 as its least and its greatest elements and $\vee$, $\wedge$, $'$ its join, meet and complementation, its **Stone space** $\mathbb{S}(B)$ is constructed as follows: on the set of all ***filters*** $\mathcal{F}$ on B (i.e., sets $\mathcal{F} \subseteq B$ such that $0 \notin \mathcal{F}$ and $b_1 \wedge b_2 \in \mathcal{F}$ whenever $b_1, b_2 \in \mathcal{F}$ and $b \vee c \in \mathcal{F}$ whenever $b \in \mathcal{F}$) maximal with respect to the set inclusion (i.e., ***ultrafilters*** or prime filters), a topology is imposed by declaring the system $\{s(b) : b \in B\}$ to be the ***base for the closed sets*** in $\mathbb{S}(B)$ where $s(b)$ denotes the set of all utrafilters $\mathcal{F}$ with $b \in \mathcal{F}$. Then $\mathbb{S}(B)$ is a **Boolean space** (i.e., a ***compact Hausdorff zero-dimensional*** space; some authors use the name "**Stone space**") and $\{s(b) : b \in B\}$ is precisely the system of all its clopen subsets. Since, clearly, $s(0) = \emptyset$, $s(b_1 \vee b_2) = s(b_1) \cup s(b_2)$ and $s(b') = \mathbb{S}(B) \setminus s(b)$, the Boolean algebra of all clopen subsets of $\mathbb{S}(B)$ is isomorphic to B. This all is proved in the very comprehensive basic Stone's paper [12]. However, Stone used in [12] an alternative description: **Boolean ring**s (= rings in which every element a satisfies $a \cdot a = a$) and their maximal ideals. (Also, his notion of a Boolean space was a little broader.) It was H. Wallman [18] who used maximal filters: he imposed the topology in the above described way on the set of all maximal filters of a bounded distributive lattice L (in fact, he created the above description of the construction), and proved that the construction gives a compact ***T_1-space*** and, if L satisfies the condition that for any distinct $l_1, l_2 \in L$ there exists $c \in L$ such that precisely one of the elements $l_1 \wedge c, l_2 \wedge c$ equals zero (clearly, this condition is satisfied in any Boolean algebra), then L is isomorphic to a base of closed sets of this space. In 1970, H.A. Priestley [10] modified the constructions starting from the following observation: while in Boolean algebras maximal filters coincide with **prime filters** (i.e., those filters $\mathcal{F}$ in which $b \vee c \in \mathcal{F}$ implies $b \in \mathcal{F}$ or $c \in \mathcal{F}$), in bounded distributive lattices maximal filters are prime filters but not vice versa. Given a bounded distributive lattice L, she imposed a topology on the set X of all prime filters of L partially ordered by their inclusion by declaring the system $\{s(l), X \setminus s(l) : l \in L\}$ to be an ***open subbase*** of X and she obtained a Boolean space $\mathbb{P}(L)$ satisfying the condition that

> if $\mathcal{F}_1 \nsubseteq \mathcal{F}_2$, then there exists a clopen increasing set d of $\mathbb{P}(L)$ with $\mathcal{F}_1 \in d$ and $\mathcal{F}_2 \notin d$,

where "increasing set" means a set containing all elements larger than any element of it. (Such partially ordered Boolean space is called a **Priestley space** as well as every partially ordered Boolean space homeomorphic and isomorphic to it.) Her construction gives a representation of any bounded distributive lattice by all the clopen increasing sets of the corresponding Priestley space.

The constructions of Stone space and Priestley space fit into a general framework of constructions (and Priestley already did her construction in this way) using a special object carrying two structures (called "schizophrenic object" by some authors, see, e.g., [3, 7]). Let us explain it on the first construction. Here, the special object is $\{0, 1\}$ which is simultaneously a Boolean algebra and a Boolean space. For a Boolean space X, the Boolean algebra $\mathbb{B}(X)$ of all its clopen subsets is in fact the set of all ***continuous maps*** of X into the Boolean space $\{0, 1\}$, i.e., a subset of the set $\{0, 1\}^X$ of all maps of X into $\{0, 1\}$. At this moment, we use that $\{0, 1\}$ is also a Boolean algebra, so that its power $\{0, 1\}^X$ is also a Boolean algebra; and the set $\mathbb{B}(X)$, being a subset of $\{0, 1\}^X$, inherits its structure. The passing from a Boolean algebra B to its Stone space $\mathbb{S}(B)$ is precisely the same procedure, only the roles of Boolean spaces and Boolean algebras are interchanged: ultrafilters on B can be regarded as homomorphisms of B into the Boolean algebra $\{0, 1\}$, hence the set $\mathbb{S}(B)$ is a subset of the set $\{0, 1\}^B$ of all maps of B into $\{0, 1\}$. At this moment, we use that $\{0, 1\}$ is also a Boolean space and $\mathbb{S}(B)$, being a subset of the ***topological power*** $\{0, 1\}^B$, inherits the topology from it. The Priestley's construction also has its "schizophrenic object" on $\{0, 1\}$. Now, $\{0, 1\}$ is viewed as a bounded distributive lattice on the one hand and as a Priestley space (with the order $0 < 1$) on the other. The fruitful idea of constructions based on an object carrying two structures is developed in the monograph [3] (where also the original references are presented). Here, e.g., Heyting algebras, median algebras, Stone algebras, Kleene algebras, De Morgan algebras and others are investigated on the one hand and Boolean spaces endowed by suitable operations and relations on the other.

Categorical terms will be used in the following sometimes without definitions. See the articles on categorical topology.

All the above constructions lead to dualities. A **duality** between categories $\mathcal{K}$ and $\mathcal{H}$ is a pair of ***contravariant functors***

$F:\mathcal{K}\to\mathcal{H}$ and $H:\mathcal{H}\to\mathcal{K}$ which are mutually "almost inverse" in the sense that every object X of $\mathcal{K}$ is isomorphic to $H(F(X))$ via an isomorphism $i_X:X\to H(F(X))$ such that the isomorphisms are "coherent with $\mathcal{K}$-morpisms", i.e., for every $\mathcal{K}$-morpism $f:X\to X'$, the composite $i_{X'}\circ f$ equals $[H(F(f))]\circ i_X$; and vice versa, every $\mathcal{H}$-object Y is isomorphic to $F(H(Y))$ by an isomorphism j_Y such that $j_{Y'}\circ g=[F(H(g))]\circ j_Y$ for every $\mathcal{H}$-morphism $g:Y\to Y'$. (For variants of this notion see [3, 7].) **Representations** within it are the isomorphisms of the $\mathcal{K}$-objects X to $H(F(X))$ or of the $\mathcal{H}$-objects Y to $F(H(Y))$. Historically, the representations were usually found earlier than the duality and inspired its discovery.

The **Stone duality** $\mathbb{B}:\mathcal{BS}\to\mathcal{BA}$ and $\mathbb{S}:\mathcal{BA}\to\mathcal{BS}$ between the categories $\mathcal{BS}$ of Boolean spaces and continuous maps and $\mathcal{BA}$ of Boolean algebras and homomorphisms is determined on objects by the above constructions; if $f:X\to X'$ is a $\mathcal{BS}$-morphism (or a $\mathcal{BA}$-morphism) then $\mathbb{B}(f):\mathbb{B}(X')\to\mathbb{B}(X)$ is given by the formula $[\mathbb{B}(f)](h)=h\circ f$ for every $\mathcal{BS}$-morphism $h:X'\to\{0,1\}$ (and $\mathbb{S}(f):\mathbb{S}(X')\to\mathbb{S}(X)$ is given by the same formula $[\mathbb{S}(f)](h)=h\circ f$ for every $\mathcal{BA}$-morphism $h:X'\to\{0,1\}$).

The **Priestley duality** between the category of all Priestley spaces and monotone continuous maps and the category of all bounded distributive lattices and all their lattice homomorphisms preserving the smallest and the last elements is determined on objects by the above constructions and on morphisms by an analogous formula using the composition of morphisms; analogously for all constructions presented in the monograph [3].

In the famous **Pontryagin duality** [9], $\mathcal{K}=\mathcal{H}$ is the category of all ***locally compact*** Abelian groups and $F=H$ sends every locally compact Abelian group G to the group $F(G)$ of all continuous homomorphisms of G into the unit circle of the complex plane, with the group $F(G)$ endowed by the ***compact-open topology***. If G is ***discrete***, then $F(G)$ is compact and vice versa. This gives a representation of Abelian groups (regarded as discrete ***topological groups***).

The **Gelfand duality** has its origin in the late thirties and the beginning of forties, see the basic and historically important paper [5] of I.M. Gelfand, D.A. Raikov and G.E. Shilov and Refs. [5, 6, 10] in it. A **commutative normed ring** in the sense of [5] is a ***Banach space*** over the field of complex numbers together with a multiplication, forming with the addition a commutative ring with unit in which always $\|a\cdot b\|\leqslant\|a\|\cdot\|b\|$, and $\|u\|=1$ for the ring unit u (i.e., a **commutative Banach algebra with unit** in an alternative terminology). By [5], if R is such a ring and I is its maximal ideal, then R/I is isomorphic to (and can be identified with) the field of complex numbers. Hence the factor-homomorphism $f_I:R\to R/I$, which is a unit-preserving algebra homomorphism (= simultaneously a linear map and a ring homomorphism) not increasing the norm, sends every $a\in R$ to a complex number. This determines an algebra homomorphism h of R into the Banach algebra of all bounded complex-valued functions on the set X of all maximal ideals of R (with the operations defined pointwise and the norm given by $\|g\|=\sup\{|g(I)|\colon I\in X\}$), by the formula $[h(a)](I)=f_I(a)$ for all $I\in X$. Imposing a topology on X by declaring the system $\{B(I,\varepsilon,a)\colon\varepsilon>0,\ a\in R\}$ to be a ***local subbase*** of any point $I\in X$, where $B(I,\varepsilon,a)=\{J\in X\colon|\tilde{a}(J)-\tilde{a}(I)|<\varepsilon\}$ with $\tilde{a}=[h(a)]$, one obtains a compact Hausdorff space such that every $h(a)$ with $a\in R$ is a continuous function. The **Gelfand representation theorem** states that if R admits an involution $a\rightsquigarrow a^*$ such that $\|a\cdot a^*\|=\|a\|\cdot\|a^*\|$, $(a\cdot b)^*=b^*\cdot a^*$ and $(\lambda a+\mu b)^*=\bar{\lambda}a^*+\bar{\mu}b^*$, where $\bar{\lambda},\bar{\mu}$ denote the conjugates of the complex numbers λ,μ (i.e., R is a **commutative C^*-algebra with unit**), then the homomorphism h is an ***isometric*** isomorphism of R onto the C^*-algebra of all complex-valued continuous functions on X. Sending any commutative C^*-algebra R with unit to the compact Hausdorff space X of all its maximal ideals and, vice versa, sending any compact Hausdorff space X to the C^*-algebra of all complex-valued continuous functions on X can be completed to the duality between the category of all compact Hausdorff spaces and continuous maps and the category of all commutative C^*-algebras with unit and all the unit-preserving algebra homomorphisms, the Gelfand duality. For these and other facts about Banach algebras, C^*-algebras and the relevant parts of mathematical analysis see [6] and the long list of references there.

The real-valued variant of the Gelfand duality (with the Stone's characterization of the rings of all real-valued continuous functions on compact Hausdorff spaces) is presented in [7].

The monograph [7] is also a standard reference for "pointless topology", an investigation of **frame**s (= complete lattices in which arbitrary joins distribute over finitary meets). Which frames can be represented as the frames of all open subsets of topological spaces? Those which have "enough points". Here, points of a frame F are frame homomorphisms (= the lattice homomorphisms preserving all joins) $F\to\{0,1\}$ (where $\{0,1\}$ with $0<1$ is a frame, of course) and "enough" means that if $a,b\in F$, $a\nleqslant b$, then there exists a point $p:F\to\{0,1\}$ with $p(a)=1$ and $p(b)=0$. Such frames (called **spatial frames**) and all the frame-homomorphisms form a category dual to the category of all **sober space**s (= the topological spaces X in which every nonvoid closed subset not expressible as the union of two of its proper closed subsets is the ***closure*** of a unique point of X) and continuous maps. The duality is built through the above scheme with "schizophrenic object" $\{0,1\}$ viewed as a frame and as a sober space endowed by the **Sierpiński topology** (in which $\emptyset$, $\{1\}$ and $\{0,1\}$ are precisely the open sets), see [7].

2. Representations by continuous maps

In 1959, J. de Groot proved (see [4]) that every abstract group G is isomorphic to the group of all ***homeomorphisms*** of a suitable space onto itself.

A natural generalization seems to be the representation of a **monoid** (= semigroup with a unit element) as the monoid $M(X)$ of all continuous maps of a space X into itself. However the situation here is quite different from that concerning groups, and this is because $M(X)$ contains all constant maps. They correspond precisely to the left zeros of an abstract monoid M isomorphic to $M(X)$, and they already determine the underlying set of X and the actual form of every map in $M(X)$, i.e., the whole structure of M. Abstract monoids with this property (i.e., with so many left zeros) are rather special (and this is far from being sufficient, see, e.g., the interesting results in [2]). Hence $M(X)$ is not a good candidate for sufficiently general representation results. J. de Groot himself suggested representations by nonconstant continuous selfmaps of spaces, although these are not closed with respect to the composition in general. It proved to be the right approach: for every monoid M there exists a space X such that all its nonconstant continuous selfmaps do form a monoid (that is, they are closed with respect to the composition) and the monoid is isomorphic to the given one. In fact, this result is a very special case of representation of categories. Every class $\mathcal{C}$ of spaces with all their continuous maps forms a category. Which categories $\mathcal{K}$ can be represented in it? Here, the representation is any one-to-one ***covariant functor*** $F:\mathcal{K}\to\mathcal{C}$ sending every set $\mathcal{K}(a,b)$ of all $\mathcal{K}$-morphisms with the domain a and the codomain b onto the set of all nonconstant continuous maps of the space $F(a)$ into the space $F(b)$. Note that, since the image of every unit $1_a\in\mathcal{K}(a,a)$ has to be nonconstant, no $F(a)$ is a one-point space.

If $\mathcal{K}$ is small (in the sense that the class $obj\,\mathcal{K}$ of all objects of $\mathcal{K}$ is a set), then it can be represented in the category of all ***connected metrizable spaces*** or in the category of all ***continua***. For a given monoid M, the represented category $\mathcal{K}$ has precisely one object, say a, such that $\mathcal{K}(a,a)$ is isomorphic to M, and we get the above representation of M. When $\mathcal{K}$ is chosen with $obj\,\mathcal{K}$ an arbitrarily large set and having only the unit morphisms, the representing collection $\{F(a)\colon a\in obj\,\mathcal{K}\}$ of spaces forms a **rigid collection of spaces** (the name **stiff** is used in [11]), i.e., for any spaces X and Y in the collection, every continuous map $f:X\to Y$ is either constant or $X=Y$ and f is the identity. Hence there are arbitrarily large rigid collections of nondegenerate continua or of connected metrizable spaces X with $|X|>1$. In 1974, V. Koubek constructed a representation of any **concretizable category** (= isomorphic to a subcategory, not necessarily full, of the category Set of all sets and maps) in the category of ***paracompact*** spaces. Deleting all the nonidentity maps from Set and representing the resulting category, we get a rigid proper class of paracompact spaces. All the above results are presented in the monograph [11]. Their proofs have a nontrivial combinatorial part, and the topological reasoning heavily uses **Cook continuum**, i.e., a nondegenerate ***metric*** continuum C such that, for every its subcontinuum K, every continuous map $f:K\to C$ is either constant or $f(x)=x$ for all $x\in K$. Such a continuum was constructed by H. Cook in 1967.

In a contrast to the existence of a rigid proper class of paracompact spaces, which is an absolute result, the existence of a rigid proper class of metrizable spaces or of nondegenerate continua is a set-theoretical statement: its negation is equivalent to the Vopěnka's principle ([60] in [15]).

Constant maps can be also avoided by "structural means". For instance, the category of T_1-spaces and all ***open*** continuous maps also contains a copy of every concretizable category as its full subcategory (see [11]), and the same is true about the category TopSmg of all ***topological semigroups*** ([78] in [15]).

Representations of all pairs of monoids $M_1\subseteq M_2$ (i.e., M_1 is a submonoid of M_2) by the nonconstant continuous selfmaps of a space X and of its ***Stone–Čech compactification*** βX and other analogous "simultaneous representations" are again special cases of "simultaneous representations" of categories. For more details (and more on "simultaneous representations"), see the survey in [15].

Another setting more general than the representability of monoids is the representability of clones which are important structures of universal algebra.

Let us recall that a **clone** $\mathbb{A}(X)$ of a topological space X consists of a sequence $\{A_n(X)\colon n\in\omega\}$ where $A_n(X)$ is the set of all continuous maps on the nth power X^n of X into X; in each $A_n(X)$ there are n distinguished elements, namely the product projections

$$\pi_i^{(n)}:X^n\to X,\quad i\in n,$$

moreover, for every continuous map $g\in A_m(X)$ and an m-tuple of continuous maps $f_0,\dots,f_{m-1}$ in $A_n(X)$, the map

$$S_n^m(g;f_0,\dots,f_{m-1})=g\circ(f_0\dot{\times}\cdots\dot{\times}f_{m-1})$$

is continuous hence in $A_n(X)$ (where $f_0\dot{\times}\cdots\dot{\times}f_{m-1}:X^n\to X^m$ sends any $x\in X^n$ to $(f_0(x),\dots,f_{m-1}(x))\in X^m$ and $\circ$ denotes the composition of maps). Clearly, the equations below are satisfied:

$$S_n^n\big(f;\pi_0^{(n)},\dots,\pi_{n-1}^{(n)}\big)=f\quad\text{for every } n\in\omega;$$

$$S_n^m\big(\pi_i^{(m)};f_0,\dots,f_{m-1}\big)=f_i$$

for every $n,m\in\omega$, $i\in m$;

$$S_p^n\big(g;S_p^m(f_0;h_0,\dots,h_{m-1}),\dots,S_p^m(f_{n-1};h_0,\dots,h_{m-1})\big)=S_p^m\big(S_m^n(g;f_0,\dots,f_{n-1});h_0,\dots,h_{m-1}\big)$$

for all $m,n,p\in\omega$.

(In less formal terms, the last equation says that when we substitute the maps f_i into the map g and then the maps h_j into the result, we obtain the same map as when we substitute into g the maps f_i with the maps h_j already substituted in them.)

Clones of many other structures (groups, semigroups, lattices, posets, ***uniform spaces*** and so on) are created similarly. By abstraction, the notion of an **abstract clone** $\mathbb{A}$ is obtained. It consists of the following data: a sequence $\{A_n: n \in \omega\}$ of (abstract) sets; in each A_n, n distinguished elements $\pi_i^{(n)}$, $i \in n$, are specified and, for any $n, m \in \omega$, a map

$$S_n^m : A_m \times (A_n)^m \to A_n$$

is given such that the above equations (required as axioms) are satisfied.

Which abstract clones can be represented as clones of topological spaces? This question is far from being resolved. Since every abstract clone $\mathbb{A}$ contains the monoid $(A_1, S_1^1, \pi_0^{(1)})$ (the third axiom guarantees that the binary operation $S_1^1 : A_1 \times A_1 \to A_1$ is associative, the first and the second one guarantee that $\pi_0^{(1)}$ is a right and left unit with respect to S_1^1), this problem cannot be easier than that for monoids. However, if $\mathbb{A}[\Sigma] = \{A_n: n \in \omega\}$ is an abstract clone freely generated by a sequence of disjoint sets $\Sigma = \{\Sigma_n: n \in \omega\}$ – in the sense of the free generation of the Birkhoff's many-sorted algebras, i.e., $\mathbb{A}[\Sigma]$ is determined uniquely (up to an isomorphism) by the requirements that Σ_n is a subset of A_n and, for any clone $\mathbb{A}' = \{A'_n: n \in \omega\}$, every sequence of maps $\varphi_n : \Sigma_n \to A'_n$ uniquely extends to a clone homomorphism $h : \mathbb{A}[\Sigma] \to \mathbb{A}'$, where the clone homomorphism h is a sequence of maps $h_n : A_n \to A'_n$ preserving the distinguished elements and such that

$$h_n\big(S_n^m(g; f_0, \dots f_{m-1})\big) = S_n^m\big(h_m(g); h_n(f_0), \dots, h_n(f_{m-1})\big),$$

then $\mathbb{A}[\Sigma]$, though having somewhat complicated internal structure, can be represented as the clone $\mathbb{A}(X)$ of a metrizable space X whenever $|\Sigma_0| \geqslant \mathfrak{c} + |\bigcup_{n=1}^{\infty} \Sigma_n|$. For a closer description and some relations to similar topics, see the survey paper [15].

In an analogy to monoids, what can be said about the representation of abstract clones by nonconstant continuous maps $X^n \to X$? Such maps generally do not form a clone (because they are not closed with respect to the S_n^m's); but for which abstract clones are there spaces X such that the nonconstant continuous maps $X^n \to X$, $n \in \omega$, do form a clone isomorphic to a given abstract one? Not every abstract clone has such a representation; counterexamples are easy. However, for every monoid M, the smallest abstract clone $\mathbb{A}$ such that $(A_1, S_1^1, \pi_0^{(1)})$ is isomorphic to M, does have such a representation, by a metrizable space X (see [16]). The two clone representation results lead to interesting classes of spaces. The space X representing the smallest clone with a given monoid M by the nonconstant continuous maps must be **coconnected** which means that, for every $n \in \omega$, every continuous map $f : X^n \to X$ has the form $f = g \circ \pi_i^{(n)}$ for some $i \in n$, $g : X \to X$. In the former result, the spaces X representing the free clones $\mathbb{A}[\Sigma]$ are **disciplined** (which means that, for every $n \in \omega$, every continuous map $f : X^n \to X$ has the form $f = h \circ \pi^{(M)}$ where $M \subseteq n$, $\pi^{(M)} : X^n \to X^M$ is the projection and $h : X^M \to X$ is a homeomorphism of X^M onto a closed subspace of X), and have no absolute fix-point (which means that, for every $x \in X$, there exists nonconstant continuous $f : X \to X$ such that $f(x) \neq x$). For more about such spaces see the survey paper [15] and the references therein.

3. Representations by products and coproducts of spaces

In 1978, J. Ketonen proved [8] that every countable commutative semigroup can be represented by products of countable Boolean algebras, i.e., in the dual form (by means of the Stone duality), that for every countable commutative semigroup $(S, \circ)$ there exists a collection $\{X(s): s \in S\}$ of non-homeomorphic metrizable Boolean spaces such that, for any $s_1, s_2 \in S$,

(+) $X(s_1 \circ s_2)$ is homeomorphic to the coproduct of $X(s_1)$ and $X(s_2)$ (let us denote it $X(s_1) + X(s_2)$),

where **coproduct**s (= ***topological sum***s) are disjoint unions as clopen subspaces. This result solved the long-open Tarski's cube problem originating in 1957 when W. Hanf constructed a Boolean space X homeomorphic to the coproduct $X + X + X$ but not to $X + X$ (i.e., $X(1) = X$ and $X(0) = X + X$ form a representation in the above sense of the cyclic group $c_2 = \{0, 1\}$ with $1 + 1 = 0$, $1 + 0 = 0 + 1 = 1$). The Hanf's space is rather simple: in the Stone–Čech compactification $\beta\omega$, every ***isolated point*** is split into a pair of isolated points and, in the resulting space, one isolated point is deleted (see [4] in [13]). Since the Hanf's space is non-metrizable, the Boolean algebra of its clopen subsets is uncountable. The Tarski's cube problem asks for a countable Boolean algebra B isomorphic to $B \times B \times B$ but not to $B \times B$.

Can every commutative semigroup $(S, \circ)$ be represented by coproducts of spaces? The difficult construction of Ketonen seems to be impossible to generalize to higher cardinalities. The Hanf's idea was generalized by J. Adámek, V. Koubek and V. Trnková (see [1] in [13]), but only representations of Abelian groups were obtained. The answer to the above question is unknown.

In a contrast to this, every commutative semigroup can be represented by ***products*** of topological spaces, i.e., for every commutative semigroup $(S, \circ)$ there exists a collection $\{X(s): s \in S\}$ of non-homeomorphic spaces such that

(×) $X(s_1 \circ s_2)$ is always homeomorphic to $X(s_1) \times X(x_2)$.

A locally compact metrizable ***separable space*** X homeomorphic to $X \times X \times X$ but not to $X \times X$ was constructed by V. Trnková in 1973. Analyzing the construction, she observed that it is based on the following statement (see [15] in [13]):

(s) For every infinite cardinal α every commutative semigroup $(S, \circ)$ with $|S| \leqslant \alpha$ can be embedded into the semigroup $[N^\alpha]^\alpha$,

where N denotes the additive semigroup of all non-negative integers, N^α is its αth direct power, i.e., for $f, g : \alpha \to N$, the operation $+$ on N^α is defined by $(f+g)(i) = f(i) + g(i)$ for all $i \in \alpha$; $[N^\alpha]^\alpha$ consists of all $A \subseteq N^\alpha$ with $|A| = \alpha$ and, finally, the operation $+$ on $[N^\alpha]^\alpha$ is defined by $A + B = \{f + g\colon f \in A, g \in B\}$. Hence, by (s), it suffices to represent the semigroups $[N^\alpha]^\alpha$. The general scheme for this is as follows: we start from a collection $\mathcal{X} = \{X(i)\colon i \in \alpha\}$ of spaces and, for every $f \in N^\alpha$, we put

$$X(f) = \prod_{i\in\alpha} \bigl(X(i)\bigr)^{f(i)}.$$

Then, clearly, $X(f) \times X(g)$ is homeomorphic to $X(f + g)$. For every $A \in [N^\alpha]^\alpha$, we let $X(A)$ be the coproduct of α copies of every $X(f)$ with $f \in A$. Hence $X(A) \times X(B)$ is homeomorphic to $X(A+B)$ for every $A, B \in [N^\alpha]^\alpha$. To get a representation of $[N^\alpha]^\alpha$, it suffices to construct the starting collection $\mathcal{X}$ of spaces such that

$(*)$ $A \neq B \Longrightarrow X(A)$ is not homeomorphic to $X(B)$.

Simple reasoning shows that a rigid collection of spaces satisfies $(*)$. Since there exist arbitrarily large rigid collections of compact Hausdorff spaces (see the previous part 2), every commutative semigroup has a representation by products of spaces which are coproducts of compact Hausdorff spaces. J. Vinárek constructed in [17] the starting collection $\mathcal{X}$ consisting of 0-***dimensional metric spaces*** such that, investigating $X(f)$ endowed by the ***topology of uniform convergence***, he obtained the representations of all $[N^\alpha]^\alpha$ by products of 0-dimensional metrizable spaces. Other variants of the above procedure give the representations of $[N^\omega]^\omega$ by products of countable paracompact spaces [14], and by countable Hausdorff spaces with countable ***weight*** [HvM, 4.14 on p. 393]. Also, every subsemigroup S of $[N^\omega]^\omega$ with countable $\bigcup S$ can be represented by the products of $F_{\sigma\delta}$***-set***s and $G_{\delta\sigma}$***-set***s of the ***Cantor discontinuum*** $\mathbb{C}$ [HvM, 4.14 on p. 393]. The last case cannot be strengthened to closed subsets of $\mathbb{C}$: V. Trnková proved [8] that if a closed subset X of $\mathbb{C}$ is homeomorphic to $X \times X \times X$, it is already homeomorphic to $X \times X$. The situation with F_σ***-subsets*** of $\mathbb{C}$ remains unclear.

The above method in which the representing spaces $X(A)$ are coproducts of infinitely many spaces avoids the representations by products of compact spaces. However an "additional procedure" allows constructions of suitable ***compactifications*** in some cases. Thus every finite Abelian group can be represented by products of compact metric spaces ([14] in [13]) or of ***separable Boolean space***s [HvM, 4.13 on p. 393, where a nonexistent paper is quoted; it should be Fund. Math. 126 (1985), 45–61]. A. Orsatti and N. Rodino represented every finite cyclic group by products of metric continua admitting a structure of compact Abelian groups [HvM, 4.12 on p. 393]. Their result is the only one which does not use the statement (s) above. Instead of this, they used algebraic constructions and translated them into topology by means of Pontryagin duality. The question of whether every commutative semigroup can be represented by products of compact spaces remains unresolved.

Representations of ordered commutative semigroups $(S, \circ, \leqslant)$ (i.e., $(S, \circ)$ is a commutative semigroup, $(S, \leqslant)$ is a partially ordered set and $(a \leqslant b) \& (c \leqslant d) \Longrightarrow a \circ c \leqslant b \circ d$) were introduced by J. Adámek and V. Koubek [1] in the following sense: a representation of $(S, \circ, \leqslant)$ is a collection $\{X(s) \mid s \in S\}$ of spaces satisfying $(\times)$ above and, moreover,

$(\leqslant)$ $s_1 \leqslant s_2$ if and only if $X(s_1)$ is embeddable onto a clopen subset of $X(s_2)$.

They enriched the above algebraic statement (s) by the observation that $[N^\alpha]^\alpha$, ordered by the inclusion, forms an ordered commutative semigroup and they embedded every $(S, \circ, \leqslant)$ with $|S| \leqslant \alpha$ into it by a homomorphism h such that $s_1 \leqslant s_2$ if and only if $h(s_1) \subseteq h(s_2)$. Then almost all the above results obtained by the use of (s) give directly the representability of $(S, \circ, \leqslant)$ in the corresponding classes of spaces.

Which semirings can be represented simultaneously by products and coproducts? For a space X, let us denote by n_X (or m_X) the smallest natural number $k \geqslant 2$ such that X is homeomorphic to its kth power (or kth copower) and it is ∞ if no such finite k exists. Hence $(m_\mathbb{C}, n_\mathbb{C})$ is $(2, 2)$, $(m_\mathbb{R}, n_\mathbb{R})$ is (∞, ∞) (where $\mathbb{R}$ is the real line). For the Hanf's space X, (m_X, n_X) is $(3, \infty)$, for the space of Orsatti and Rodino it is $(\infty, 3)$, for a space obtained by (s) it can be $(2, 3)$. Does there exist a space X with (m_X, n_X) equal to $(3, 2)$? More generally, which pairs of elements of $\{2, 3, \ldots, \infty\}$ can be obtained in this way? Although published in 1982 in [13] and then again in [HvM, 4.16 on p. 394], this problem remains open.

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J: Influences of other fields

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j-1 Descriptive Set Theory

Descriptive set theory is the study of definable sets of real numbers. More generally, we could consider subsets of any Polish space, i.e., a separable complete metric space.

In fact, it is often most convenient to work in the Baire space, ω^ω. The symbol ω denotes set of nonnegative integers or equivalently the first infinite ordinal number, $\omega = \{0, 1, 2, 3, \ldots\}$. Elements of ω^ω can be thought of either as infinite sequences of elements of ω or as functions $f : \omega \to \omega$. The metric on ω^ω is defined by

$$d(f, g) = \begin{cases} \frac{1}{n+1} & \text{where } n \text{ is minmal with } f(n) \neq g(n), \\ 0 & \text{if } f = g. \end{cases}$$

Baire showed that under this topology the Baire space ω^ω is homeomorphic to the irrational numbers with their usual topology.

The first person to consider definable sets of real number was probably Borel. Borel reasoned that basic open sets should be considered definable and if we allow countably many bits of information, then the family of definable sets should be closed under taking countable unions and countable intersections. This is the family of **Borel sets**.

The classical hierarchy on Borel sets is defined as follows.

(1) $\utilde{\Sigma}^0_1$ is the family of open sets.
(2) $\utilde{\Pi}^0_1$ is the family of closed sets.
(3) $\utilde{\Sigma}^0_2$ is the family of sets which are the countable unions of $\utilde{\Pi}^0_1$-sets.
(4) $\utilde{\Pi}^0_2$ is the family of sets which are the countable intersections of $\utilde{\Sigma}^0_1$-sets.
(5) In general, for each countable ordinal α, $\utilde{\Sigma}^0_\alpha$ is the family of sets which are the countable unions of sets each of which is $\utilde{\Pi}^0_\beta$ for some $\beta < \alpha$ and $\utilde{\Pi}^0_\alpha$ is the family of sets which are the countable intersection of sets each of which is $\utilde{\Sigma}^0_\beta$ for some $\beta < \alpha$.
(6) $\utilde{\Delta}^0_\alpha = \utilde{\Sigma}^0_\alpha \cap \utilde{\Pi}^0_\alpha$.

By De Morgan's laws it is easy to see that the $\utilde{\Pi}^0_\alpha$-sets are precisely the sets whose complement is $\utilde{\Sigma}^0_\alpha$. Lebesgue proved that each of these classes is nontrivial, i.e., for any countable ordinal α there are sets in $\utilde{\Sigma}^0_\alpha$ which are not in $\utilde{\Pi}^0_\alpha$ and hence vice-versa. We can think of the Borel sets as being closed under quantification over ω, so the next higher classes would be those in the projective hierarchy, i.e., closure under quantification over ω^ω. Since the class $\utilde{\Sigma}^0_\alpha$ is closed under countable unions its elements are called sets of **additive class** α; likewise the sets in $\utilde{\Pi}^0_\alpha$ are said to be of **multiplicative class** α. Furthermore a separable metric space that is a Borel set in its ***completion*** is said to be an **absolute Borel set**; this is equivalent to saying that it is Borel in every space in which it is embedded.

Suppose X is a Polish space and $A \subseteq X$. Then A is $\utilde{\Sigma}^1_1$ in X iff there exists a Borel set $B \subseteq \omega^\omega \times X$ such that

$$y \in A \quad \text{iff} \quad \exists y \in \omega^\omega (x, y) \in B.$$

Thus the **$\utilde{\Sigma}^1_1$-set**s or **analytic set**s are precisely those which are the projection of a Borel set. The family of **$\utilde{\Pi}^1_1$-set**s consists of the complements of $\utilde{\Sigma}^1_1$-sets and are therefore also called **co-analytic**. This is the beginning of the hierarchy of **projective sets**: for $n \geqslant 2$ a **$\utilde{\Sigma}^1_n$-set** is the projection of a $\utilde{\Pi}^1_{n-1}$-set and a **$\utilde{\Pi}^1_n$-set** is the complement of a $\utilde{\Sigma}^1_n$-set. The classes $\utilde{\Sigma}^1_n$ and $\utilde{\Pi}^1_n$ are known as **projective class**es. Figure 1 shows how some of these classes of sets are arranged.

Analytic sets can also be obtained by applying the **Souslin operation** (or **A-operation**) to a families $\{A_s : s \in \bigcup_n \mathbb{N}^n\}$ of closed sets indexed by finite sequences of natural numbers: a subset A of a Polish space is analytic iff it can be written as $\bigcup_{x \in \mathbb{N}^\omega} \bigcap_n A_{x \restriction n}$.

Analytic sets have been studied in the context of general topological spaces by C.A. Rogers, L.E. Jayne, A.H. Stone, and many others, see [12]. Continuing up the projective hierarchy, we take $\utilde{\Sigma}^1_2$ to be the family of projections of $\utilde{\Pi}^1_1$-sets and $\utilde{\Pi}^1_2$ to be the complements of $\utilde{\Sigma}^1_2$, etc. It is a classical result that the class of Borel sets coincides exactly with the **$\utilde{\Delta}^1_1$-set**s, i.e., those sets which are both $\utilde{\Sigma}^1_1$ and $\utilde{\Pi}^1_1$.

Early descriptive set theorists were concerned with questions about the regularity of projective sets of reals. For example, it was shown that any uncountable $\utilde{\Sigma}^1_1$-set must contain a perfect subset, i.e., a homeomorphic copy of the Cantor space, $2^\omega = \{0, 1\}^\omega$. They showed that $\utilde{\Sigma}^1_1$-sets are Lebesgue measurable and have the Property of Baire. They also proved that any $\utilde{\Sigma}^1_2$ is the ω_1-union of Borel sets. See [6–8, 11].

In the 1930s Kurt Gödel as a consequence of his work on the consistency of Continuum Hypothesis, showed that it was consistent with the usual axioms of set theory that there is a $\utilde{\Delta}^1_2$-set which neither contains nor is disjoint from a perfect set. Such a set cannot have the property of Baire or be Lebesgue measurable.

Fig. 1. The Borel hierarchy and a little beyond.

In 1960s Robert Solovay using the forcing technique of Paul Cohen, was able to show (relative to the consistency of an inaccessible cardinal) that it is consistent that every projective set of reals is Lebesgue measurable, has the property of Baire, and has the perfect subset property.

The strongest known regularity property is called determinacy, it arose from the study of infinite two person games in the 1950s (see [5]). It implies the perfect set property, Lebesgue measurability and the Baire property, as well as, many other natural properties of the projective sets. It was a celebrated theorem of D.A. Martin that Borel sets are determined. In fact, one of the reasons this proof was difficult to find, was proved earlier by Harvey Friedman who showed that a proof of Borel determinacy must necessarily use uncountable many uncountable cardinals. The axiom of determinacy for projective sets was established by D.A. Martin and John Steel using large cardinal axioms, specifically infinitely many Woodin cardinals. The large cardinal assumption was also shown to be necessary. This confirmed a conjecture of Solovay who was the first to connect large cardinal theory with the axiom of determinacy. Kanomori's book [3] contains many of the results on large cardinal theory in set theory.

In 1970s, Jack Silver proved the following theorem about Borel equivalence relations, or actually Π^1_1 equivalence relations. Namely, every Π^1_1 equivalence relation with uncountably many equivalence classes must contain a perfect set of inequivalent members. John Burgess established a similar theorem for Σ^1_1 equivalence relations, namely any such equivalence relation with more than ω_1 equivalence classes must have a perfect set of inequivalent elements. Leo Harrington using a topology invented by Robin Gandy gave a simpler proof of Silver's Theorem. This technique achieved great success at proving a number of other results using effective descriptive set theory. For example, Louveau's theorem, the Borel version of Dilworths Theorem, and the Glimm–Effros–Kechris–Harrington Dichotomy Theorem were all proved using this technique. See [2, 4, 5, 9, 10].

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j-2 Consistency Results in Topology, I: Quotable Principles

A noteworthy feature of general topology – as contrasted with geometry, number theory, and most other fields of mathematics – is that many of its fundamental questions are not decided by "the usual axioms of set theory". What this means is that if one codifies the set-theoretic assumptions used implicitly in ordinary mathematics and consolidates these in a list of axioms – the usual axioms of set theory – one is left with many statements that can neither be proved nor refuted on the basis of these axioms alone. In some sense such undecidability results are commonplace: everyone knows that, say, the statement $\varphi = (\forall x)(\forall y)(xy = yx)$ cannot be decided by "the usual axioms of Group Theory". Indeed, the group of integers is a **model** for "Group Theory plus φ", which makes this conjunction consistent and φ non-refutable; the permutation group S_3 does the same for $\neg\varphi$.

A convenient list of axioms for Set Theory, one which has become standard is "ZFC" – Zermelo–Fraenkel set theory, including the Axiom of Choice. See, e.g., [Ku] for a detailed exposition. Thus there are important topological statements φ such that neither φ nor its negation follow from ZFC. Establishing this is not as easy as in the case of Group Theory. While it is easy to prove the consistency of Group Theory by providing models (groups) for it, the same is, by Gödel's Second Incompleteness Theorem, in principle impossible for ZFC (or any other useful collection of axioms for set theory). Gödel's theorem says that a consistency proof of a theory as strong as ZFC cannot be formalized in the theory itself. This explains why many results are formulated as "if ZFC is consistent then so is ZFC plus φ" instead of simply "φ is consistent with ZFC" and why, formally, we should be speaking of *relative* consistency proofs. Even if one is not interested in consistency results per se, it is nonetheless prudent to be aware of them, lest one waste effort trying to prove a proposition that has a consistent negation.

There are two ways in which consistency results in topology are obtained. The first consists of proving implications between statements, where the antecedent is a statement previously proven consistent with ZFC. Thus, the consequent itself is proven consistent with ZFC as well. The present article deals with results like these and in particular with the best-known combinatorial principles that occur as antecedents. In the second kind of consistency result will be dealt with in the companion article; here one needs an intimate knowledge of *how* one actually proves (relative) consistency results. The principal subjects in that article are forcing and large cardinals.

1. The Continuum Hypothesis

The best-known quotable principle is undoubtedly the **Continuum Hypothesis**, abbreviated as CH, which states that the real line has minimum possible cardinality, i.e., $\mathfrak{c} = |\mathbb{R}| = 2^{\aleph_0} = \aleph_1$. It was proved consistent by Gödel in [5] (see [Ku] for a proof) but before that numerous consequences were derived from it, see, e.g., [7].

It derives its strength from the facts that so many sets have cardinality $\mathfrak{c}$ and the initial segments of ω_1 are all countable; this makes for relatively easy transfinite constructions. To give the flavour we construct a **Lusin set**: an uncountable subset of $\mathbb{R}$ that meets every nowhere dense set in a countable set. It suffices to take an enumeration $\langle N_\alpha : \alpha < \omega_1 \rangle$ of the family of all closed nowhere dense sets and apply the ***Baire Category Theorem*** each time to choose l_α outside $\bigcup_{\beta<\alpha} N_\alpha$; then $\{l_\alpha : \alpha < \omega_1\}$ is the Lusin set. A similar construction will yield a ***strongly infinite-dimensional*** subspace of the ***Hilbert cube*** all of whose finite-dimensional subsets are countable. In [KV, Chapter 8, § 4] one finds more intricate constructions, involving CH, of ***hereditarily separable*** non***Lindelöf*** spaces (***S*-spaces**) as well as ***hereditarily Lindelöf*** non***separable*** spaces (***L*-space**s).

On occasion CH helps simply by counting: in combination with ***Jones' Lemma*** CH (or even its consequence $2^{\aleph_0} < 2^{\aleph_1}$) implies that in a separable normal space closed and discrete subsets are countable and hence that separable normal ***Moore space***s are metrizable. By contrast, CH implies there is a non-metrizable normal Moore space [KV, Chapter 16].

Another counting example comes from the theory of ***uniform spaces***. It is well-known that in a uniform space the finite uniform covers generate a uniformity again as do the countable uniform covers. The proofs can be generalized to show that any $\aleph_1$-sized uniform cover has a uniform star refinement of cardinality $\mathfrak{c}$. Thus, CH implies that the $\aleph_1$-sized uniform covers generate a uniformity. In general, the **Generalized Continuum Hypothesis** (GCH), which states that for all cardinals κ one has $2^\kappa = \kappa^+$, implies that for every κ the uniform covers of cardinality less than κ generate a uniformity.

The Continuum Hypothesis also implies that there is an almost disjoint family of uncountable subsets of ω_1 of cardinality $2^{\aleph_1}$ or, equivalently, that the ***cellularity*** of the space $U(\omega_1)$ of uniform ultrafilters is the maximum possible $2^{\aleph_1}$. Here 'almost disjoint' means that intersections are countable and an ultrafilter u on a set X is **uniform** if every element has full cardinality, i.e., $|U| = |X|$ for all $U \in u$.

See also the article on $\beta\mathbb{N}$ and $\beta\mathbb{R}$ for other applications of CH.

2. Martin's Axiom

Every consequence of CH is a potential theorem, as it can no longer be refuted. Some consequences are in fact equivalent to CH, e.g., the existence of the special infinite-dimensional space above; such consequences are then automatically non-provable because Cohen proved the consistency of the negation of CH (see the next article for an indication of the proof and [Ku] for a full treatment). To decide other consequences one would need proofs that avoid CH or principles that imply their negations.

Martin's Axiom (MA) is such a principle. It can be viewed as an extension of the Baire Category Theorem: it states that if X is a ***compact Hausdorff*** space with the ***countable chain condition*** (the ***ccc*** for short) then the union of fewer than $\mathfrak{c}$ many nowhere dense in X still has a dense complement. As such it is a consequence of CH but the conjunction $\mathsf{MA} + \neg\mathsf{CH}$ is also consistent.

Its original formulation, though more extensive, is ultimately more useful: if $\mathbb{P}$ is a partially ordered set (a poset) with the ccc and $\mathcal{D}$ is a family of fewer than $\mathfrak{c}$ dense sets then there is a filter G on $\mathbb{P}$ that meets all members of $\mathcal{D}$. Two elements p and q of $\mathbb{P}$ are **compatible** if there is an element r with $r \leqslant p, q$, and **incompatible** otherwise. An **antichain** is a set of mutually incompatible elements and "$\mathbb{P}$ has the ccc" means every antichain is countable. A set $D \subseteq \mathbb{P}$ is a **dense set** if for every p there is $d \in D$ with $d \leqslant p$. Finally, a **filter** on $\mathbb{P}$ is a subset G that satisfies: if $p, q \in G$ then there is $r \in G$ with $r \leqslant p, q$, and if $p \in G$ and $q \geqslant p$ then $q \in G$.

The equivalence between the two formulations becomes somewhat apparent when one thinks of $\mathbb{P}$ as representing the open sets of the space X – a proof may be devised along the lines of the ***Stone Representation Theorem*** of Boolean algebras. The usefulness of the poset formulation may be illustrated by a proof of the equality $2^{\aleph_0} = 2^{\aleph_1}$ from $\mathsf{MA} + \neg\mathsf{CH}$. We shall construct an injective map from the power set $\mathcal{P}(\omega_1)$ of ω_1 into $\mathcal{P}(\mathbb{N})$, using an ***almost disjoint family*** $\{x_\alpha : \alpha < \omega_1\}$ on $\mathbb{N}$. Given $A \subseteq \omega_1$ we find $B_A \subseteq \mathbb{N}$ satisfying "$B_A \cap x_\alpha$ is infinite iff $\alpha \in A$", which makes $A \mapsto B_A$ one-to-one. For this we use a poset of approximations to B_A. An element of $\mathbb{P}$ is a ordered pair $p = \langle F_p, b_p \rangle$, where F is a finite subset of $\omega_1 \setminus A$ and b_p a finite subset of $\mathbb{N}$. We say $p \leqslant q$ if $F_p \supseteq F_q$, $b_p \supseteq b_q$ and $b_p \setminus b_q \cap \bigcup_{\alpha \in F_q} x_\alpha = \emptyset$. We interpret b_p as an approximation of B_A, with the promise that $B_A \cap x_\alpha = b_p \cap x_\alpha$ for $\alpha \in F_p$. For every $\alpha \in \omega_1 \setminus A$ the set $D_\alpha = \{p : \alpha \in F_p\}$ is dense in $\mathbb{P}$, as is $E_{\alpha,n} = \{p : |b_p \cap x_\alpha| > n\}$ for each $\alpha \in A$ and $n \in \mathbb{N}$. To see that $\mathbb{P}$ has the ccc, note that two elements with the same second coordinate are compatible: $\langle F_p \cup F_q, b \rangle \geqslant \langle F_p, b \rangle, \langle F_q, b \rangle$. Finally then if G is a filter that meets the dense sets above then $B_A = \bigcup \{b_p : p \in G\}$ is the required set.

The conjunction $\mathsf{MA} + \neg\mathsf{CH}$ has often been advertised as 'an alternative to the Continuum Hypothesis' and indeed, many consequences of CH become false if $\mathsf{MA} + \neg\mathsf{CH}$ is assumed. On the other hand, 'small cardinals' – those smaller than $\mathfrak{c}$ – behave like $\aleph_0$ under MA, e.g., sets of reals of size less than $\mathfrak{c}$ are ***meager*** and of measure zero. We discuss the consequences of CH mentioned above.

Lusin sets no longer exist as every set of reals of cardinality less than $\mathfrak{c}$ is of first category. It becomes harder to find S- and L-spaces. Indeed, $\mathsf{MA} + \neg\mathsf{CH}$ denies the existence of compact such spaces and it is even consistent with $\mathsf{MA} + \neg\mathsf{CH}$ that no ***S-spaces*** exist.

On the positive side: $\mathsf{MA} + \neg\mathsf{CH}$ implies that separable normal nonmetrizable Moore spaces exist. It also implies that there is a uniform space with a uniform cover of cardinality $\aleph_1$ without an $\aleph_1$-sized uniform star refinement, to wit the subspace $\{x : \|x\| = 1\ (\forall\alpha)(x_\alpha \geqslant 0)\}$ of $\ell_\infty(\aleph_1)$ [6]. This paper does not mention $\mathsf{MA} + \neg\mathsf{CH}$ directly but the proof needs a family $\mathcal{A}$, of size $\aleph_2$, of uncountable subsets of ω_1 such that for some fixed cardinal $\kappa \leqslant \aleph_2$ every subfamily $\mathcal{A}'$ of $\mathcal{A}$ of size κ has a finite intersection. To make such a family one starts with an almost disjoint family $\mathcal{B}$, of size $\aleph_2$, of uncountable subsets of ω_1 (almost disjoint means distinct elements have a countable intersection); by repeated application of [4, 42I] one shrinks the elements of $\mathcal{B}$ to produce the desired family $\mathcal{A}$ (with $\kappa = 2$).

The ccc is the weakest in a line of properties, the best-known of these are σ-centered (corresponding to separable compact spaces) and countable (corresponding to compact metrizable spaces). These in turn give rise to weakenings of MA that have generated interest of their own, since they provide the possibility of denying some of MA's consequences while retaining others.

Martin's Axiom for σ-centered partially ordered sets was shown to be equivalent to the purely combinatorial statement known variously as $P(\mathfrak{c})$ or $\mathfrak{p} = \mathfrak{c}$: if $\mathcal{F}$ is a family, of cardinality less than $\mathfrak{c}$, of subsets of $\mathbb{N}$ with the **strong finite intersection property**, i.e., the intersection of every finite subfamily is infinite, then there is an infinite subset A of $\mathbb{N}$ with $A \setminus F$ finite for all $F \in \mathcal{F}$.

Martin's Axiom for countable partially ordered sets is equivalent to the strong Baire Category Theorem for $\mathbb{R}$: if $\mathcal{U}$ is a family of fewer than $\mathfrak{c}$ dense open sets in $\mathbb{R}$ then $\bigcap \mathcal{U}$ is dense.

Another way to vary Martin's Axiom is to restrict the number of dense sets. Thus $\mathsf{MA}(\aleph_1)$ means MA for families of $\aleph_1$ many dense sets. This version is strong enough to ensure there are no ***Souslin tree***s (see below for the definition).

Fremlin's book [4] and Weiss' survey [KV, Chapter 19] are good places to start exploring the consequences and versions Martin's Axiom.

3. The proper forcing axiom

The **Proper Forcing Axiom** (PFA) is a strengthening of MA, and a considerable one at that. Its formulation is quite similar, replacing 'ccc' by 'proper' and 'fewer than $\mathfrak{c}$' by '$\aleph_1$ many'. The notion of a **proper partial order** is more involved than that of a ccc partial order; it was developed in connection with iterations of forcing, for which see part II of this article. Given a set X, a subset of $[X]^{\aleph_0}$ (the family of countable subsets of X) is a **closed and unbounded set** if it is cofinal and closed under unions of countable chains. A **stationary set** is one that intersects every closed and unbounded set. A proper poset preserves stationary sets, i.e., $\mathbb{P}$ is proper means that for every set X and every stationary set in $[X]^{\aleph_0}$ remains stationary in any ***generic extension*** by $\mathbb{P}$ – this is not a given: $[X]^{\aleph_0}$ will most likely grow and so will the family of closed and unbounded sets. There is a combinatorial characterization of properness in terms of games: player I starts with $p \in \mathbb{P}$ and at move n chooses a maximal antichain A_n in $\mathbb{P}$ while player II the counters by choosing a countable subset B_i^n of A_i for each $i \leqslant n$. In the end we look at $B_i = \bigcup_{n \geqslant i} B_i^n$ and declare II the winner is there is a $q \leqslant p$ such that every B_i is **predense** below q, i.e., every $r \leqslant q$ is compatible with an element of B_i. The partial order $\mathbb{P}$ is proper precisely when II has a winning strategy for this game.

From this it follows easily that ccc partial orders are proper – simply take $B_n^n = A_n$ and $q = p$ – as are **countably closed** partial orders: at move n II picks p_n and $a_n \in A_n$ with $p_n \leqslant a_n$ (and $p_n \leqslant p_{n-1}$ when $n \geqslant 1$), she then plays $B_n^n = \{a_n\}$. In the end there is a q below all p_n by countable closedness; this q witnesses II's victory. The definition of properness implies that the forcing composition (and indeed iteration) of proper partial orders is again proper and in practice this is often how applications of PFA go. One has a candidate partial order for the problem at hand; this partial order is usually not proper, but after some preparatory forcing one can get a better version (even ccc) of the candidate. This preparation is itself often countably closed, so that the composition is proper – one applies PFA to this composition. The foregoing discussion should make clear that working with PFA is more involved than applying MA. The results obtained from PFA are generally much stronger than those obtainable from MA. Among (many) others, PFA implies: there are no S-spaces, every compact space of ***countable tightness*** is ***sequential***, there are no $\aleph_2$-Aronszajn trees. This last result implies that PFA harbours ***large cardinal*** strength, as the nonexistence of $\aleph_2$-Aronszajn trees implies $\aleph_2$ is a ***weakly compact cardinal*** in L (Gödel's constructible universe).

As it stands PFA implies MA($\aleph_1$) but it actually implies the full MA *because it implies* $2^{\aleph_0} = \aleph_2$, see [1].

Baumgartner's survey [KV, Chapter 21] is recommended as a first introduction to the Proper Forcing Axiom. Even stronger forcing axioms are coming into prominence: Martin's Maximum [3] and $\mathbf{P}_{\max}$ [9].

4. The Diamond principle

This is the first in a range of so-called prediction principles. It is denoted by $\diamondsuit$ and it states that there is a sequence $\langle A_\alpha \colon \alpha < \omega_1 \rangle$ (a **$\diamondsuit$-sequence**) of sets such that $A_\alpha \subseteq \alpha$ for all α and for every subset A of ω_1 the set $\{\alpha \colon A \cap \alpha = \alpha\}$ is stationary. This is intended to capture the essence of Jensen's proof that there is a **Souslin tree** in Gödel's constructible universe L (see [Ku, VII, B9]).

For much more on what follows, see [KV, Chapter 6]. A **tree** is a partially ordered set T in which for every x the set $\hat{x} = \{y \colon y \prec x\}$ of predecessors is well-ordered. The αth level of T is the set of x for which $\hat{x}$ has order type α. A **branch** in a tree is a maximal chain and an **antichain** is a set of mutually incomparable elements. A **κ-tree** is a tree of height κ (i.e., $T_\kappa = \emptyset$ and $T_\alpha \neq \emptyset$ for $\alpha < \kappa$) all of whose levels have cardinality less than κ. An **Aronszajn tree** is an $\aleph_1$-tree without branches of length ω_1 and a Souslin tree is an Aronszajn tree with no uncountable antichains. A **Kurepa tree** is an $\aleph_1$-tree with more than $\aleph_1$ branches of length ω_1. From a Souslin tree one can make a **Souslin line**, i.e., a linearly ordered set that is ccc but not separable in its order-topology and vice versa; thus, a Souslin tree/line refutes **Souslin's hypothesis**. We have indicated above that MA $+ \neg$CH denies the existence of Souslin trees, in fact it implies that Aronszajn trees are **special**, which means that they can be covered by countably many antichains. The definitions of Aronszajn and Souslin trees carries over easily to larger cardinal numbers: $\aleph_1$ is replaced by the desired κ and 'countable' by 'smaller than κ'.

The construction of a Souslin tree is still one of the best introductions to the use of $\diamondsuit$. One constructs a tree-order $\prec$ on ω_1 in such a way that the interval $I_\alpha = [\omega \cdot \alpha, \omega \cdot (\alpha + 1))$ becomes the α-th level of the tree. Thus, $I_0 = [0, \omega)$ is left totally unordered. If $\alpha = \beta + 1$ then I_α is unordered but it supplies two direct $\prec$-successors for each point of I_β. If α is a limit and A_α is a maximal antichain in the ordering on $[0, \omega \cdot \alpha)$ constructed thus far then we choose countably many branches that cover the set and such that each passes through a point of A_α; we put the points of I_α on top of these branches (one point for each branch) – this ensures that A_α is even maximal in the final tree. In the end if A is a maximal antichain in the tree $(\omega_1, \prec)$, then the set of those α with $\alpha = \omega \cdot \alpha$ *and* $A \cap \alpha$ is maximal in $(\alpha, \prec)$ is closed and unbounded. There is therefore such an α with $A \cap \alpha = A_\alpha$, but A_α was to remain maximal, hence $A = A_\alpha$ and so A is countable.

One uses $\diamondsuit$ if the property under consideration allows **reflection**, as in the case above, where a maximal antichain intersects initial segments of the tree in maximal antichains – the $\diamondsuit$-sequence enables one to capture and deal with such reflections. There have been strengthenings of $\diamondsuit$ that assert that more is captured or more often; these are denoted by $\diamondsuit^*$, $\diamondsuit^+$, $\diamondsuit$ for stationary systems, etc. The last mentioned version implies that normal and first-countable spaces are $\aleph_1$-collectionwise Hausdorff, see [KV, Chapter 15].

It is clear that $\diamondsuit$ implies CH and, in fact $\diamondsuit = \mathsf{CH} + \clubsuit$, where $\clubsuit$ is a weakening of $\diamondsuit$: there is a sequence $\langle S_\alpha \colon \alpha < \omega_1 \rangle$ (a **$\clubsuit$-sequence**) such that S_α is a cofinal ω-sequence in $\omega \cdot \alpha$ and every uncountable subset of ω_1 contains some S_α. The $\clubsuit$-principle can be used to construct an S-space topology on ω_1 – make sure each initial segment is open and that $[\omega \cdot \alpha, \omega_1) \subseteq \operatorname{cl} S_\alpha$ – and a simple ***Dowker space*** on $\omega_1 \times \omega$ [KV, Chapter 17]. From the stronger $\diamondsuit$ one gets more: ***Ostaszewski's space*** and Fedorchuk's compact S-space of cardinality $2^{\mathfrak{c}}$ with no convergent sequences.

5. The open colouring axiom

The **Open Colouring Axiom** (OCA) is a Ramsey-type statement. It states that given a ***separable metrizable space*** X and an open subset K_0 of $[X]^2$ (the two-element subsets of X with the ***Vietoris topology***) there either is an uncountable subset Y of X with $[Y]^2 \subseteq K_0$ or $X = \bigcup_n X_n$ with $[X_n]^2 \cap K_0 = \emptyset$ for all n, see [8, Chapter 8].

It has a remarkable effect on the theory of the space $\mathbb{N}^*$. Very few of the results on this space proved from CH remain when OCA is assumed. Its status as a universal space is simply demolished: the ***Stone space*** of the ***measure algebra***, the square $\mathbb{N}^* \times \mathbb{N}^*$, the space $\mathbb{R}^*$, and many others are no longer continuous images of $\mathbb{N}^*$. The conjunction OCA + MA implies that all autohomeomorphisms of $\mathbb{N}^*$ are induced by bijections between cofinite subsets of $\mathbb{N}$. The proofs of these results follow a by now well-established pattern: one proves that a potential map cannot have too much structure and one also proves that OCA implies (invariably via its second alternative) that a potential map must have a lot of structure.

A purely topological application of OCA is the following: if X is a cometrizable space then either X has a countable ***network***, or an uncountable discrete subspace, or it contains an uncountable subspace of the ***Sorgenfrey line***. A space is **cometrizable** if there is a weaker ***metrizable*** topology on it such that each point has a neighbourhood base consisting of sets which are closed in the metric topology

The memoir [2] contains many results related to OCA and gives lots of historic information.

In conclusion

We have barely scratched the surface of the use of quotable principles in General Topology; the volumes [KV] and [HvM] contain many applications of such principles.

A few words on the consistency of the principles. As mentioned above, CH holds in Gödel's Constructible Universe as do $\diamondsuit$ and its strengthenings. There is no such canonical model for the other principles discussed. The consistency of Martin's Axiom was established using the method of ***iterated forcing***, as was the consistency of PFA, though the latter required a ***supercompact cardinal*** in the initial model. The Open Colouring Axiom follows from PFA but its consistency may be established in an 'ordinary' iteration eschewing large cardinals.

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j-3 Consistency Results in Topology, II: Forcing and Large Cardinals

When Quotable Principles fail one will have to turn to the machinery of consistency proofs itself for a solution of the problem.

Many topologists are familiar with the forcing method, which we will describe in the first section of this article. This method works fairly well when the problems involve sets of bounded cardinality but it tends to fail when one wants to prove something like "*all* spaces that are such and such are so and so". A prime example is the ***Normal Moore Space Conjecture***, which we now know cannot be proved consistent assuming the consistency of ZFC alone – the proof requires additional, stronger, axioms that most set theorists regard as safe to assume. Such axioms assert the (consistency of the) existence of **large cardinal**s; these will be described in the second section. In the last section we give a small sample of consistency results.

This article needs to be read in conjunction with the preceding one in this volume, in order to get a reasonable picture of the state of consistency results in topology. In particular, almost all of the propositions listed as consistent here are known to be undecidable, but the consistency of their negations is often not mentioned, since it is most easily derived from a combinatorial principle such as the ones discussed in the preceding article.

Note to the reader: there is an extended version of this article, with references for all the results mentioned and more, available on Topology Atlas, see [6].

1. Forcing

Forcing is a method for producing a new model of ZFC from a given one, called "the ground model". Proving that the method works requires much attention to metamathematical details (see [Ku], where an accessible introduction to the forcing method can be found). The actual applications of forcing, however, mainly boil down to verifying combinatorial properties of partial orders, orders either "taken off the shelf" or specially constructed for the problem at hand. Forcing is used to prove the consistency of both existential and universal propositions.

To prove the consistency of an existential proposition one constructs a partial order consisting of approximations to the desired object, with the approximations (which are usually smaller in size than the desired object) being ordered by $p \leqslant q$ if p is a "better" approximation than q – this is like the discussion of MA in the previous article but with a major difference, as will become clear momentarily. For example, to prove it consistent that $2^{\aleph_0} \geqslant \aleph_2$, one could consider finite approximations to a listing of $\aleph_2$ distinct functions from ω into 2, i.e., finite partial functions from $\omega_2 \times \omega$ into 2. These approximations (forcing conditions) are then ordered by extension. A filter (i.e., an upward closed consistent subset) for that partial order $\mathcal{P}$ naturally yields a partial function from $\omega_2 \times \omega$ into 2. The trick is to make that function total. For that to happen, it would suffice to show that for each $\alpha \in \omega_2$ and $n \in \omega$, the filter met

$$D_{\alpha,n} = \{p \in P\colon \langle\alpha, n\rangle \in \operatorname{dom} p\}.$$

Note $D_{\alpha,n}$ is a **dense set**, i.e., every element of the partial order has an extension in $D_{\alpha,n}$.

Given any countable collection of dense sets, it is easy to construct a filter that meets each of them; a metamathematical argument establishes that in fact when forcing, one only has to consider those dense subsets of the partial order which lie in a fixed countable model M. Thus one gets a filter that simultaneously meets all the $D_{\alpha,n}$'s in M and thus gets the *generic* function to be total. The requirement that the induced map from ω_2 to $\mathcal{P}(\omega)$ be one-to-one can actually be dispensed with by meeting those $D_{\alpha,\beta}$'s, $\alpha, \beta \in \omega_2$, which are in M, where

$$D_{\alpha,\beta} = \{p \in P\colon (\exists n)\big(p(\alpha, n) \neq p(\beta, n)\big)\}.$$

This shows the difference with the MA approach: there the filter meeting the 'good' dense sets belongs to M, in the present example, if CH holds in M there is no filter in M that meets all $D_{\alpha,\beta}$'s; we adjoin the filter to M and use it to construct an extension $M[G]$. This extension is a model of ZFC and in it there is an injective map from ω_2 into $\mathcal{P}(\omega)$. If the ground model M satisfies the GCH and we use ω_3 instead of ω_2 then in $M[G]$ all almost disjoint families of uncountable subsets of ω_1 have cardinality at most $\aleph_2$, which is strictly less than $2^{\aleph_0} = 2^{\aleph_1}$.

A ***Souslin tree*** can be constructed by ***countably closed*** or ccc forcing. The existence of a Souslin tree shows that MA + ¬CH fails but one can do much better: Baumgartner constructed a model where MA($\aleph_1$) fails completely in that for *every* ccc partial order there is a family of $\aleph_1$ many dense sets such that *no* filter meets them all.

To prove the consistency of a universal proposition, one usually uses **iterated forcing** (or repeated forcing), see [Ku] and [1]. For example, to prove the consistency of "all perfectly marvelous subsets of $\mathbb{R}^2$ are splendiferous", one would iterate the process of forcing one perfectly marvelous subset to be splendiferous. Alternatively, one could iteratively force to "kill" non-splendiferous perfectly marvelous subsets. Of course, inevitable difficulties arise. The process of iterating the forcing could introduce new perfectly marvelous non-splendiferous sets, so one has to arrange that the new ones are either made splendiferous or are killed, i.e., made not perfectly marvelous. This is done by an "*initial stage*" argument – e.g., if the Continuum Hypothesis is assumed, subsets of $\mathbb{R}^2$ have cardinality $\aleph_1$; if one iterates $\aleph_2$ times, one argues that every perfectly marvelous non-splendiferous subset appears at some stage and is taken care of there. One also needs, e.g., that perfectly marvelous sets one has forced to make splendiferous stay that way. Thus one needs "preservation arguments".

A typical example of a universal proposition proved consistent by iterated forcing is the ***Souslin hypothesis***. Here we take as our partial orders the Souslin trees themselves – old and new – with the order being the reverse of the tree order. Actually, we first prune the trees so that each element of the tree has successors at all levels beyond it. Then, noting that compatibility = comparability in a tree, we observe that a filter meeting each

$$D_\alpha = \{t\colon \text{ the height of } t \text{ is at least } \alpha\}$$

is a ***branch*** of length ω_1 and therefore destroys Souslinity. Once a tree has a branch all the way through it, it stays that way, so one iterates $\aleph_2$ times; the Souslin trees have cardinality $\aleph_1$ so appear at an initial stage, and, when killed, stay dead.

This proof and others like it gave us both the formulation and consistency proof of $\mathsf{MA} + \neg\mathsf{CH}$; a study of this proof, as presented in [Ku] for example, will reveal the basic technical problems one usually encounters in iterated forcing constructions.

A complication that arises in dealing with topological spaces rather than with algebraic objects is that, after forcing to produce a new (larger) model of set theory (called the **forcing extension**, the **generic extension** or just the **extension**), a topological space in the ground model is no longer a topological space; the best one can do is use the former topology as a basis for a topology on the original set. This preserves the separation axioms up to ***complete regularity*** but properties like normality can be destroyed or created, see [4].

2. Large cardinals

Although set theorists have investigated a plethora of large cardinals, we will confine ourselves here to several that appear most often in topological contexts, namely inaccessible, weakly compact, measurable, strongly compact, supercompact, and huge ones. In the definitions we shall implicitly assume all the large cardinals are uncountable.

A note on the term **large cardinal** is in order. A cardinal number is 'large' if the assumption of its existence, when added to the axioms of ZFC, proves the consistency of ZFC. This works as follows, for any cardinal κ one can consider the set H_κ – the set of all sets which have size less than κ and whose members and members of members and ... all have size less than κ. Loosely speaking κ is large if H_κ is a model of ZFC.

A cardinal number κ is an **inaccessible cardinal** (also a **strongly inaccessible cardinal**) if it is regular and $2^\lambda < \kappa$ whenever $\lambda < \kappa$ is a cardinal. κ is a **weakly compact cardinal** if it is inaccessible and, whenever T is a tree of height κ with levels of size less than κ, then T has a branch of length κ. κ is a **measurable cardinal** if there is a non-principal **κ-complete ultrafilter on** κ (i.e., closed under intersections of size less than κ). A cardinal κ is a **strongly compact cardinal** if every κ-complete filter can be extended to a κ-complete ultrafilter.

Sometimes measurability is defined using countable completeness (i.e., ω_1-completeness) rather than κ-completeness. Let us call such cardinals **Ulam measurable**. The least Ulam-measurable cardinal is in fact measurable.

All of these cardinals have several equivalent formulations. The easiest to state are often in terms of ultrafilters, but the most useful involve elementary embeddings:

An **inner model** is a class $M = \{x\colon \varphi(x)\}$, for some formula φ, such that ZFC holds in M. An **elementary embedding** $j: V \to M$, where V is the universe of sets, is a function such that for every $a_1, \dots, a_n \in V$, and for every formula $\psi(x_1, \dots, x_n)$, $\psi(a_1, \dots, a_n)$ holds if and only if $\psi\big(j(a_1), \dots, j(a_n)\big)$ holds in M. M is **closed under λ-sequences** if ${}^\lambda M$, the class of all λ-sequences of members of M, is a subclass of M.

One can prove that κ is measurable if and only if there is an inner model M closed under κ-sequences and an elementary embedding $j: V \to M$ such that $j(\kappa) > \kappa$.

We now define supercompactness as a stronger version of measurability: κ is a **supercompact cardinal** if for every $\lambda \geqslant \kappa$, there is an inner model M_λ closed under λ-sequences, and an elementary embedding $j_\lambda: V \to M_\lambda$, such that $j_\lambda(\kappa) > \lambda$.

As for measurability, there is an equivalent formulation, which we omit, which avoids the apparent difficulty of quantifying over formulas.

I have listed these cardinals in order of increasing strength, i.e., every supercompact cardinal is strongly compact, every measurable cardinal is weakly compact, and so forth. Finer analyses of large cardinals consider a hierarchy of **consistency strength**, i.e., the consistency of a cardinal with property P implies the consistency of a cardinal with property Q. For example, define huge cardinals as another generalization of measurable ones.

A cardinal κ is a **huge cardinal** if there is an inner model M and elementary embedding $j: V \to M$ such that $j(\kappa) > \kappa$ and M is closed under $j(\kappa)$-sequences.

Huge cardinals need not be supercompact, but their consistency strength is strictly stronger than supercompactness. For more information about large cardinals, see Kanamori's book [5]. Large cardinals may either be used directly in a proof or may be used to construct a model of set theory in which a desired proposition holds.

Very often a large cardinal property is a generalization of a property of $\aleph_0$ to the uncountable. For instance, $\aleph_0$ is clearly inaccessible: it is regular and $2^n < \aleph_0$ for all $n \in \omega$. For another example, consider trees. A κ-**Aronszajn tree** is a tree of height κ whose levels are of cardinality less than κ and with no κ-branch. For definitions, see Todorčević's survey [KV, Chapter 6]. **König's Tree Lemma** says there is no $\aleph_0$-Aronszajn tree. Generalizing this plus inaccessibility gives us weak compactness. Likewise the existence of ultrafilters on ω says that $\aleph_0$ is 'measurable'.

Large cardinals sometimes appear in purely topological circumstances. We know from ***Jones' Lemma*** that $2^{|D|} \leqslant 2^{d(X)}$ whenever D is a closed discrete subset of a normal space X, where $d(X)$ denotes the ***density*** of X. The ***extent*** of X, denoted $e(X)$, is the supremum of the cardinalities of the closed discrete subsets of X and this suggests the natural question whether also $2^{e(X)} \leqslant 2^{d(X)}$ for normal spaces. This leads to inaccessible cardinals: if $2^{e(X)} > 2^{d(X)}$ then $e(X)$ is a **weakly inaccessible cardinal** (a regular limit cardinal) and from an inaccessible cardinal one can prove the consistency of the existence of a normal space satisfying the above inequality.

An inaccessible cardinal is weakly compact if and only if its ***absolute*** is normal – here the cardinal carries the ***order topology***.

Measurable cardinals date back to the 1930s and have a number of significant – although isolated – direct applications to general topology. For example, a discrete space of size κ is ***realcompact*** if and only κ is not Ulam-measurable. It is also easy to prove that if X is a Lindelöf space with points G_δ, then $|X|$ is less than the first measurable cardinal. On a different tack the existence of a measurable cardinal is equiconsistent with the existence of a ***Baire space*** without isolated points which is **irresolvable**, i.e., any two dense sets meet.

Strongly compact and weakly compact cardinals can be equivalently formulated topologically: κ is strongly compact if and only if the κ**-box product** of κ**-compact space**s is κ-compact, wherein one takes the ***Tychonoff Product Theorem*** and replaces "finite" by "$< \kappa$" everywhere; κ is weakly compact if the ordinary product of κ-compact spaces is again κ-compact.

There are other straightforward applications, e.g., if κ is weakly compact and X is $< \kappa$-collectionwise Hausdorff and $\chi(X) < \kappa$, then X is κ-collectionwise Hausdorff. A more difficult direct application is due to Watson, who proved that if there is a strongly compact cardinal, there is a σ***-discrete*** hereditarily normal ***Dowker space***, see the corresponding article in this volume.

The most significant uses of large cardinals in topology occur in contexts in which one is proving the consistency of universal statements about objects of unbounded cardinality, for example, the **Normal Moore Space Conjecture**: all normal Moore spaces are metrizable, or the **Moore–Mrówka problem**: compact spaces of countable tightness are sequential. The latter is an application of the ***Proper Forcing Axiom*** (see the previous article), which is proved consistent from the consistency of a supercompact cardinal, and applications of which – in contrast to those of Martin's Axiom – often require the practitioner to actually do some forcing. Frequently, finer analyses of PFA consequences reveal that in fact one need only consider objects of bounded cardinality, in particular $\aleph_1$. In such cases, a more delicate forcing argument enables one to avoid large cardinals. That is the case with the Moore–Mrówka problem referred to above.

The Normal Moore Space Conjecture – more generally the assertion that normal spaces of character less than the continuum are collectionwise normal – was first shown consistent by Nyikos who derived it in ZFC from the ***Product Measure Extension Axiom***, which had been proved consistent by Kunen from the existence of a strongly compact cardinal (see [KV, Chapter 16]). More general results that do not depend on measures were established in [3], wherein a general framework was set up for proving the consistency of universal topological assertions involving objects of unbounded cardinality and spaces of small character from the consistency of supercompact (in special cases, strongly compact) cardinals. Applying the machine to, e.g., getting a model in which all silly spaces of character $< \kappa$ (the supercompact cardinal which will become small, e.g., $2^{\aleph_0}$) are ridiculous will reduce the problem to showing an appropriate forcing notion **preserves** non-ridiculousness. When we say a forcing *preserves* a topological property, we mean that if a space satisfies the property in the ground model, then the space it generates in the extension satisfies the property there.

Balogh noticed that in the particular context of normality vs. collectionwise normality, this framework could be modified, weakening the character restriction, in order to obtain that if it is consistent that there is a supercompact cardinal, it is consistent that all normal spaces of ***pointwise countable type*** – in particular all locally compact normal spaces and all first-countable spaces – are collectionwise normal; a space is of **pointwise countable type** (or of **point-countable type**) if every point is contained in a compact set with a countable neighbourhood base. Grunberg, Junqueira and Tall [4] then extended the method of [3] to obtain a general method for proving the consistency of universal topological assertions involving objects of unbounded cardinality and spaces of small pointwise type, obtaining in particular a more useful proof of Balogh's result.

Supercompact cardinals are the most useful of large cardinals for topologists, because they yield the Proper Forcing Axiom, and, in contrast to weakly compact, measurable, and huge cardinals, not only affect the cardinal itself, but all larger cardinals. This phenomena is referred to as **reflection**. Roughly speaking it tells us that if there is a counterexample to some universal statement φ, there is one of size less than the supercompact cardinal. For particular φ, one can then try to make the supercompact cardinal κ small (e.g., $\aleph_1$, $\aleph_2$ or $2^{\aleph_0}$) by forcing, and perhaps be able to prove that the reflection phenomena for φ at κ still hold.

In a similar vein one can take a property holding at a large cardinal and bring it down so that some small cardinal such as $\aleph_1$, $\aleph_2$, or $2^{\aleph_0}$ has that property. In this case – where one is not so interested in what happens for cardinals larger than that particular small cardinal – one can usually make do with a weakly compact or measurable cardinal rather than a supercompact cardinal. For example, a weakly compact cardinal may be collapsed to $\aleph_2$ to create a model in which there are no $\aleph_2$-Aronszajn trees, see [KV, Chapter 6]. A more topological example is Shelah's theorem that if it is consistent that there is a weakly compact cardinal then it is consistent that there are no Lindelöf spaces of size $\aleph_2$ with all points G_δ, which brings down the fact there is no Lindelöf space of size κ with all points G_δ, if κ is weakly compact.

The first use of supercompact cardinals in topology was due to Shelah who proved that if it is consistent there is a supercompact cardinal, then it is consistent that locally separable first-countable $\aleph_1$-collectionwise Hausdorff spaces are collectionwise Hausdorff. The proof (or its progenitor concerning reflection of ***stationary sets***) is a prototypical argument; other applications include Axiom R and its consequences, see [2] and [HvM, Chapter 1].

Whenever one uses large cardinals to establish (the consistency of) a topological statement, one wonders whether they are actually necessary. As mentioned earlier, consequences proved from PFA can often be proved consistent without large cardinals. The usual way one shows that a statement φ requires large cardinals for its proof is to show that if φ holds, then there is an inner model which has a large cardinal. For example, having noted [Ku, VIII, §3] that the consistency of the existence of an inaccessible cardinal enables one to prove the consistency of there being no Kurepa trees, one then shows that if there is no ***Kurepa tree***, then $\aleph_2$ is an inaccessible cardinal in Gödel's constructible universe L (see [Ku, VII, B9]), and hence that it is consistent that there is an inaccessible cardinal.

L is an **inner model for inaccessible cardinals**, which means that if κ is an inaccessible cardinal, then κ is inaccessible in L. There are more complicated inner models for a measurable cardinal, for "many" measurable cardinals (e.g., if there is a sequence of measurable cardinals, they are all still measurable in the inner model), etc. These inner models however do have L-like characteristics which are useful in showing that large cardinals are required in order to obtain the consistency of certain propositions. Finding inner models for supercompact cardinals is an ongoing area of research in set theory; at present there are no good techniques for showing that the use of a supercompact cardinal in a consistency proof is necessary. In practice, topologists have not as yet actually engaged in inner model theory, but rather have shown that topological statements imply combinatorial statements of known large cardinal strength. A typical example is Fleissner's proof (see [KV, Chapter 16]) that the Normal Moore Space Conjecture has large cardinal strength. He proved that the NMSC entailed the failure of the **Covering Lemma**. The Covering Lemma (for an inner model M) asserts that every uncountable set is included in some member of M of the same cardinality. The failure of the Covering Lemma for, e.g., M, an inner model for "many" measurables implies that in fact there are "many" measurables in M. Another example of this technique is due to C. Good, who showed that the Covering Lemma entails the existence of *a first-countable Dowker space* with sundry additional properties (see the article on Dowker spaces). Note: the Covering Lemma for L is often referred to via an equivalent formulation as "$0^\#$ does not exist".

3. Methods and models

In this section we shall briefly discuss some of the most useful models employed in consistency results, equivalently, the most useful partial orders.

Cohen reals

The simplest non-trivial forcing employs the partial order of finite partial functions from ω into 2 ordered by reverse inclusion. It adds a new $f:\omega \to 2$ and hence a new real, called a **Cohen real**. Interestingly, from a single Cohen real one can already construct a Souslin tree.

Using finite partial functions from $\kappa \times \omega$ into 2 instead, as mentioned previously, one shows that κ new reals are added. Since $|\kappa| = |\kappa \times \omega|$, one may as well use κ instead of $\kappa \times \omega$. This forcing is referred to as "adding κ Cohen reals" and the corresponding extension is commonly called the **Cohen model**. It is particularly useful when κ is supercompact since reflection phenomena persist. For example, every space of character $< 2^{\aleph_0}$ in which subspaces of size $< 2^{\aleph_0}$ are metrizable is metrizable [3]. Aside from general machinery for handling large cardinals and forcing, the key lemma is that adding Cohen reals preserves non-metrizability. Typically, one gets some weaker results without large cardinals, e.g., upon adding $\aleph_2$ Cohen reals, if a first-countable space of weight $\aleph_1$ has all its subspaces of size $\aleph_1$ metrizable, then it is metrizable.

Countably closed forcing

Countably closed forcing (every countable descending sequence of conditions has a lower bound) covers a wide variety of models, so it is somewhat misleading to group them together since they may exhibit incompatible behaviors. Nonetheless they do have common features such as L-like phenomena (e.g., $\diamondsuit$ is forced if new subsets of ω_1 are added) and the use of an ω_1-descending sequence of forcing conditions deciding the properties of a function from ω_1 into V in the extension. Most of the time one is interested in such orders with countable conditions. The simplest example is forcing with countable partial functions from κ into 2, where κ is a regular cardinal. For $\kappa \geqslant \aleph_2$, this yields a model in which normal spaces of character $< \kappa$ are $\aleph_1$-collectionwise Hausdorff. This forcing is called "adding κ Cohen subsets of ω_1".

A very useful one alluded to earlier is the Lévy-collapse, which makes a large cardinal κ into $\aleph_2$ by creating maps from ω_1 onto each uncountable cardinal smaller than κ. Countably closed partial orders using countable conditions to create a subset of ω_2 have been employed to construct complicated Lindelöf spaces, e.g., ones of size $\aleph_2$ with points G_δ or ones of that size with no Lindelöf subspaces of size $\aleph_1$.

Random real forcing

This is forcing with the product measure algebra on $\{0,1\}^\kappa$. We say we are "adding κ random reals". See [KV, Chapter 20] for a description. The new **random real**s are obtained from a generic filter G as follows: let F denote the set of finite partial functions from κ to 2 whose associated clopen subset of $\{0,1\}^\kappa$ belongs to G. The union $\bigcup F$ is a total function from κ to 2 and gives rise to new reals via a bijection between κ and $\kappa \times \omega$.

Adding at least $\aleph_2$ random reals yields a model in which $\mathfrak{d} = \aleph_1$ yet $2^{\aleph_0} > \aleph_1$, see the article on $\beta\mathbb{N}$ for the definition of $\mathfrak{d}$. It is particularly interesting to force to add $\aleph_2$ or more random reals over a model of $\mathsf{MA} + \neg\mathsf{CH}$, since much of that axiom is preserved. For instance, in contrast with just one Cohen real, no Souslin tree is created. Adding strongly compact many random reals produces a model of the Product Measure Extension Axiom mentioned earlier, see [KV, Chapter 16].

Other forcings

There are many other ways of adding new reals; the 'total failure of MA' alluded to above was obtained by adding Sacks reals.

The Proper Forcing Axiom, has many strong consequences mentioned above. To prove a proposition φ is undecidable by $\mathsf{MA} + \neg\mathsf{CH}$, the standard stratagem is to show PFA implies φ, but that a carefully constructed example of $\neg\varphi$ constructed from CH or $\Diamond$ remains an example under countable chain condition forcing. A noteworthy example is Szentmiklóssy's ***S-space*** that is consistent with $\mathsf{MA} + \neg\mathsf{CH}$. Another one is a Baire space of size $\aleph_1$ without isolated points, consistent with $\mathsf{MA} + \neg\mathsf{CH}$.

It is often useful to have some example that contradicts $\mathsf{MA} + \neg\mathsf{CH}$, while retaining as much of that axiom as possible. The standard procedure is to construct the example using CH or $\Diamond$ and then iterate forcing with partial orders having a strong form of countable chain condition that preserves the example. The first such example was a Souslin line consistent with Martin's Axiom for partial orders with **property** K (every uncountable set contains an uncountable subset of pairwise compatible elements).

Consistency results in topology have come a long way from the initial applications of $\mathsf{MA} + \neg\mathsf{CH}$ and $V = L$. A recent trend in particular is to obtain models in which consequences of these two contradictory axioms hold simultaneously. The key idea is to first force to construct a particularly nice ***Souslin tree*** S, then force as much of $\mathsf{MA} + \neg\mathsf{CH}$ as can possibly be compatible with the existence of S, and then force with S. This stratagem enabled Larson and Todorčević to solve an old problem of Katětov by showing it consistent that whenever X^2 is compact hereditarily normal, then X is metrizable. In this model there are no S- or L-subspaces of compact first-countable spaces – a strong consequence of $\mathsf{MA} + \neg\mathsf{CH}$, yet there are also no ***Q-set***s, contradicting $\mathsf{MA} + \neg\mathsf{CH}$.

Extending the method to PFA, rather than MA, yields that subspaces of compact spaces with countable tightness are hereditarily Lindelöf if and only if they are hereditarily separable (Todorčević). By doing preliminary forcing one can also get normal first-countable spaces to be collectionwise Hausdorff. This was used by Larson and Tall to obtain the consistency (relative to a supercompact cardinal) of every locally compact perfectly normal space being paracompact.

By using proper partial orders that don't add reals, one can get some consequences of PFA consistent with CH. For example, that every compact space of countable tightness is sequentially compact and has points of character $\leqslant \aleph_1$. It is possible to obtain some of the consequences of PFA that imply $\neg\mathsf{CH}$ and yet still have $2^{\aleph_0} < 2^{\aleph_1}$. This was accomplished by Eisworth, Nyikos and Shelah in order to obtain the consistency of there being no separable, hereditarily normal, locally compact space of size $\aleph_1$.

As with any other field of mathematics that has reached a certain level of maturity, particular difficult problems require either a new method apparently unique to them or an intricate combination of known methods. An example of the latter is the difficult forcing construction by Gruenhage and Koszmider of a locally compact normal metacompact space which is not paracompact.

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j-4 Digital Topology

1. Digital topology in computer image analysis

In image processing and computer graphics, objects in Euclidean n-space $\mathbb{R}^n$ are frequently represented in an n-dimensional binary image array (each of whose elements has a value of 1 or 0). Each object is represented by a set of 1's in the array. Usually $n = 2$ or 3, but there is also some interest in higher-dimensional cases. (Indeed, a 4-dimensional image array can be used to represent a time-varying 3D scene. Moreover, since the state space of a system with n degrees of freedom is a subset of $\mathbb{R}^n$, n-dimensional binary image arrays are of practical interest even for $n > 4$.) We assume the array is unbounded in all directions, but also assume that only a finite number of its elements have a value of 1. More precisely, we define an **n-dimensional binary image array** to be a map $I : \mathbb{Z}^n \to \{0, 1\}$ such that $I(p) = 1$ for only finitely many points $p \in \mathbb{Z}^n$. A point $p \in \mathbb{Z}^n$ is called a 1 or a 0 of I according to whether $I(p) = 1$ or $I(p) = 0$.

Digital topology deals with properties and features of a binary image array that correspond to simple topological properties (e.g., ***connectedness***) and topological features (e.g., boundaries) of objects represented in the array and of the complements of those objects. Concepts and results of digital topology are used to specify and justify some important low-level image processing algorithms, including algorithms for thinning, boundary extraction, object counting, and contour-filling.

Digital topology was first studied in the late 1960s, by leading computer image analysis researcher Azriel Rosenfeld, whose early journal articles on the subject (e.g., [19, 20]) played a major role in establishing the field. See [8] for more on these seminal papers. The term 'Digital Topology' was itself invented by Rosenfeld, who used it in [20].

We use the term **pixel** to mean an upright closed unit square centered at a point in $\mathbb{Z}^2$, and use the term **voxel** to mean an upright closed unit cube centered at a point in $\mathbb{Z}^3$. (Thus a pixel is an interval $[i_1 - \frac{1}{2}, i_1 + \frac{1}{2}] \times [i_2 - \frac{1}{2}, i_2 + \frac{1}{2}]$ in $\mathbb{R}^2$, and a voxel is an interval $[i_1 - \frac{1}{2}, i_1 + \frac{1}{2}] \times [i_2 - \frac{1}{2}, i_2 + \frac{1}{2}] \times [i_3 - \frac{1}{2}, i_3 + \frac{1}{2}]$ in $\mathbb{R}^3$, where the i's are integers.) Two pixels are said to be **8-adjacent** and two voxels are said to be **26-adjacent** if the two are distinct but share at least a vertex. Two pixels are said to be **4-adjacent** and two voxels are said to be **18-adjacent** if the two are distinct but share at least an edge. Two voxels are said to be **6-adjacent** if the two are distinct but share a 2D face. Thus in each of these cases κ-adjacency is defined in such a way that a pixel or voxel is κ-adjacent to exactly κ others.

Let $n = 2$ or 3, let $p, q \in \mathbb{Z}^n$, and let $\kappa \in \{4, 8\}$ or $\kappa \in \{6, 18, 26\}$ according to whether $n = 2$ or $n = 3$. Then p is said to be **κ-adjacent** to q if the pixels or voxels centered at those two points are κ-adjacent. This irreflexive symmetric binary relation on $\mathbb{Z}^n$ is frequently denoted just by the integer κ. A **κ-path** is a nonempty sequence of points in which each point (except the last) is κ-adjacent to the next point; if its first and last points are respectively p and q, then it is called a κ-path from p to q. Now let $S \subseteq \mathbb{Z}^n$. For all $x, y \in S$, x is said to be **κ-connected in** S to y if there is a κ-path in S from x to y. This is an equivalence relation on S; each of its equivalence classes is called a **κ-component** of S. We say S is **κ-connected** if S consists of just one κ-component, or if $S = \emptyset$.

Rosenfeld's approach to digital topology, which is now the standard approach to the subject in computer image analysis, is based on the use of a pair of binary adjacency relations (κ_1, κ_0). In the 2D case, $(\kappa_1, \kappa_0) = (8, 4)$ or $(4, 8)$; in the 3D case, (κ_1, κ_0) = (26,6), (18,6), (6,26), or (6,18). We use κ_1-connectedness as our definition of connectedness for sets of 1's, but use κ_0-connectedness as our definition of connectedness for sets of 0's.

Ref. [19], the first journal article on digital topology, convincingly demonstrated that this seemingly bizarre idea of using different definitions of connectedness on 1's and 0's actually leads to a workable and useful mathematical theory. For an expository introduction to some of the basic ideas of Rosenfeld's approach to digital topology, see, e.g., [12] and [8].

The following theorem, an analog for $\mathbb{Z}^2$ of the ***Jordan Curve Theorem*** for $\mathbb{R}^2$, is one of the earliest and best known results of digital topology. It well illustrates the need to use different notions of connectedness on 1's and 0's, because the theorem is false when $\kappa_1 = \kappa_0 = 4$ (in which case I's set of 0's may have more than two κ_0-components) and also false when $\kappa_1 = \kappa_0 = 8$ (in which case I's set of 0's may have just one κ_0-component). For $\kappa = 4$ or 8, a **Jordan κ-curve** is a nonempty finite κ-connected set $C \subset \mathbb{Z}^2$ each of whose points is κ-adjacent to exactly two other points of C.

THEOREM 1 (Rosenfeld [19, 20]). *Let $(\kappa_1, \kappa_0) = (8, 4)$ or $(4, 8)$. Let I be a 2-dimensional binary image array with more than four 1's, whose set of 1's is a Jordan κ_1-curve. Then I's set of 0's has exactly two κ_0-components, and every 1 of I is κ_0-adjacent to at least one 0 in each of those two κ_0-components.*

Another approach to digital topology was developed in the 1980s by Kovalevsky [18]. This approach is quite different from that of Rosenfeld, and makes no direct use of adjacency relations on 1's and 0's. Instead, we subdivide $\mathbb{R}^n$ into a cubical ***cell complex***, and choose **face-membership rules** which associate, with each set S of open n-cells of the complex, certain lower-dimensional open cells of the complex that lie in the closure of $\bigcup S$. (An **open cell** of the complex is a set obtained from a cell of the complex by removing the points on the cell's proper faces. Thus an open n-cell of the complex is open in $\mathbb{R}^n$, and a lower-dimensional open cell is open in its own affine hull in $\mathbb{R}^n$.) We identify the points of $\mathbb{Z}^n$ with the open n-cells of the complex. Given any n-dimensional binary image array I, we take the union of the set of open n-cells that correspond to 1s of I with the set of lower-dimensional open cells which our chosen face-membership rules associate with that set of open n-cells. On the resulting (finite) set of open cells Kovalevsky defines a topology; his topology can be obtained as a quotient of the usual Euclidean metric topology on the union of those open cells, by identifying the points within each of the open cells. Properties of the topological space that is obtained in this way are regarded as "topological properties" of the image I. The more recent "lighting function" theory of Ayala, Domínguez, Francés, and Quintero may be regarded as an extension of Kovalevsky's theory. (See, e.g., their paper in [22], though Kovalevsky's topology on open cells is not explicitly used in the paper.)

A third approach to digital topology was suggested by Khalimsky (also in the 1980s) [5]. **Khalimsky's topology** on $\mathbb{Z}^n$ is the product of n copies of the topology on $\mathbb{Z}$ for which $\{\{2i-1, 2i, 2i+1\} \mid i \in \mathbb{Z}\}$ is a subbase. For each n-dimensional binary image array I, we consider the relativization of Khalimsky's topology on $\mathbb{Z}^n$ to the set of 1s of I. Properties of this topology are regarded as topological properties of I. An interesting result relating to this approach to digital topology is the 3D Jordan surface theorem of [14].

Of the three approaches to digital topology that have been mentioned, Rosenfeld's is by far the most widely used in practice.

Ref. [12] is a survey of work on digital topology through the mid-1980s. The appendix of the book [13] has a bibliography of over 350 references, which includes most publications relating to digital topology that appeared before 1995 in the computer image analysis literature. For more recent work, see, e.g., the journal special issues [21] and [22], each of which includes a number of papers on digital topology. Other digital topology references are given in the bibliography of [8]. A particularly significant recent reference is the book [3]. The forthcoming book [6] will include chapters relating to digital topology.

2. Comparing digital and metric spaces

Among the approaches to digital topology discussed in the first section, only Khalimsky's (based on Khalimsky's space $\mathbb{Z}^n$) is purely topological in that concepts used in digital topology, such as "connected" and "boundary", are given their topological meanings. It is not difficult to show that the Khalimsky topology on $\mathbb{Z}$ is (up to homeomorphism) the only one such that the resulting space is a **COTS**: a connected space such that among any three distinct points is one whose deletion from the space leaves the others in distinct components of the remainder (see [10]). Intervals in $\mathbb{R}$ are also COTS.

In this section we explain why any such purely topological approach must be based on this space. Such an explanation, and much more, results from the notion of metric analog, defined in [9] by Kong and Khalimsky. Our definition uses the following terminology: As usual, given spaces with base points (A, a_0) and (M, m_0), a (**base point preserving**) map $f:(A, a_0) \to (M, m_0)$ is a continuous function such that $f(a_0) = m_0$. Given two such maps f and g, a **homotopy** $F: A \times [0,1] \to M$ from f to g is a continuous function such that, whenever $a \in A$ and $t \in [0,1]$, $F(a,0) = f(a)$, $F(a,1) = g(a)$, and $F(a_0, t) = m_0$.

Given a quotient $q: M \to X$, and two maps $f, g: A \to M$ such that $qf = qg$, a homotopy F from f to g such that $t \to qF(x,t)$ is constant for each x is said to *be ignored by* q. We then say f, g are *q-quotient homotopic* (via F) and write $f \overset{q}{\simeq} g$.

DEFINITION 1. A **metric analog** of a topological space with base point (X, x_0) is a metric space with base point (M, m_0), with an open quotient map $q: M \to X$, such that, if (A, a_0) is any metric space with base point, then:

- for any map $f: A \to X$ there is a map $\hat{f}: A \to M$ such that $f = q\hat{f}$, and
- any two maps $f, g: A \to M$ such that $qf = qg$ are q-quotient homotopic.

Composition by the open quotient q induces a bijection between the path components of M and those of X, and this composition induces isomorphisms between the homotopy groups of M and those of X; that is to say, q is a weak homotopy equivalence between M and X.

As a central case of the above, let $q: \mathbb{R} \to \mathbb{Z}$ be defined by setting, for $n \in \mathbb{Z}$, $q(2n) = 2n$ and $q(x) = 2n+1$ whenever $x \in (2n, 2n+2)$; this is clearly an open quotient. Also let $r_0 \in \mathbb{R}$, $z_0 \in \mathbb{Z}$, be base points such that $q(r_0) = z_0$. We show that $(\mathbb{R}, r_0)$, with q, is a metric analog of $(\mathbb{Z}, z_0)$:

To see the two special properties of the definition, note that if $((A,d), a_0)$ is a metric space with base point and $f: A \to \mathbb{Z}$, then the function $\hat{f}$, defined by

$$\hat{f}(x) = 2n + \frac{2d(x, f^{-1}\{2n\})}{d(x, f^{-1}\{2n\}) + d(x, f^{-1}\{2n+2\})}$$

if $f(x) \in \{2n, 2n+1\}$ is continuous; it certainly satisfies $f = q\hat{f}$. For the other property, notice that if g is another such map and $qf = qg$, then $f(x), g(x)$ are always in the same interval of the form $[2n, 2n]$ or $(2n, 2n+2)$, thus $H(x,t) = (1-t)f(x) + tg(x)$ is also in the same one of these intervals, which is to say that, for each x and t, $qH(x,t) = qf(x) = qg(x)$. H is clearly continuous, so a homotopy from f to g ignored by q.

The map $q^n: \mathbb{R}^n \to \mathbb{Z}^n$ also has the above properties, as seen coordinatewise, and similar considerations (see [11]) show that each Alexandroff T_0 space has a metric analog. The following summarizes key facts about the existence and uniqueness (up to homotopy equivalence) of metric analogs. (Below, let $\mathbf{1}_A$ denote the identity map on A.)

THEOREM 2.

(a) *Each T_0 countable join of Alexandroff topologies has a metric analog, so all second-countable T_0 spaces have them* (*see* [11]).

(b) [9] *Suppose (M, q) is a metric analog of a space X. If (N, r) is another metric analog of X, then there are maps $f: M \to N$ and $g: N \to M$ such that $gf \overset{q}{\simeq} \mathbf{1}_M$ and $fg \overset{q}{\simeq} \mathbf{1}_N$. Conversely, if N is a metric space, $r: N \to X$ is an open quotient, and there are maps $f: M \to N$, $g: N \to M$ so that $gf \overset{q}{\simeq} \mathbf{1}_M$ and $fg \overset{q}{\simeq} \mathbf{1}_N$, then (N, r) is metric analog of X.*

The converse part of (b) is useful in creating other metric analogs from a given one. In particular, it is used to show the existence, for each locally finite space, of a **polyhedral analog**: one whose space is a geometric realization of the abstract simplicial complex which has a vertex identified with each point in the finite space, and whose simplices are its specialization order chains, with the quotient map which takes each point of this metric space into the specialization-largest vertex of the smallest simplex whose geometric realization contains the point. The proof in [14] of a Jordan surface theorem for three-dimensional digital spaces uses polyhedral analogs.

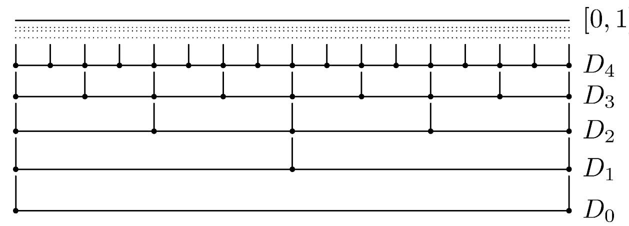
The definition of 4-adjacency in $\mathbb{Z}^2$ (6-adjacency in $\mathbb{Z}^3$) is easily extended to obtain $2n$-adjacency in $\mathbb{Z}^n$, and that of 8-adjacency in $\mathbb{Z}^2$ (26-adjacency in $\mathbb{Z}^3$) is easily extended to get $(3^n - 1)$-adjacency in $\mathbb{Z}^n$. In [7], the fact that $\mathbb{R}^n$ is a metric analog of $\mathbb{Z}^n$ is used to establish the following result, which is the result we promised at the beginning of the section.

THEOREM 3. *The Khalimsky topology on $\mathbb{Z}^n$ is, up to translation, the only simply connected topology on $\mathbb{Z}^n$ whose connected sets include all $2n$-connected sets but no $(3^n - 1)$-disconnected sets.*

3. Approximating compact Hausdorff spaces with finite spaces

The above shows how certain familiar infinite spaces can be used to study finite spaces. In fact thinking in terms of polyhedral and metric analogs, diagrams of polytopes with a finite number of simplices can be seen as representing finite spaces by viewing features as singletons, and sets that "look" open to be open (such diagrams can be seen in [10] and [5]). So it is not difficult to gain intuition and knowledge about finite spaces. In this section, we reverse the above flow of knowledge, by using finite spaces to approximate others.

The next diagram (from [17]) suggests a way to approximate the unit interval by finite spaces. Its top horizontal line represents the unit interval, but those at the bottom are meant to be finite COTS: $D_n = \{\frac{i}{2^n} \mid 0 \leqslant i \leqslant 2^n\} \cup \{(\frac{i}{2^n}, \frac{i+1}{2^n}) \mid 0 \leqslant i < 2^n\}$, with $2^{n+1} + 1$ points and the quotient topology induced from $[0, 1]$. The vertical lines indicate maps going down, for which a closed point is the image of the one directly above it, while an open point is that of the three above it. *Caution*: To read the diagram correctly, note particularly that (counter to our intuition and training) each open real interval $(\frac{i}{2^n}, \frac{i+1}{2^n})$ is an *element*, rather than an infinite subset, of D_n:



To formalize the above, we define a **Hausdorff reflection** of a topological space X: it is a Hausdorff space X_H with a continuous map $h: X \to X_H$ such that whenever $f: X \to Z$ is continuous and Z is an Hausdorff space, then for some unique $\hat{f}: X_H \to Z$, $f = \hat{f}h$. The Hausdorff reflection is unique up to homeomorphism; also, h is known to be surjective. We then have:

THEOREM 4 [17]. *A T_2 space is the Hausdorff reflection of an inverse limit of finite T_0-spaces and quotient maps if and only if it is compact.*

[17] *It is the Hausdorff reflection of an inverse limit of connected finite T_0-spaces and quotient maps if and only if it is connected and compact.*

Hausdorff reflections may be hard to visualize in many cases, but we can do much better. We need two further classes of maps. A continuous map $f: X \to Y$ is:

normalizing if inverse images of disjoint closed sets are contained in disjoint open sets,

chaining if whenever $y, z \in \text{cl}\{x\}$, then $f(y) \in \text{cl}(f(z))$ or $f(z) \in \text{cl}(f(y))$.

calming if chaining and whenever $C \subseteq f[\text{cl}(x)]$ is a chain, then $C = f[\text{cl}(y)]$ for some $y \leqslant x$.

These definitions are essentially from [15, 16], but continuity is not part of the definitions there; further, a map is calming if and only if a certain associated map is closed, and that is used in its definition there. Note that a topological space is normal if and only if $\mathbf{1}_X$ is normalizing. Also, the maps at the beginning of this section are chaining maps. Then we have that:

THEOREM 5. *A topological space (X, τ) is*:

(a) [4] *a chainable continuum if and only if it is the space of minimal points of an inverse limit of COTS and calming maps.*

(b) [16] *a k-dimensional continuum if and only if it is the space of minimal points of an inverse limit of k-dimensional finite connected spaces and calming maps.*
(c) [15] *a continuum if and only if it is the space of minimal points of an inverse limit of connected finite spaces and chaining maps.*
(d) [15] *a compact Hausdorff space if and only if it is the space of minimal points of an inverse limit of finite spaces and chaining maps.*

Indeed, the Hausdorff reflections of the inverse limits of chaining maps and finite T_0-spaces referred to in these results, are inverse limits of the polyhedral analogs of the spaces. Theorem 5(b) is closely related to a 75-year old result of Alexandroff (see [1]).

The above has led to the discovery that members of an important class of non-Hausdorff topological spaces are normal. First, we use bitopology to define a collection of topological spaces large enough to include the finite T_0-spaces and the compact Hausdorff spaces, but which, like the latter class, is a complete category under the appropriate maps. This is the class of skew compact spaces (see [17] for further discussion):

A bitopological space (X, τ, τ^*) is **pseudoHausdorff** (pH) if whenever $x \notin \mathrm{cl}_\tau\{y\}$ then there are $T \in \tau$ and $T^* \in \tau^*$ which are disjoint and such that $x \in T$ and $y \in T^*$. It is a **joincompact space** if $\tau \vee \tau^*$ is compact and T_0, and both (X, τ, τ^*) and (X, τ^*, τ) are pH. A topological space (X, τ), is **skew compact** if there is a topology, τ^* on X, such that (X, τ, τ^*) is joincompact.

Inverse limits of skew compact spaces, with maps that are continuous with respect to both topologies, are skew compact [17]. Also, it is not difficult to see that compact Hausdorff spaces are skew compact with $\tau^* = \tau$, and finite T_0-spaces are skew compact, with τ^* the collection of τ-closed sets (a topology since X is finite). By these results, functions continuous with respect to compact Hausdorff or finite T_0-space topologies, are continuous with respect to τ^*, so inverse limits of these spaces, with continuous maps, are skew compact.

By (iv) below, the minimal point subspaces in the previous result are precisely the Hausdorff reflections of the spaces, discussed earlier:

THEOREM 6 [15]. *Let be X a skew compact space. The following are then equivalent*:

(i) *X is normal,*
(ii) *X is an inverse limit of finite spaces and normalizing maps,*
(iii) *each point of X has a unique closed point in its closure,*
(iv) *$m : X \to X$, defined by $m(x)$ is the unique closed point in $\mathrm{cl}(x)$, is a retract from (X, τ) to its subspace $\mu(X)$, of closed points.*

If any of these hold, $\mu(X)$, together with m, is the Hausdorff reflection of (X, τ).

The proof of (d) in [15] above brings together many of these issues and some classical results: The spectral spaces are precisely the inverse limits of finite T_0-spaces and continuous functions (proved in our notation, with further references given, in [17]). These are so named since the prime spectrum of a commutative ring with identity with the ***hull-kernel topology*** is a spectral space whose specialization is easily seen to be reverse set inclusion. If X is compact Hausdorff, then it is homeomorphic to the subspace of maximal ideals, which is the space of specialization-minimal elements of the prime spectrum $\mathrm{Spec}(C(X))$ of the ring $C(X)$ with the hull-kernel topology (see [2]; 7M). Since the ideals in $C(X)$ which include a given prime ideal form a chain under set inclusion (see [2]; 14.8), they form a specialization chain, so $\mathrm{Spec}(C(X))$ is a completely normal spectral space.

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j-5 Computer Science and Topology

The field of study known as Computer Science involves research on the mechanisms, technologies and logic by which information is handled, as well as the development of machines that embody the results of that research in concrete form. Such machines are currently referred to as computers. The prototype of today's computer was the ENIAC (Electronic Numerical Integrator and Calculator) – the world's first electronic calculator, created at the University of Pennsylvania in 1946. The following year, 1947, saw the creation of the EDSAC (Electronic Delay Storage Automatic Calculator) – the world's first means of storing programs internally at Cambridge University. This idea for the computer was put forth by John von Neumann.

Johann (John) Ludwig von Neumann (1903–1957) was a mathematician. He proposed ideas that would later form the basis of computer architecture and thereby determine the future direction of computer development. The computer built based on this proposed architecture became known as the **von Neumann computer**.

Computers after ENIAC can be classified into four developmental stages or generations based on the nature and number of devices used in these machines. First-generation computers (1946–1958) used vacuum tubes, while in second-generation computers (1958–1963) vacuum tubes were replaced by transistors. Integrated circuits were incorporated in third-generation machines (1964–1980), and today's computers are fourth-generation machines realized using LSIs that are backed by pattern desing technology in which hundreds of transistors are connected by means of wiring. Computer science encompasses a wide range of fields, but is centered primarily around electronics. Today's flourishing era of computer science can be directly attributed to the invention of the transistor and the development into the LSI.

Next-generation computers will be classified not by the conventional standards used for classification but rather by the qualitative elements used to generate the functions realized by the computer. This indicates that the development of computers has shifted from an electronics-centered orientation and is moving into a new direction. Simply put, this new direction focuses on Artificial Intelligence. For instance, the pattern recognition technology shown in the following example, method (b) enebles intelligent processing, and the development of that technology is one of the principal aims of computer science in the future.

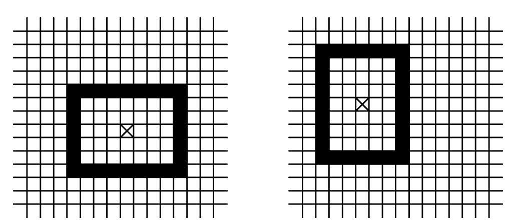
Example

When using a computer to judge whether or not two figures match, the following two methods can be considered.

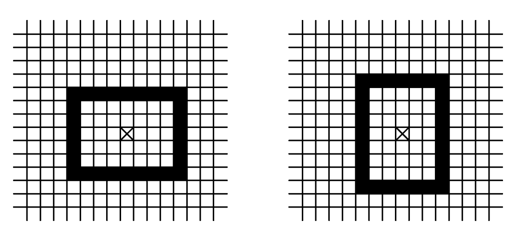
Method (a) is a digital geometric approach that is part of image analysis. In step (1) the center of gravity of a group of black pixels (indicated by $\times$ in the figure) is determined for each of the digital images. In step (2) one of the digital images is shifted in a parallel direction so that the coordinates for the centers of gravity are the same. In step (3) the interval $[0, 2\pi)$ is quantified (divided into eight equal parts in the present example) and one of the digital images is rotated around the center of gravity. If for some level (a digital value expressing the angle of rotation) the rotated digital image matches the other one then the decision is made that the two figures are congruent.

Method (b) is reduced to a problem of computational geometry. In step (1), using the Hough transform, the digital images are recognized as a polygon. In step (2) the orientation of the polygons are corrected so that the sum of the internal angles is $(n-2)\pi$, where n is the number of vertices of the polygon. In step (3) the numerical vectors of the internal angles and the lengths of edges are determined and one set of the numerical vectors is shifted cyclically. If in some stage of shifting the two numerical vectors match then the decision is made that the two figures are congruent.

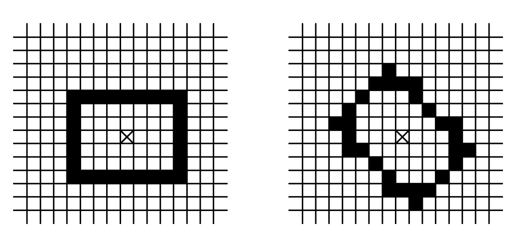
Method (a) is illustrated below.



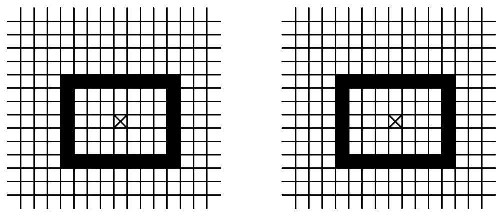
The digital image on the right is shifted by $(3, -2)$



The digital images on the left and right do not match, so the one on the right is rotated $\pi/4$

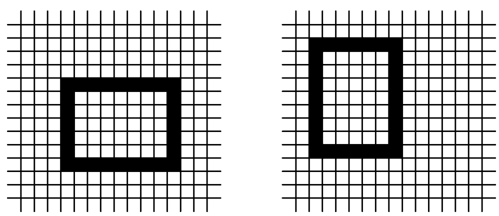


The digital images on the left and right do not match, so the original digital image on the right is rotated again, this time by $\pi/2$



The digital images on the left and right now match, so the decision is made that the figures are congruent

Next we illustrate Method (b). Each of the polygons is



recognized from the digital image:

P_0	P_1	P_2	P_3	:	P_0	P_1	P_2	P_3
(4, 3)	(12, 3)	(12, 9)	(4, 9)	:	(2, 4)	(2, 12)	(8, 12)	(8, 4)

The polygon on the right is corrected by reversing its orientation because $\sum \angle(P_{i-1}P_iP_{i+1}) = 6\pi \neq 2\pi$, where $\angle(P_{i-1}P_iP_{i+1})$ is the angle measured from $\overline{P_iP_{i+1}}$ to $\overline{P_iP_{i-1}}$, counterclockwise.

P_0	P_1	P_2	P_3	:	P_0	P_1	P_2	P_3
(4, 3)	(12, 3)	(12, 9)	(4, 9)	:	(8, 4)	(8, 12)	(2, 12)	(2, 4)

The angle and distance vectors now become

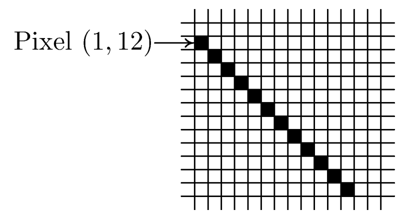
$\angle(P_{i-1}P_iP_{i+1})$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	:	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
$d(P_i, P_{i+1})$	8	6	8	6	:	6	8	6	8

The vectors on the left and right do not match, so the vectors on the right are shifted (cyclically) to the left by one unit.

$\angle(P_{i-1}P_iP_{i+1})$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	:	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
$d(P_i, P_{i+1})$	8	6	8	6	:	8	6	8	6

The vectors on the left and right now match, so the decision is made that the figures are congruent.

Recognizing figures using the **Hough transform** goes as follows. (1) A table is created for (θ, c) based on quantification of the angle θ and the (oriented) distance c. (2) For each of the black pixels, when the straight line $-x\sin\theta + y\cos\theta + c = 0$ determined by θ an c passes the pixel within the quantification tolerance, the frequency of (θ, c) is increased by one. (3) The digital image is recognized as the straight line determined by the (θ, c) with the highest frequency in the table. For instance, if the following digital image is processed with $[0, \pi)$ being quantified by the partial width $\pi/4$ and $[-20, 20]$ by the partial width 1, then the frequency of $(3\pi/4, 9)$ will be 12, which will be the highest value and the figure will be recognized as the straight line $-\frac{1}{2}\sqrt{2}x - \frac{1}{2}\sqrt{2}y + 9 = 0$, or $y = -x + 9\sqrt{2}$. In this process, the pixel (1, 12)



increases the frequency of each of $(0, -12)$, $(\pi/4, 0)$, $(\pi/2, 1)$ and $(3\pi/4, 9)$ by one.

Topology

Topology had its beginnings in the research of the non-quantitative properties of geometric figures and it currently serves as the fundamental field underlying mathematics and computer science.

Long before, the need for such a field of study was first indicated by Leibniz, in passing through the pioneering research of Euler, Möbius and others; the field was later given form by Cantor, Poincaré and others. The Euler polyhedron theorem (1752) is an early result on non-quantitative properties of geometric figures. The Jordan Curve Theorem (1893) and the ***Brouwer Fixed Point Theorem*** (1910) are results from the period when topology was being establishing itself. Each result was later to make a substantial contribution to computer science.

Leonhard Euler (1707–1783) was a mathematician. The **Euler Polyhedron Theorem** is as follows. Let Φ be a convex polyhedron, v the number of vertices of Φ, e the number of edges of Φ and f the number of faces, then $v - e + f = 2$. This result of Euler's can also be interpreted as a result in Graph Theory. A famous problem on the border of Graph Theory and Topology is the **Four Colour Problem**: can every planar or spherical map be coloured with four colours? This problem was solved affirmatively by Appel and Haken in 1976 using a computer.

(Marie-Ennemond) Camille Jordan (1838–1922) was a mathematician. A curve homeomorphic to the circle S^1 is called a **Jordan curve**. The **Jordan Curve Theorem** (1893) states that every Jordan curve J in the plane $\mathbb{R}^2$ divides the plane into two domains: $\mathbb{R}^2 \setminus J$ is the union of two disjoint open sets A_1 and A_2, one bounded and one unbounded, with $\partial A_1 = \partial A_2 = J$. This enables us to define a **simple polygon**. Let $P_0, \ldots, P_{n-1}$ be points in $\mathbb{R}^2$ so that the closed polygonal line $J = \bigcup_i \overline{P_iP_{i+1}}$ (with $P_n = P_0$) is a Jordan curve. Then the union of J and the bounded component of $\mathbb{R}^2 \setminus J$

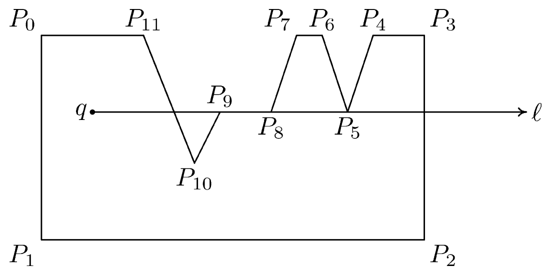


Fig. 1. The line segment P_2P_3, the polygonal line $P_7P_8P_9P_{10}$ and the line segment $P_{10}P_{11}$ each cross the half line ℓ once.

is the **simple n-polygon** determined by the points $P_0, \ldots, P_{n-1}$. If Φ is a simple n-polygon and $q \notin \partial\Phi$ then $q \in \operatorname{Int}\Phi$ iff some half line with q as its starting point is crossed by $\partial\Phi$ an odd number of times. (Crossing means that it meets both open sets that make up the complement of the line, see Figure 1.)

This theorem is fundamental when handling polygons with a computer.

The polygon is an important figure in computer science. The fundamental function of working with polygons is incorporated in the Windows operating system, and when geometric problems are solved using computers, polygons can be effectively used for the formulation. For example, polygons can be used to obtain a solution of the Four Colour Problem above and problems such as determining the optimal positioning of cameras in an art gallery.

The art gallery problem falls within the scope of a field called Computational Geometry. This field emerged in the latter half of the 1970s and is subsumed under the field of computer science, in which research is conducted on the use of computers to solve geometric problems. The primary focus of this research is the evaluation of computational complexity and the development of efficient algorithms based on those evaluations, but using means such as conversion to visualization, etc., the proposed methods are now being applied to other fields as well. Many of the problems dealt with in computational geometry are the quest for efficiency, and its focus is far from a primitive geometrical basis. But when it comes to problems involving topological properties, research is still in the initial stages, some of them have a primitive geometrical interest. The art gallery problem is among these.

Often problems dealt with in computational geometry are expressed in terms of everyday issues such as the **art gallery problem** of calculating the number of cameras and their locations needed to provide optimum surveillance in an art gallery. When research is conducted on such problems using a scientific approach, the technical term ‘polygon’ is used in place of ‘art gallery’ and ‘surveillance’ would be described as ‘observing any given point in the interior of the polygon with some camera’ (the line segment joining the point and some camera is contained in the interior of the polygon). Thus, the problem would be expressed as follows: for a given polygon Φ determine a set of points $\{a_0, a_1, \ldots, a_n\}$ inside the polygon that satisfies the condition

$$(\forall q \in \operatorname{Int}\Phi)(\exists a_i)(\overline{a_i q} \subseteq \operatorname{Int}\Phi),$$

where $\operatorname{Int}\Phi$ denotes the ***interior*** of Φ. Research is advancing in the problem mainly through a method called triangulation.

Luitzen Egbertus Jan Brouwer (1881–1966) was a mathematician. According to the Brouwer Fixed Point Theorem (1910), any continuous map from an ***n-simplex*** to itself has a ***fixed point***. Circles and polygons are ***homeomorphic*** to triangles and a triangles are 3-simplexes. Therefore the theorem is valid for circles and polygons. The Brouwer Fixed Point Theorem expresses a logical source for research on methods of solving equations based on numerical calculations called fixed-point algorithms. Since the time when the first fixed-point algorithm was obtained by Scarf, striking advances have been made in research on the value of fixed-point algorithms. Unique methods of calculation unlike anything found in Newton’s method or other methods are emerging from the research.

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j-6 Non Standard Topology

Non standard methods have been introduced in 1960 by A. Robinson [11] who used ideas of logic to construct an extension $^*\mathbb{R}$ of $\mathbb{R}$ containing infinitesimals and infinite elements. The main tool in these methods, which are often as said by L. Haddad in [7] "a new way to look at old things", is the notion of enlargement.

First of all, let us recall the definition of an enlargement.

Let $\mathcal{M}$ be a full structure on a set X, i.e., a set containing X, $\mathcal{P}(X)$, the set of all binary, ternary, ..., relations and their power sets.

(1) A **model** of $\mathcal{M}$ is a mathematical structure $^*\mathcal{M}$ in which $\mathcal{M}$ is embedded and such that, if $i : x \mapsto {}^*x$ denotes the one-to-one map from $\mathcal{M}$ into $^*\mathcal{M}$, then for each sentence $P(A, B, \ldots, C)$ of $\mathcal{M}$, this sentence holds in $\mathcal{M}$ if and only if the corresponding sentence $P(^*A, {}^*B, \ldots, {}^*C)$ holds in $^*\mathcal{M}$ (the transfer principle) – in model-theoretic terms: $x \mapsto {}^*x$ is an elementary embedding. The entities of $^*\mathcal{M}$ of the form $^*x$ are the **standard entities**, the others **non-standard**. All the elements of $^*\mathcal{M}$ are said to be **internal**. Other objects, such as $\mathcal{P}(^*X)$, are called **external**.

(2) Let be X a set, $\mathcal{F}$ a filter on X and $^*\mathcal{M}$ a model of $\mathcal{M}$. The intersection of all $^*F$, where $F \in \mathcal{F}$, is called the **monad of the filter** $\mathcal{F}$ and denoted $\mu(\mathcal{F})$ (let us notice that $\mu(\mathcal{F})$ may be empty or even external).

Moreover, if there exists an internal element I of $^*\mathcal{F}$ such that $I \subset \mu(\mathcal{F})$, such an element is called by M. Machover and J. Hirschfeld [10] and also L. Haddad [7], an **infinitesimal member** of $\mathcal{F}$.

(3) Let $R \subset A \times B$ be a binary relation in $\mathcal{M}$. This relation R is said to be **concurrent** on A whenever for every finite subset $\{a_1, \ldots, a_n\}$ of A, there exists an element b of B such that $(a_i, b) \in R$ for all i. For example, if we consider a filter $\mathcal{F}$ on a set X, the binary relation $R \subset \mathcal{F} \times \mathcal{F}$ defined by $(F, G) \in R$ if and only if $F \supset G$ is concurrent on $\mathcal{F}$.

We define an **enlargement** of $\mathcal{M}$ to be a model $\mathcal{E}$ of $\mathcal{M}$ satisfying any one of the following equivalent statements:

(i) For every binary relation $R \subset A \times B$ in $\mathcal{M}$ that is concurrent on A, there exists an internal element ω of $\mathcal{E}$ such that $\omega \in {}^*B$ and $(^*x, \omega) \in {}^*R$ for all x of A.
(ii) Every filter $\mathcal{F}$ on X has an infinitesimal member in $\mathcal{E}$.
(iii) The monad of any filter on X of nonempty in $\mathcal{E}$.

Now, compare the non standard treatment and the standard treatment for usual topological properties.

First of all, consider the special case where $X = \mathbb{R}$. Let $\mathcal{E}$ be an enlargement of a full structure on $\mathbb{R}$. Since $^*\mathbb{R}$ is totally ordered and $^*\mathbb{N} \subset {}^*\mathbb{R}$, the set $^*\mathbb{N}$ is also totally ordered. Consequently, since the relation $x < y$ is concurrent on $\mathbb{N}$, there exists an internal ω of $^*\mathbb{N}$ such that $^*n < \omega$ for any n of $\mathbb{N}$. Such an element ω is called an **infinite integer**. Let then Ω be the set of all infinite integers (notice that Ω has no first element). The elements of $^*\mathbb{N} \setminus \Omega$ are then called **finite integers**. Let x be an element of $^*\mathbb{R}$. This element is said to be an **infinitesimal element** if $|x| < {}^*\varepsilon$ for all $\varepsilon > 0$, a **finite element** if there exists $n \in \mathbb{N}$ such that $|x| \leqslant {}^*n$, and an **infinite element** otherwise.

Let then $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $(x_n)_{n\in{}^*\mathbb{N}}$ be the corresponding standard sequence in $^*\mathbb{R}$. It follows from the previous definitions that:

(i) $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence if and only if x_ω is finite for each infinite integer ω,
(ii) $(x_n)_{n\in\mathbb{N}}$ is a ***convergent sequence***, with limit ℓ if and only if $x_\omega - {}^*\ell$ is infinitesimal for each infinite integer ω,
(iii) $(x_n)_{n\in\mathbb{N}}$ is a ***Cauchy sequence*** if and only if $x_\omega - x_\tau$ is infinitesimal for all infinite integers ω and τ.

Let now X be a ***topological space*** and $\mathcal{E}$ be an enlargement of a full structure on X. For any x in X, the monad of the ***neighbourhood filter*** of x is called the **monad of** x and denoted by $\mu(x)$ (notice that if $X = \mathbb{R}$ then $\mu(x)$ is the set of all elements a of $^*X$ such that $a - {}^*x$ is infinitesimal).

Then, it follows from this definition of the monad and from the transfer principle that, for every x in X, every subset A of X and every filter $\mathcal{F}$ on X:

(i) x belongs to the ***interior*** of A if and only if $\mu(x) \subset {}^*A$,
(ii) x is a ***cluster point*** of A if and only if $\mu(x) \cap {}^*A \neq \emptyset$,
(iii) x is a ***limit point*** of $\mathcal{F}$ if and only if $\mu(\mathcal{F}) \subset \mu(x)$,

and that $f : X \to Y$ is a ***continuous*** map if and only if $^*f(\mu(x)) \subset \mu(f(x))$ for every x in X.

More generally, we can define the monad of any element of $^*X$. Let a be an element of $^*X$ and $\mathcal{U}(a)$ be the filter on X generated by the open subsets A of X such that a belongs to $^*A$. The monad of the filter $\mathcal{U}(a)$ is called the **monad** (or **halo**) of a and denoted by $\mu(a)$. Moreover, the binary relation $\mu \subset {}^*X \times {}^*X$ defined by $(a, b) \in \mu$ if and only if $b \in \mu(a)$ is called the **monad of the topological space** X (notice that this monad defines a preorder on $^*X$). Lastly, an element a of $^*X$ is called **near standard** if there exists an element x of X such that $a \in \mu(x)$.

It follows from these definitions that a topological space X is:

(i) ***Hausdorff*** if and only if for every $(x, y) \in X \times X$, $\mu(x) \cap \mu(y) \neq \emptyset$ implies $x = y$,
(ii) ***regular*** if and only if $\mu^{-1}[\mu(x)] = \mu(x)$ for every x in X,
(iii) ***normal*** if and only if $\mu^{-1}\mu \subset \mu\mu^{-1}$,
(iv) ***compact*** if and only if $^*X = \mu(X)$, i.e., if and only if every point of $^*X$ is near standard.

With these different characterizations, we can prove again all classical topological properties. There are some other applications.

Let X be a Hausdorff space. It follows from i) that for every near standard element a of ${}^*X$, there exists a unique element x of X such that $a \in \mu(x)$. That x is called the **standard part** of a and denoted ${}^\circ a$. Moreover, if I is an internal subset of ${}^*X$, the set of standard parts of all near standard elements of I is called the standard part of I and is denoted ${}^\circ I$. It then follows from the previous results that, if I is an internal subset of ${}^*X$, its standard part is closed in X and that, if X is regular and if all the points of I are near standard in ${}^*X$, then its standard part is compact. Notice that, since $\bar{A} = {}^\circ({}^*A)$ for every subset A of X, this last result implies that, for every subset A of a regular space X, A is ***relatively compact*** if and only if all elements of ${}^*A$ are near standard. (See [9] for an application to a proof of Ascoli's theorem).

Now we discuss the continuity of maps. Let X and Y be two topological spaces with monads μ and ν respectively and let f be a map from X into Y. If f is continuous then $({}^*f(a), {}^*f(b)) \in \nu$ for every $(a, b) \in {}^*X \times {}^*X$ such that $(a, b) \in \mu$. In particular, if X and Y are Hausdorff and f is continuous, then for any **infinitely close** elements a and b of ${}^*X$ (two elements are said to be infinitely close if they are near standard and have the same standard part), ${}^*f(a)$ and ${}^*f(b)$ are infinitely close. Conversely (see [4, Proposition III.2.6]), if X is ***locally compact***, Y is regular and if F is an internal function map infinitely close elements of ${}^*X$ to infinitely close elements of ${}^*Y$ (such an F is sometimes called **S-continuous**), then the **standard part** of F, the function f of X into Y defined by $f(x) = {}^\circ F({}^*x)$, is continuous.

Notice that one can also give (see [2]) a non standard definition of the ***topological dimension***. This new dimension which is called **thickness** and denoted by ep (for épaisseur in French). It is defined as follows: for an element a of ${}^*X$ and a base $\mathcal{B}$ for X put $h_{\mathcal{B}}(a) = \bigcap\{{}^*B\colon B \in \mathcal{B},\ a \in {}^*B\}$ (the $\mathcal{B}$-halo of a). Next, for $x \in X$ we denote by $\operatorname{ep}(x, \mathcal{B})$ the supremum of sizes of finite chains of $\mathcal{B}$-halos strictly contained in $h_{\mathcal{B}}(x)$. The thickness of $\mathcal{B}$ is $\operatorname{ep}\mathcal{B} = \sup\{\operatorname{ep}(x, \mathcal{B})\colon x \in X\}$ and, finally, $\operatorname{ep} X = \inf\{\operatorname{ep}\mathcal{B}\colon \mathcal{B}$ is a base for $X\}$.

This dimension is namely such that, for any topological space X and any subset A of X, $\operatorname{ep} A \leqslant \operatorname{ep} X$, and that, for any topological spaces X and Y, $\operatorname{ep}(X + Y) \leqslant \operatorname{ep} X + \operatorname{ep} Y$ (in contrast to the classical definitions of ***small inductive dimension***, ***large inductive dimension*** and ***covering dimension*** [4], there is no need for special hypothesis for the previous statements to be true). Moreover, for every ***linearly ordered topological space*** X, $\operatorname{ep} X = 0$ if and only if X is ***totally disconnected*** and $\operatorname{ep} X = 1$ if not. Lastly, if ind, Ind and dim denote respectively the small inductive dimension, the large inductive dimension and the covering dimension respectively, then we have $\operatorname{ind} X \leqslant \operatorname{ep} X$ for every topological space X and $\operatorname{ep} X \leqslant \operatorname{Ind} X = \dim X$ for every metric space X, which implies that $\operatorname{ep} X = \operatorname{ind} X = \operatorname{Ind} X = \dim X$ for every ***separable*** metric space X.

Consider now a ***quasi-uniform space*** $(X, \mathcal{U})$. The monad of the filter $\mathcal{U}$ is called the **monad of the quasi-uniform space** $(X, \mathcal{U})$. Notice that this monad is the graph of a preorder (an equivalence relation if $(X, \mathcal{U})$ is a ***uniform space***) on ${}^*X$. Moreover, if we denote by $\mu_{\mathcal{U}}$ the monad of the quasi-uniform space $(X, \mathcal{U})$ and by τ the monad of its topology, we have $\mu_{\mathcal{U}}(x) = \tau(x)$ for all x in X. With these different results, one can prove again that every topological space is quasi-uniformisable and that every uniform space is regular. We can also characterize uniform spaces. Let $(X, \mathcal{U})$ be a uniform space. Call **pre-near standard** any point a of ${}^*X$ such that there exists a minimal ***Cauchy filter*** $\mathcal{F}$ such that $\mu(\mathcal{F}) = \mu_{\mathcal{U}}(a)$. Then (see [9]), $(X, \mathcal{U})$ is ***complete*** if and only if every pre-near standard element of ${}^*X$ is near standard, and $(X, \mathcal{U})$ is ***precompact*** if and only if any element of ${}^*X$ is pre-near standard. Notice that these last characterizations imply easily the classical result that a uniform space is compact if and only if it is complete and precompact.

To conclude, let us give two other applications of non standard methods in Topology. First of all, let (X, d) be a metric space. Denote by ${}^*d$ the corresponding *metric on ${}^*X$ and by $\widetilde{X}$ the set of all pre-near standard elements of ${}^*X$. On $\widetilde{X}$ consider the equivalent relation $\simeq$ defined by $a \simeq b$ if and only if ${}^*d(a, b)$ is infinitesimal. Then, if we put $\widehat{X} = \widetilde{X}/\simeq$ and if we consider on $\widehat{X}$ the metric $\hat{d}$ defined by $\hat{d}(\hat{a}, \hat{b}) = {}^\circ({}^*d(a, b))$, the metric space $(\widehat{X}, \hat{d})$ is the ***completion*** of (X, d).

Let now X be a ***completely regular*** space. On ${}^*X$ consider the equivalence relation $\sim$ defined by $a \sim b$ if and only if, for every real bounded continuous function f, the numbers ${}^*f(a)$ and ${}^*f(b)$ are infinitely close and put $\widehat{X} = {}^*X/\sim$. Then, if we denote, for any bounded continuous function f, by $\bar{f}$ the function defined on $\widehat{X}$ by $\bar{f}(\hat{a}) = {}^\circ({}^*f(a))$ and if we consider on $\widehat{X}$ the weakest topology for which the functions $\bar{f}$ are continuous, $\widehat{X}$ with its topology is the ***Stone-Čech compactification*** of X.

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j-7 Topological Games

Examples of topological games appear explicitly in The Scottish Book. In these games there are two players, whom we will call ONE and TWO. The games are of length ω. These games never end in draws. Examples of such games appear earlier implicitly: In [7] Hurewicz introduced the following covering property for a space X: For each sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n: n \in \mathbb{N})$ of finite sets such that for each n we have $\mathcal{V}_n \subseteq \mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X. This property is said to be the **Menger property** because for metric spaces it is equivalent to a basis property that K. Menger introduced; although such a space is also called a **Hurewicz space**. In [16] R. Telgarsky defined the following infinite two-person game: In the nth inning ONE first chooses an open cover O_n of X; then TWO responds with a finite subset $T_n \subseteq O_n$. A play $(O_1, T_1, \ldots, O_n, T_n, \ldots)$ is won by TWO if $\bigcup_{n\in\mathbb{N}} T_n$ is an open cover of X; otherwise, ONE wins. Hurewicz proved that a space X has the Menger property if, and only if, ONE has no winning strategy in this game, and Telgarsky proved that a metric space X is σ-compact if, and only if, TWO has a winning strategy in this game.

Sierpiński showed that ***Lusin set***s of real numbers have the Menger property. Since Lusin sets are not σ-compact they are spaces where neither player has a winning strategy. The existence of a Lusin set is independent of ZFC. Fremlin and Miller show that there is a set of real numbers for which neither player has a winning strategy.

This example illustrates some of the trends in the study of topological games. There are generally four basic questions to consider: (1) Does some player have a winning strategy? (2) Is some well-known topological property characterized by the existence/non-existence of a winning strategy of some player of the game under study? When the answer is yes, the game is usually a powerful tool in analysing the corresponding topological property. Many well-known topological concepts that were introduced and studied long before the games were invented, are now characterized by games. (3) In games where neither player has a winning strategy, does this situation change when the length of the game is increased? (4) In games where some player has a winning strategy, how much memory does that player really need to win the game?

There are now numerous examples of topological games. Some are surveyed in [17]. We briefly describe two particularly important classes of games: **nested chain games** and **diagonalization games**.

1. Nested chain games: The Banach–Mazur game

The classical **Banach–Mazur game** from the Scottish Book is probably the best known example of an infinite topological game. There are several other nested chain games in the literature. In the Banach–Mazur game $\mathsf{BM}(X)$ on a space X, players ONE and TWO play a game which has an inning per positive integer. In the nth inning ONE chooses a nonempty open subset O_n of X and TWO responds by choosing a nonempty open subset T_n of O_n. In the $(n+1)$th inning ONE chooses a nonempty open set $O_{n+1} \subseteq T_n$, and so on. TWO wins the play $O_1, T_1, \ldots, O_n, T_n, \ldots$ if $\bigcap_{n\in\mathbb{N}} T_n \neq \emptyset$; else, ONE wins.

A space is a ***Baire space*** if any sequence of dense open subsets has dense intersection. Answering a question of Mazur, Banach proved for subspaces of the real line, and Oxtoby later for general spaces, that the space (X, τ) is a Baire space if, and only if, ONE has no winning strategy in $\mathsf{BM}(X)$.

Spaces for which TWO has a winning strategy in the Banach–Mazur game have the Baire property in a strong sense: if TWO has a winning strategy in $\mathsf{BM}(X)$, then all powers of X, endowed with the ***box topology***, are Baire spaces.

There is a rich literature of examples of spaces for which some box powers are Baire spaces, while other box powers are not Baire spaces. In the early 1970s in unpublished work F. Galvin made the following beautiful conjecture: *if all powers of X, endowed with the box topology, are Baire spaces, then TWO has a winning strategy in* $\mathsf{BM}(X)$.

This conjecture in conjunction with the Banach–Oxtoby result and the result on box-products above would give a complete description of existence of winning strategies in the Banach–Mazur games in terms of Baireness of spaces. The current state of this conjecture is: If it is consistent that there is a proper class of measurable cardinals, then Galvin's conjecture is consistent. There is no consistency result known to imply the negation of Galvin's conjecture.

There are some beautiful characterizations of spaces for which TWO has a winning strategy in $\mathsf{BM}(X)$. Here are two examples. In [8] Kenderov and Revalski consider for completely regular spaces X the set $C^*(X)$ of bounded continuous real-valued functions on X, endowed with the ***topology of uniform convergence***. They prove that TWO has a winning strategy in $\mathsf{BM}(X)$ if, and only if, the set $\{f \in C^*(X): f$ has a minimum value on $X\}$ contains a dense G_δ-subset of $C^*(X)$.

In [9] Ma considers for completely regular locally compact spaces the set $C(X)$ of continuous real-valued functions on X, endowed with the ***compact-open topology***. Let $C_k(X)$ denote this space; then TWO has a winning strategy in $\mathsf{BM}(C_k(X))$ if, and only if, X is paracompact.

In connection with memory requirements: In all results mentioned here the default meaning of "**strategy**" is "perfect information strategy". A **perfect information strategy** is a function which has as input the sequence of all prior moves of the opponent, and as output the response of the strategy owner. Fleissner and Kunen brought attention to the fact that when player ONE has a winning strategy in the Banach–Mazur game, then in fact player ONE has a winning strategy depending on only the most recent move of TWO. A strategy which uses as input only the most recent move of the opponent is said to be a **tactic** (or 1**-tactic**). They asked if it is also the case that when TWO has a winning strategy in $\mathsf{BM}(X)$, then TWO has a winning tactic. G. Debs gave counterexamples. In Debs' counterexamples TWO does not have a winning tactic, but has a winning strategy depending on the most recent two moves of ONE. For a fixed k, call a strategy that uses as input only the most recent $\leqslant k$ moves of the opponent a k**-tactic**. Telgársky conjectured that *for each* $k \geqslant 1$ *there is a space in which TWO does not have a winning k-tactic, but does have a winning* $(k+1)$*-tactic in the Banach–Mazur game*.

Currently there are no examples known of spaces for which TWO has a winning 3-tactic, but not a winning 2-tactic, in the Banach–Mazur game.

In [5] Galvin and Telgársky study memory requirements in the Banach–Mazur game. They prove among other things that if TWO has a winning strategy in the Banach–Mazur game, then TWO has a winning strategy which uses as information only the most recent move of ONE and the most recent move of TWO.

2. Diagonalization games

The class of *diagonalization games* is at least as important as the nested chain games and also has a long history. An unusually large number of games in the literature can be reformulated as diagonalization games.

Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of an infinite set S. The symbol $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ denotes the game which has an inning per positive integer, and in the nth inning ONE chooses an element O_n of $\mathcal{A}$, and TWO responds by choosing a finite set $T_n \subset O_n$. A play $(O_1, T_1, \dots, O_n, T_n, \dots)$ is won by TWO if $\bigcup_{n\in\mathbb{N}} T_n \in \mathcal{B}$. One can consider $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ as a game-theoretic version of the selection property $S_{fin}(\mathcal{A},\mathcal{B})$, which is defined as follows: For each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{V}_n \subset \mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a member of $\mathcal{B}$.

The symbol $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ denotes the game which has an inning per positive integer, and in the nth inning ONE chooses an element O_n of $\mathcal{A}$, and TWO responds by choosing an element $T_n \in O_n$. A play $(O_1, T_1, \dots, O_n, T_n, \dots)$ is won by TWO if $\{T_n\colon n \in \mathbb{N}\} \in \mathcal{B}$. Now $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ is a game-theoretic version of the selection property $S_1(\mathcal{A},\mathcal{B})$, which is defined as follows: For each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(V_n\colon n \in \mathbb{N})$ such that for each n, $V_n \in \mathcal{U}_n$, and $\{V_n\colon n \in \mathbb{N}\}$ is a member of $\mathcal{B}$.

The existence of winning strategies for player TWO received much attention in the literature. If we let $\mathcal{O}$ denote the collection of open covers of a space X, then the game $\mathsf{G}_{fin}(\mathcal{O},\mathcal{O})$ is Telgarsky's game discussed in the introduction of this article, and $S_{fin}(\mathcal{O},\mathcal{O})$ is the Menger property discussed there. Thus Telgarsky's theorem above states that for metric spaces TWO has a winning strategy in $\mathsf{G}_{fin}(\mathcal{O},\mathcal{O})$ if, and only if, the space is σ-compact. With $\mathcal{O}$ as above Galvin introduced the game $\mathsf{G}_1(\mathcal{O},\mathcal{O})$; Rothberger introduced $S_1(\mathcal{O},\mathcal{O})$ in [12]. In [4] Galvin proved that for a first-countable space X TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O},\mathcal{O})$ if, and only if, X is countable.

In [15] Telgársky generalized this as follows: The games he introduced there can all be reformulated as games of the form $\mathsf{G}_1(\mathcal{A},\mathcal{O})$ where $\mathcal{A}$ is a family of special open covers. For example: Let X be a $T_{3\frac{1}{2}}$-space and let $\mathcal{K}$ be a collection of closed proper subsets of X such that $\mathcal{K}$ contains all one-element subsets of X, and is closed-hereditary. Then an open cover $\mathcal{U}$ of X is a $\mathcal{K}$-cover if there is for each $C \in \mathcal{K}$ a $U \in \mathcal{U}$ with $C \subset U$, and $X \notin \mathcal{U}$. Let $\mathcal{O}(\mathcal{K})$ denote the $\mathcal{K}$-covers of X. The symbol $\mathcal{DK}$ denotes the collection of subsets of X which are representable as a union of a discrete family of sets in $\mathcal{K}$. It is evident that $\mathcal{K} \subset \mathcal{DK}$. Here are some examples of such $\mathcal{A}$ considered by Telgársky and collaborators:

- Ω: The ω**-cover**s of X. An open cover $\mathcal{U}$ of a space is an ω-cover if $X \notin \mathcal{U}$ and there is for each finite set $F \subset X$ a set $U \in \mathcal{U}$ with $F \subset U$.
- K: The κ**-cover**s of X. An open cover $\mathcal{U}$ of a space is a κ-cover if $X \notin \mathcal{U}$ and there is for each compact proper subset $C \subset X$ a set $U \in \mathcal{U}$ with $C \subset U$.
- $\check{\mathsf{C}}$: The **č-cover**s of X. An open cover $\mathcal{U}$ of a space is a č-cover if $X \notin \mathcal{U}$ and there is for each closed Čech-complete proper subset $C \subset X$ a set $U \in \mathcal{U}$ with $C \subset U$.
- D: The **d-cover**s of non-discrete space X. An open cover $\mathcal{U}$ of a space is a d-cover if $X \notin \mathcal{U}$ and there is for each closed discrete proper subset $C \subset X$ a set $U \in \mathcal{U}$ with $C \subset U$.
- Dim_n: The $\dim_n$**-cover**s of space X. An open cover $\mathcal{U}$ of a space is a $\dim_n$-cover if $X \notin \mathcal{U}$ and there is for each closed proper subset $C \subset X$ which is normal and of covering dimension $\leqslant n$ a set $U \in \mathcal{U}$ with $C \subset U$.

Telgarsky proved that if every element of $\mathcal{K}$ is a G_δ-set then for completely regular X, TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O}(\mathcal{K}),\mathcal{O})$ if, and only if, X is a union of countably many elements of $\mathcal{K}$.

Call a ***paracompact*** space X a **Tamano space** if for each paracompact space Y also $X \times Y$ is paracompact. An old question of Tamano asks to characterize the Tamano spaces. Games of the form $\mathsf{G}_1(\mathcal{A},\mathcal{O})$ have been useful in characterizing some subclasses of the class of Tamano spaces. In particular, let $\mathcal{C}$ denote the collection of compact proper subsets of a space X. If X is a paracompact space such that TWO has a winning strategy in the game $\mathsf{G}_1(\mathcal{O}(\mathcal{DC}),\mathcal{O})$, then X is a Tamano space (Telgarsky).

The analogues of Tamano's problem for ***subparacompact*** spaces and ***metacompact*** spaces have also been considered, and Yajima [MN, Chapter 13] proved: (1) If X is a regular subparacompact space and if TWO has a winning strategy in the game $\mathsf{G}_1(\mathcal{O}(\mathcal{DC}), \mathcal{O})$, then for each subparacompact space Y also $X \times Y$ is subparacompact. (2) If X is a regular metacompact P-space and if TWO has a winning strategy in the game $\mathsf{G}_1(\mathcal{O}(\mathcal{DC}), \mathcal{O})$, then for each metacompact space Y also $X \times Y$ is metacompact.

Notice that the class of Tamano spaces is closed under finite products. The same is not clear about the subclass characterized by these games. Specifically it is an open problem of Telgársky whether TWO has a winning strategy in the game $\mathsf{G}_1(\mathcal{O}(\mathcal{DC}), \mathcal{O})$ on $X \times Y$ if TWO has a winning strategy in this game on each of the $T_{3\frac{1}{2}}$-spaces X and Y. Some partial results are know: If X and Y are subparacompact T_3-spaces then the answer is yes.

Several games that have been introduced by Gruenhage also can be reformulated as games of the form $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$. We now briefly survey some beautiful results from this research.

A family $\mathcal{M}$ of nonempty compact subsets of a space X is said to be a **moving off family** if there is for each compact subset $C \subset X$ a set $M \in \mathcal{M}$ such that $M \cap C = \emptyset$. Let $\mathcal{A}$ denote the collection of moving off families for a space X. Let $\mathcal{F}$ denote the set of families $\mathcal{G}$ where each member of $\mathcal{G}$ is a nonempty compact subset of X, and $\mathcal{G}$ is locally finite. For a locally compact (but non-compact) Hausdorff space X, TWO has a winning strategy in $\mathsf{G}_1(\mathcal{A}, \mathcal{F})$ if, and only if, X is paracompact.

Let a space X and a subset H of X be given and define $\Omega_H = \{S \subset X \setminus H$: (for each open $U \supseteq H)(S \cap U \neq \emptyset)\}$ and $\Gamma_H = \{S \subset X \setminus H$: (for each open $U \supseteq H)(S \setminus U$ is finite)$\}$. Also, recall that for a space X the symbol Δ denotes $\{(x, x): x \in X\}$. Gruenhage proved the following for a compact space X and a subset H: (1) If X is countably tight then TWO has a winning strategy in $\mathsf{G}_1(\Omega_H, \Gamma_H)$ if, and only if, $X \setminus H$ is ***meta-Lindelöf***. (2) If X is scattered then TWO has a winning strategy in $\mathsf{G}_1(\Omega_H, \Gamma_H)$ if, and only if, $X \setminus H$ is ***metacompact***. (3) TWO has a winning strategy in $\mathsf{G}_1(\Omega_\Delta, \Gamma_\Delta)$ (on X^2) if, and only if, X is ***Corson compact***.

Initially authors were occupied with the existence of winning strategies for TWO in these diagonalization games. Some nice theorems were missed by not instead considering the non-existence of winning strategies of ONE. Two fundamental results about the non-existence of winning strategies for ONE are Hurewicz's theorem above that a space has property $S_{fin}(\mathcal{O}, \mathcal{O})$ if, and only if, ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{O}, \mathcal{O})$, and the following theorem of Pawlikowski, proved in [11]. A space X has property $S_1(\mathcal{O}, \mathcal{O})$ if, and only if, ONE has no winning strategy in $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$.

The nonexistence of a winning strategy for ONE is often a Ramsey-theoretic statement. For positive integers n and k, the symbol $\mathcal{A} \to (\mathcal{B})^n_k$ denotes the Ramseyan statement

> For each $A \in \mathcal{A}$, for each function $f : [A]^n \to \{1, \ldots, k\}$, there is a $B \subseteq A$ such that $B \in \mathcal{B}$, and f is constant on $[B]^n$.

Ramsey's Theorem is this statement for the case when $\mathcal{A} = \mathcal{B}$ and $\mathcal{A}$ is the collection of infinite subsets of the integers.

The symbol $\mathcal{A} \to \lceil \mathcal{B} \rceil^2_k$ denotes the statement:

> For each $A \in \mathcal{A}$ and for each $f : [A]^2 \to \{1, \ldots, k\}$ there is a $B \subset A$ with $B \in \mathcal{B}$, and a partition $B = \bigcup_{n<\infty} B_n$ of B into disjoint finite sets, and an $i \in \{1, \ldots, k\}$ such that $f(\{x, y\}) = i$ whenever x and y are from different B_n's.

A theorem of Baumgartner and Taylor states that if $\mathcal{F}$ is a non-principal ultrafilter on the integers, then $\mathcal{F}$ is a P-point if, and only if, for all k, $\mathcal{F} \to \lceil \mathcal{F} \rceil^2_k$. Scheepers proved: (1) ONE has no winning strategy in $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$ if, and only if, $\Omega \to (\mathcal{O})^2_2$. (2) ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{O}, \mathcal{O})$ if, and only if, $\Omega \to \lceil \mathcal{O} \rceil^2_2$.

There are now numerous examples of such results where ONE has no winning strategy in a game of the form $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$ (respectively $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$) if, and only if the corresponding selection hypothesis $S_{fin}(\mathcal{A}, \mathcal{B})$ (respectively $S_1(\mathcal{A}, \mathcal{B})$) holds, if and only if, the corresponding Ramseyan statement $\mathcal{A} \to \lceil \mathcal{B} \rceil^2_k$ (respectively $\mathcal{A} \to (\mathcal{B})^n_k$) holds for appropriate k and n. A large number of important topological properties have also been characterized by nonexistence of winning strategies of ONE in such games – for example, countable fan tightness in $C_p(X)$, countable strong fan tightness, being a Lusin subset of the real line, being a strong measure zero subset of a sigma-compact metric space, and so on.

In connection with the length of games, Daniels and Gruenhage define in [3] the **point open type** of a set of real numbers to be the least $\alpha \leqslant \omega_1$ such that if the game $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$ is allowed to run for α innings, then TWO has a winning strategy. For example: Let L be a Lusin set of real numbers. The point-open type of L is $\omega + \omega$. To see this, note that during the first ω-innings TWO can cover a dense countable subset of L. The uncovered part of L now left is nowhere dense, and as L is a Lusin set, is countable. The remaining ω innings are used to cover these points. Following the initial results of [3], Baldwin proved in [2] that it is consistent that there is for each limit ordinal $\alpha < \omega_1$ a set of real numbers with point-open type α. By examples presented in [13] there are topological spaces with infinite successor point-open type, but it is not known if there can be an infinite set of real numbers with successor point-open type. In this connection we have the following conjecture: *The point-open type of any infinite set of reals is a limit ordinal.*

And games introduced by Berner and Juhász can be reformulated as games of the form $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$. For let $\mathcal{D}$ denote the collection of dense subsets of X. One can show that in the **open-point** game of Berner and Juhasz, ONE has a winning strategy, if an only if, in the game $\mathsf{G}_1(\mathcal{D}, \mathcal{D})$, TWO has a winning strategy, and in their game TWO has a winning strategy if, and only if, in the game $\mathsf{G}_1(\mathcal{D}, \mathcal{D})$, ONE has a winning strategy. In the case of metrizable spaces X there is then a beautiful duality theory between the game $\mathsf{G}_1(\mathcal{D}, \mathcal{D})$ on $C_p(X)$, and the game $\mathsf{G}_1(\Omega, \Omega)$ on X, as explored in [14].

Finally, as to memory requirements in diagonalization games: In [1] it was shown that some games of Gruenhage introduced in [6] can be reformulated as games of the form $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$. Then some of Gruenhage's results translate to statements of the form that for locally compact T_2-spaces metacompactness is equivalent to TWO having a particular type of limited memory strategy in the corresponding game, and for compact spaces ***Eberlein compactness*** is also characterized by the existence of a type of limited memory strategy of TWO in the corresponding game.

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j-8 Fuzzy Topological Spaces

1. Basic concepts

In order to define what a fuzzy topological space is one requires the notion of fuzzy set. Given a set X and an ordered set L a function from X to L is called an **L-fuzzy set**. The set L is often referred to as a set of **truth values**. A large amount of research has been performed on the nature of this set of truth values and on the relation with logic. If $L = \{0, 1\}$ then an L-fuzzy set is the characteristic function of an ordinary subset of X. There are restrictions on the type of ordered sets L considered in the literature. L is at least a complete lattice with an order reversing involution, which gives rise, by pointwise extension, to a so-called **pseudo-complement** on fuzzy sets. The most widely used lattice of truth values is the one originally introduced by L.A. Zadeh, namely the unit interval $I := [0, 1]$, and it is also from that setting that the name fuzzy set originated. An overwhelming majority of published work on fuzzy sets and fuzzy topology is situated in the original setup with the unit interval as set of truth values. In this case reference to I is dropped from all terminology. It is beyond the scope of this paper to deal with more general types of fuzzy sets and their associated topological structures. We refer to the edited volume [13] for more information on that area of research.

A **fuzzy topological space** is a set X equipped with a **fuzzy topology** [10], which is a collection Δ of fuzzy sets on X satisfying the following properties, where the same symbol is used for a value and the constant function with that value:

(FT1) $\forall \alpha \in I\colon \alpha \in \Delta$,
(FT2) $\forall \alpha, \beta \in \Delta\colon \alpha \wedge \beta \in \Delta$,
(FT3) $\forall \Gamma \subset \Delta\colon \bigvee \Gamma \in \Delta$.

The elements of Δ are called **open fuzzy sets**. Fuzzy topological spaces form the objects of a category, the morphisms of which are defined as follows. If (X, Δ_X) and (Y, Δ_Y) are fuzzy topological spaces and $f : X \to Y$ is a function, then f is called **continuous** if for any $\alpha \in \Delta_Y$ we have $\alpha \circ f \in \Delta_X$. The function $\alpha \circ f$ is called the **preimage of a fuzzy set** and is often denoted by $f^{-1}(\alpha)$. The category of fuzzy topological spaces and continuous functions is denoted **FTS**.

As topological spaces, fuzzy topological spaces can be characterized in several different ways. Making use of the order reversing involution $c : I \to I : \alpha \mapsto 1 - \alpha$, closed fuzzy sets are defined with properties dual to those of the open fuzzy sets. A fuzzy set α is called a **closed fuzzy set** if $1 - \alpha$ is an open fuzzy set. A **fuzzy closure operator** and a **fuzzy interior operator** are defined by

$$\mathrm{cl} : I^X \to I^X : \alpha \mapsto \inf\{\beta\colon \alpha \leqslant \beta,\ 1 - \beta \in \Delta\},$$

and

$$\mathrm{int} : I^X \to I^X : \alpha \mapsto \sup\{\beta\colon \beta \leqslant \alpha,\ \beta \in \Delta\}.$$

The relations which hold among topological closure and interior operators and open and closed sets, mutatis mutandis also hold in the case of their fuzzy counterparts and appropriate sets of axioms ensure that all these concepts are equivalent to one another. Continuous maps between fuzzy topological spaces can also be characterized by means of, e.g., fuzzy closure operators. If (X, Δ) and (X', Δ') are fuzzy topological spaces with corresponding fuzzy closure operators cl and cl$'$, and $f : X \to X'$, then f is continuous if and only if for any fuzzy set α on X: $f(\mathrm{cl}(\alpha)) \leqslant \mathrm{cl}'(f(\alpha))$. Here the **image of a fuzzy set** is defined by $f(\alpha)(x) := \sup\{\alpha(y)\colon f(y) = x\}$.

2. Some fundamental results

Fuzzy topological spaces, as topological spaces, form a so-called ***topological category*** [7]. Especially, this implies that arbitrary initial and final structures exist. Given fuzzy topological spaces $(X_j, \Delta_j)_{j \in J}$ and **source**

$$\big(f_j : X \to (X_j, \Delta_j)\big)_{j \in J}$$

in **FTS**, then the **initial fuzzy topology** on X is the smallest fuzzy topology containing the collection

$$\Gamma := \{\alpha \circ f_j\colon j \in J,\ \alpha \in \Delta_j\}.$$

If we have a ***sink***

$$\big(f_j : (X_j, \Delta_j) \to X\big)_{j \in J}$$

in **FTS** then the **final fuzzy topology** on X is given by

$$\Delta := \{\alpha\colon \forall j \in J : \alpha \circ f_j \in \Delta_j\}.$$

Given a topological space $(X, \mathcal{T})$, a natural fuzzy topological space is associated with it by considering the fuzzy topology $\Delta_{\mathcal{T}}$ consisting of all lower semi-continuous functions from $(X, \mathcal{T})$ to I, where I is equipped with the usual topology. This fuzzy topology can also be characterized as the smallest one containing all characteristic functions of open sets. A function between topological spaces then is continuous if and only if it is continuous between the associated fuzzy topological spaces. The functor

$$\begin{aligned} &\mathbf{TOP} \to \mathbf{FTS}, \\ &(X, \mathcal{T}) \mapsto (X, \Delta_{\mathcal{T}}), \\ &\quad f \mapsto f, \end{aligned}$$

is a full concrete embedding of **TOP** into **FTS**. **TOP** is actually embedded as a concretely **bireflective** (***epireflective*** and ***monoreflective***) and **bicoreflective** (defined similarly) subcategory of **FTS**.

For any space $(X, \Delta) \in |\mathbf{FTS}|$, its **TOP**-bicoreflection is determined by $\mathrm{id}_X : (X, \Delta^t) \to (X, \Delta)$, where Δ^t is the fuzzy topology generated by the initial topology on X for the source

$$(\alpha : X \to I)_{\alpha \in \Delta}.$$

The **TOP**-bireflection is given by $\mathrm{id}_X : (X, \Delta) \to (X, \Delta_t)$, where Δ_t is the fuzzy topology generated by the topology $\{G\colon 1_G \in \Delta\}$ (where for any set G, 1_G denotes the characteristic function of G).

TOP is not the only subcategory of **FTS** which is at the same time bireflectively and bicoreflectively embedded. This situation is unusual in topology, for instance, neither **TOP** nor **UNIF** have subcategories which are at the same time bireflective and bicoreflective. Topological properties give rise to isomorphism-closed subcategories of **TOP** and conversely any isomorphism-closed subcategory determines a topological property. Many such subcategories of **FTS**, and hence fuzzy topological properties which generalize topological properties, have been studied in the literature and it is quite beyond the scope of the present paper to deal with this aspect of the theory. We refer the interested reader to the survey paper by A.P. Šosták [14] and to the Ph.D. Thesis of G. Jäger [3]. However, subcategories which are bireflective and bicoreflective determine a particular type of fuzzy topological property which has no topological counterpart and which is stable under all initial and final constructions. One of the best known such subcategories, other than **TOP**, is that of so-called **fuzzy neighbourhood spaces**, denoted **FNS** [9]. A fuzzy neighbourhood space is a fuzzy topological space which, instead of property (FT1) satisfies the stronger property:

(FN) $\forall \alpha \in I,\ \forall \mu \in \Delta\colon \alpha \wedge 1_{\mu^{-1}(]\alpha,1])} \in \Delta.$

One of the reasons why these spaces have been the subject of much research is the fact that they are much easier to work with because their structure can also be characterized by means of a type of neighbourhood systems in the same way as in topology, in particular by a type of fuzzy neighbourhoods of the points of the underlying space. This is a property which does not hold in arbitrary fuzzy topological spaces, where all different local descriptions of the fuzzy topological structure which have been proposed in the literature require notions of fuzzy neighbourhoods of pairs (x, α) where x is a point of the space and $\alpha \in I$ [14].

Subcategories of **FTS** which are at the same time bireflectively and bicoreflectively embedded can be characterized in the following way [11]. A **total fuzzy topology on** I is a fuzzy topology which contains the identity and which is stable under composition. Denote by Γ_0 the total fuzzy topology consisting of all increasing lower semi-continuous fuzzy sets on I. A subcategory $\mathcal{C}$ of **FTS** then is bireflective and bicoreflective if and only if there exists a total fuzzy topology Γ coarser than Γ_0, which is such that $(X, \Delta) \in \mathcal{C}$ if and only if for all $\mu \in \Delta$ and for all $\alpha \in \Gamma\colon \alpha \circ \mu \in \Gamma$. Although this brings the search for bireflective bicoreflective subcategories of **FTS** down to determining all total fuzzy topologies coarser than Γ_0, at the moment they have not yet all been classified. It is known that there are at least $\mathfrak{c}$ different such subcategories. This is easiest seen by proving that for any $\alpha, \beta \in I$ such that $0 \leqslant \alpha \leqslant \beta < 1$ the collection of all lower semi-continuous, increasing functions which are smaller than the identity together with the function $\beta \wedge 1_{]\alpha,1]}$ generates a total fuzzy topology, and that these total fuzzy topologies generate a partially ordered set of $\mathfrak{c}$ different subcategories of **FTS**, all of which contain **TOP** and are contained in **FNS**. Recently it was shown by Li and Luo [5] that there are exactly $\mathfrak{c}$ bireflective bicoreflective subcategories lying between **TOP** and **FNS**. The lattice of these subcategories is isomorphic to the lattice of open subsets of $]0, 1[$.

3. Categorical extensions

Besides research into and classification of particularly well-behaved subcategories of **FTS** there is also research into categorically better behaved supercategories of **FTS**. In the same way that interesting supercategories of **TOP** are characterized via convergence of filters, supercategories of **FTS** are characterized by means of a notion of fuzzy convergence of fuzzy filters. The basics for this approach were laid down in the papers [7] and [8]. A **fuzzy filter** is a filter in the lattice of all fuzzy subsets. The important role of ultrafilters in topology is played by minimal prime fuzzy filters. Given a fuzzy filter $\mathfrak{F}$, a **minimal prime fuzzy filter** containing $\mathfrak{F}$ is a prime fuzzy filter which is minimal in the collection of all fuzzy filters which are finer than $\mathfrak{F}$. Each such minimal prime fuzzy filter $\mathfrak{U}$ finer than $\mathfrak{F}$ is determined by an ultrafilter $\mathcal{U}$ in the sense that $\mathfrak{U} = \{\mu \wedge 1_U\colon \mu \in \mathfrak{F},\ U \in \mathcal{U}\}$. In this way there is a one-to-one correspondence between the minimal prime fuzzy filters $\mathfrak{U}$ finer than $\mathfrak{F}$ and the ultrafilters $\mathcal{U}$ finer than the filter $\{\mu^{-1}(]0, 1])\colon \mu \in \mathfrak{F}\}$. With this set-theoretic concept as a tool a fuzzy convergence theory can be constructed [8], [13]. Basic to this is the notion of a **fuzzy limit**, which is a function assigning to each fuzzy filter and each point a real number $\lim \mathfrak{F}(x) \in I$ indicating the degree with which the fuzzy filter converges to the point. Let $\mathcal{P}_m(\mathfrak{F})$ stand for the set of all minimal prime fuzzy filters finer than $\mathfrak{F}$. In order to define a generalized theory of fuzzy convergence, several axioms can now be invoked; we mention a few (always meant to hold for all fuzzy filters and all points).

(FC1) $\lim \mathfrak{F} = \inf_{\mathfrak{G} \in \mathcal{P}_m(\mathfrak{F})} \lim \mathfrak{G}$.
(FC2) $\lim\{\mu\colon \mu(x) \geqslant \alpha\}(x) \geqslant \alpha$.
(FC3) $\lim \mathfrak{F} \leqslant \inf\{\alpha\colon \alpha \text{ constant}, \alpha \in \mathfrak{F}\}$.
(FC4) $\mathfrak{F}_1, \mathfrak{F}_2$ prime, $\mathfrak{F}_1 \subset \mathfrak{F}_2 \Rightarrow \lim \mathfrak{F}_2 \leqslant \lim \mathfrak{F}_1$.

Axioms (FC1) and (FC2) are generalizations of similar axioms in topological convergence theory, axioms (FC3) and

(FC4) have no topological counterpart, and axiom (FC4) actually is counterintuitive from the topological point of view but results from the fact that in the fuzzy theory prime filters are not necessarily maximal. By judiciously choosing the axioms of such a fuzzy convergence theory, various supercategories of **FTS** are considered. The largest one studied to date is the category **FCS** of so-called **fuzzy convergence spaces**. This is a categorically particularly well behaved category since it is a so-called **topological universe**. This means that, not only is it a topological category, but moreover it is at the same time a ***Cartesian closed*** and a **extensional category**. Cartesian closedness means that well behaved function spaces exist and extensionality means that one-point extensions with particular good categorical properties exist [7]. In [13] H. Herrlich posed three problems in this context, namely to determine the Cartesian closed topological hull of **FTS**, the extensional topological hull of **FTS** and the topological universe hull of **FTS**. The fact that **FCS** is a topological universe and a supercategory of **FTS**, proves that these hulls exist. Up to now only a first step towards finding the extensional topological hull has been determined. This was done recently by Dexue Zhang in [16] who constructed an extensional topological supercategory of **FTS** which is embedded in **FCS** and wherein **FTS** is finally dense. Similar investigations have been made for the smaller category **FNS** where also partial results were obtained [2].

4. Further research

A few, what one would call, natural fuzzy topologies have been studied in the literature [10, 13]. The most notable ones arise in the context of probability measures by a process of dualization. If X is a topological space (if necessary with some good properties) then on the space of all probability measures on the Borel σ-algebra on X, say $\mathcal{M}(X)$, various natural fuzzy topologies can be constructed which in one way or another are all generated by fuzzy sets of type

$$\alpha_G : \mathcal{M}(X) \to I : P \mapsto P(G),$$

where G is an open subset of X. The different fuzzy topologies which can thus be generated depend on the choice of some extra conditions on the functions α_G. In all cases the topological coreflection of these fuzzy topologies are well-known topologies on the space of probability measures, such as, e.g., the weak topology. In the case of $X = \mathbb{R}$ we obtain the so-called **fuzzy real line** and in the case of $X = I$ we obtain the so-called **fuzzy unit interval**. Both these spaces have been studied extensively and have turned out to be particularly useful in the study of fuzzy topology in a way not dissimilar to the way in which the real line and the unit interval are useful in topology, see, e.g., [12].

Another direction of research is concerned with uniform aspects of the theory. Thus several different notions of fuzzy uniformity have been introduced, see, e.g., [3, 4, 12]. Main results in this direction, besides the development of a coherent theory of fuzzy uniformity, are that certain fuzzy topological spaces, such as those mentioned higher up, in particular the fuzzy real line(s), are indeed, in some sense, fuzzy uniformizable. Also in this direction there is fruitful research into function spaces [3, 4] and the related categorical question of the characterization of exponential objects in the various categories of fuzzy topological spaces [15].

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K: Connections with other fields

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k-1 Banach Spaces and Topology (I)

Banach spaces were defined by S. Banach and others (notably N. Wiener) independently in the 1920s. However it was Banach's 1932 monograph [2] that made the theory of Banach spaces ("espaces du type (B)" in the book) an indispensable tool of modern analysis. The novel idea of Banach is to combine point-set topological ideas with the linear theory in order to obtain such powerful theorems as Banach–Steinhaus theorem, open-mapping theorem and closed graph theorem. Both general topology and theory of Banach spaces continue to benefit from cross-fertilization of analysis and topology, some of which can be seen in the following pages.

1. Definitions and basic properties

The material of this section is quite standard and details can be found in any textbook on functional analysis. Let X be a linear space (= vector space) over the field $\mathbb{K}$ (the field of **scalars**) where $\mathbb{K}$ is either the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers. A **norm** $\| \ \|$ on X is a function $x \mapsto \|x\|$ on X into $[0, \infty)$ satisfying the following conditions: (i) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in X$; (ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K}, x \in X$; and (iii) $\|x\| = 0$ if and only if $x = 0$. A linear space X equipped with a fixed norm $\| \ \|$ is called a **normed linear space** or **normed space** and is denoted by $(X, \| \ \|)$ or simply by X when no ambiguity is likely. When one wishes to specify $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$), the normed space is referred to as a real (or complex) normed linear space. The **unit ball** of the normed space X is the set $B_X \equiv \{x \in X\colon \|x\| \leqslant 1\}$ and the **unit sphere** of X is $S_X \equiv \{x \in X\colon \|x\| = 1\}$. The norm $\| \ \|$ on the normed space X defines the ***metric*** d (called the **norm metric**) on X by $d(x, y) = \|x - y\|$, $x, y \in X$. The topology of the norm metric is called the **norm topology**, and the normed space is always assumed to have the norm topology unless other topology is specified. Two norms $\| \ \|_1, \| \ \|_2$ on the linear space X are said to be **equivalent** if the norm topologies of the two norms coincide, and this is the case if and only if there are positive numbers a and b such that for each $x \in X$, $a\|x\|_1 \leqslant \|x\|_2 \leqslant b\|x\|_1$. A **Banach space** is a ***complete*** normed space. If Y is a linear subspace of X, where $(X, \| \ \|)$ is a normed space, then we let Y itself be normed with the restriction of $\| \ \|$ to Y. If X is a Banach space and if Y is a closed linear subspace of X, then Y is a Banach space.

A linear map between normed spaces is continuous if it is continuous at a point, and in this case the map is a ***Lipschitz map***. A scalar-valued liner map f on a normed space is called a **linear functional** on X. The linear functional f is continuous if and only if the null space $f^{-1}(0)$ is closed, and this is the case if and only if $\|f\| \equiv \sup\{|f(x)|\colon x \in B_X\} < \infty$. The linear space X^* of all continuous linear functionals on X is a Banach space with the norm $\| \ \|$ given above. The Banach space X^* is called the **dual space** (or simply 'dual') of X. One of the fundamental facts on normed spaces is the following **Hahn–Banach extension theorem**. Let X be a normed space and let Y be its linear subspace. If $g \in Y^*$ then there exists an $f \in X$ such that $g = f|_Y$ and $\|f\| = \|g\|$. The theorem in particular implies that, for each $x \in X$, $\|x\| = \sup\{|f(x)|\colon f \in B_{X^*}\}$. This formula can be interpreted as follows. Let X^{**} be the **bidual** of X, namely $(X^*)^*$ (also called the **second dual space**). Then there is a canonical linear map $x \mapsto \hat{x}$ from X into X^{**} given by $\hat{x}(f) = f(x)$ for all $x \in X$ and $f \in X^*$. The formula above says that the map $x \mapsto \hat{x}$ is an **isometry**, i.e., $\|x\| = \|\hat{x}\|$ for each $x \in X$. It is often convenient to identify x and $\hat{x}$ and regard X a subspace of X^{**} (the **canonical embedding** of X into its bidual). If $X = X^{**}$, then X, which is necessarily a Banach space, is said to be **reflexive**.

Two Banach spaces X and Y are said to be **isometric** if there is a one-to-one linear map T of X onto Y such that $\|T(x)\| = \|x\|$ for each $x \in X$. We give some examples of standard Banach spaces.

EXAMPLE 1. Let T be a ***completely regular*** topological space and let $C(T)$ (respectively $C_b(T)$) be the space of all scalar-valued continuous (respectively bounded continuous) functions on T. For each $f \in C_b(T)$ the **supremum norm** $\|f\|$ of f is defined by $\|f\| = \sup\{|f(t)|\colon t \in T\}$. Then $(C_b(T), \| \ \|)$ is a Banach space. If βT denotes the ***Stone–Čech compactification*** of T, then the Banach spaces $C_b(T)$ and $C(\beta T)$ are isometric. So, here we restrict our discussion to $C(K)$ where K is a compact Hausdorff space. Then for each $\varphi \in C(K)^*$ there is a unique scalar-valued ***Radon measure*** μ on K such that $\varphi(f) = \int_K f \, d\mu$ for all $f \in C(K)$ (**Riesz representation theorem**). In this case $\|\varphi\|$ is equal to the total variation of μ on K. Hence $C(K)^*$ is isometric to the space $M(K)$ of all scalar-valued Radon measures on K with the total variation as the norm.

EXAMPLE 2 (*Hilbert spaces*). Let H be a linear space over $\mathbb{C}$. An **inner product** on H is a complex-valued function $(x, y) \mapsto \langle x, y\rangle$ on $H \times H$ satisfying (i) $\overline{\langle y, x\rangle} = \langle x, y\rangle$ for each $(x, y) \in H \times H$; (ii) For each $y \in H$, the map $x \mapsto \langle x, y\rangle$ is linear on H; and (iii) $\langle x, x\rangle > 0$ for each $x \in H$ not equal to 0. An **inner product space** or **pre-Hilbert space** is a complex linear space H equipped with a fixed inner product $\langle \ , \ \rangle$. The inner product space H is a normed space with the norm defined by $\|x\| = \langle x, x\rangle^{1/2}$. If H is complete with respect to this norm, H is called a **Hilbert space**. If H is a Hilbert space and if $f \in H^*$, then there is a unique $y \in H$

such that $f(x) = \langle x, y \rangle$ for each $x \in H$. From this it follows that a Hilbert space is a reflexive Banach space.

EXAMPLE 3. Let Γ be an arbitrary set. For a non-negative function w on Γ, we define

$$\sum_{\gamma \in \Gamma} w(\gamma) = \sup\left\{\sum_{\gamma \in F} w(\gamma) \colon F \subset \Gamma,\ F \text{ finite}\right\}.$$

If w is a scalar-valued function on Γ such that

$$\sum_{\gamma \in \Gamma} |w(\gamma)| < \infty,$$

then w is called **summable**. Since in this case the support of w is countable, $\sum_{\gamma \in \Gamma} w(\gamma)$ is well-defined.

For each $p \in [1, \infty)$, let $\ell^p(\Gamma)$ be the space of all scalar-valued functions f such that $\|f\|_p \equiv (\sum_{\gamma \in \Gamma} |f(\gamma)|^p)^{1/p} < \infty$. Then $(\ell^p(\Gamma), \|\ \|_p)$ is a Banach space. We also let $\ell^\infty(\Gamma) = C_b(\Gamma)$ with Γ given the discrete topology. Then by Example 1, $\ell^\infty(\Gamma)$ is a Banach space with the supremum norm $\|\ \|_\infty$. Now suppose $p \in [1, \infty)$ and $\varphi \in (\ell^p(\Gamma))^*$. Then there is a unique $g \in \ell^q(\Gamma)$, where $1/p + 1/q = 1$ ($q = \infty$ if $p = 1$), such that fg is summable and $\varphi(f) = \sum_{\gamma \in \Gamma} f(\gamma)g(\gamma)$. Furthermore in this case $\|\varphi\| = \|g\|_q$. So $(\ell^p(\Gamma))^*$ and $\ell^q(\Gamma)$ are isometric. It follows that the space $\ell^p(\Gamma)$ is reflexive if $p \in (1, \infty)$. The spaces $\ell^1(\Gamma)$ and $\ell^\infty(\Gamma)$ are not reflexive unless Γ is finite. In case $p = 2$ and $\mathbb{C}$ is the scalar field, $\ell^2(\Gamma)$ is a Hilbert space with the inner product given by $\langle f, g \rangle = \sum_{\gamma \in \Gamma} f(\gamma)\overline{g(\gamma)}$. It can be shown that each Hilbert space H is isometric to $\ell^2(\Gamma)$ for a suitable Γ: if H is infinite dimensional the cardinality of Γ is the ***density*** of H. Hence the isometric classification of Hilbert spaces is simply a matter of the cardinality of Γ.

EXAMPLE 4. Let Γ be as in the previous example, and let $c_0(\Gamma)$ be the space of all scalar-valued functions f that **vanish at infinity**, i.e., for each $\varepsilon > 0$ the set $\{\gamma \in \Gamma \colon |f(\gamma)| \geqslant \varepsilon\}$ is finite. Then $c_0(\Gamma)$ is a closed linear subspace of $\ell^\infty(\Gamma)$, and so it is a Banach space with the supremum norm. If $\varphi \in (c_0(\Gamma))^*$, then there is a unique $g \in \ell^1(\Gamma)$ such that, for each $f \in c_0(\Gamma)$, $\varphi(f) = \sum_{\gamma \in \Gamma} f(\gamma)g(\gamma)$, and in this case $\|\varphi\| = \|f\|_1$. This shows that $(c_0(\Gamma))^*$ is isometric to $\ell^1(\Gamma)$ and that the bidual $(c_0(\Gamma))^{**}$ is isometric to $\ell^\infty(\Gamma)$.

2. The norm topology

In this section, we describe several topics related to the norm topology of the Banach spaces.

(A) Category theorems

Since a Banach space is a complete metric space, it is a Baire space. This fact has the following important and extremely useful consequences.

(a) ***Banach–Steinhaus Theorem*** *(or* ***uniform boundedness principle****).* Let $\{T_\alpha \colon \alpha \in A\}$ be a family of continuous linear maps of a Banach space X into normed spaces. If, for each $x \in X$, $\sup\{\|T_\alpha(x)\| \colon \alpha \in A\} < \infty$, then $\sup\{\|T_\alpha(x)\| \colon \alpha \in A,\ x \in B_X\} < \infty$. In particular, if A is a subset of the dual X^* of a Banach space X such that $\sup\{|f(x)| \colon f \in A\} < \infty$ for each $x \in X$, then A is a **bounded set** in the sense that $\sup\{\|f\| \colon f \in A\} < \infty$. Similarly if A is a subset of a *normed* space X such that $\sup\{|f(x)| \colon x \in A\} < \infty$ for each $f \in X^*$, then A is bounded.

(b) ***Open mapping theorem.*** Let T be a continuous linear map of a Banach space X onto a Banach space Y. Then T is an ***open*** map. Banach spaces X and Y are said to be **isomorphic** if there is a one-to-one continuous linear map T of X onto Y. In this case the inverse map $T^{-1} \colon Y \to X$ is also continuous by the open mapping theorem, and hence T is a linear homeomorphism.

(c) ***Closed Graph Theorem.*** Let T be a linear map on a Banach space X into a Banach space Y such that the ***graph*** of T, namely $\{(x, T(x)) \colon x \in X\}$, is closed in $X \times Y$. Then T is continuous.

(B) Maps into Banach spaces

Here we mention two theorems. For the proof and the original sources, we refer the reader to [4] and the bibliography therein.

(a) ***Michael's selection theorem.*** Let E be a ***paracompact*** space and let φ be a multivalued map on E taking values among non-empty closed convex subsets of a Banach space X, which is ***lower semi-continuous***, i.e., for each open subset U of X, the set $\{t \in E \colon \varphi(t) \cap U \neq \emptyset\}$ is open in E. Then (i) the map φ has a continuous ***selection***, i.e., a continuous function $f \colon E \to X$ such that $f(t) \in \varphi(t)$ for each $t \in E$, and (ii) if $A \subset E$ is closed, then each continuous selection for $\varphi|A$ extends to a continuous selection for φ. It follows from this theorem that if X, Y are Banach spaces and if T is a continuous linear map of X onto Y, then there is a continuous map $f \colon Y \to X$ such that $T(f(y)) = y$ for all $y \in Y$ (the **Bartle–Graves theorem**).

(b) ***Borsuk–Dugundji extension theorem.*** Let A be a closed subset of a metrizable space M and let X be a normed space. Also let $C(A, X)$ (respectively $C(M, X)$) be the space of all continuous functions on A (respectively M) into X. Then there is a linear map $L \colon C(A, X) \to C(M, X)$ such that, for each $f \in C(A, X)$, $L(f)$ is an extension of f and the range of $L(f)$ is contained in the convex hull of $f(A)$.

(C) The topological classification

Kadec proved in 1966 that all infinite-dimensional separable Banach spaces are homeomorphic to $\ell^2(\mathbb{N})$, see [4, Corollary 9.1, p. 231]; moreover Anderson, 1966, proved that

$\ell^2(\mathbb{N})$ is homeomorphic to the countable infinite product of real lines, i.e., $\mathbb{R}^{\mathbb{N}}$. Finally if X is a separable infinite-dimensional Banach space, then the spaces X, B_X and S_X are all homeomorphic to $\mathbb{R}^{\mathbb{N}}$, see [4, Corollary 5.1, p. 188]. The ultimate answer to the topological classification problem was given by Toruńczyk [13] in 1981: two infinite dimensional Banach spaces are homeomorphic if they have the same ***density***. We also mention that each infinite-dimensional compact convex subset of a Banach space is homeomorphic to the ***Hilbert cube*** $[0,1]^{\mathbb{N}}$ (Keller, 1931; see [4, p. 100]).

(D) Spaces of type $C(K)$

According to the **Banach–Stone theorem** ([2] for metrizable cases and [11] in general), a compact Hausdorff space is characterized by the Banach space of all real-valued continuous functions with the supremum norm: if K and L are compact Hausdorff spaces and if $C(K,\mathbb{R})$ is isometric to $C(L,\mathbb{R})$, then K and L are homeomorphic (see [10, Theorem 7.8.4]). On the other hand, the isomorphic type of $C(K)$ is not sufficient to characterize K, for by **Milutin's theorem**, if K and L are uncountable metrizable compact spaces, then $C(K)$ and $C(L)$ are isomorphic (see [10, 21.5.10]).

(E) Uniform and Lipschitz classification

As said in Section 2(C) the topological structure of a Banach space contains no information on its linear structure. Nonetheless, it was stated by Mazur–Ulam, 1932, that any map preserving the norm metric from a Banach space onto another one and sending 0-to-0 is a linear isometry in the sense of Section 1, [3, Theorem 14.1]. The idea of classifying Banach spaces using Lipschitz or more generally uniformly continuous non-linear maps is a relatively new and very active area of research. The Benyamini–Lindenstrauss monograph [3] is the place to start if one is interested in this subject. We cite only a few results here. Let us say that two metric spaces are **Lipschitz equivalent** (respectively **uniformly equivalent**) if there is a map of one onto the other such that the map and its inverse are both Lipschitz (respectively uniformly continuous). Such a map is called a **Lipschitz equivalence** (respectively **uniform equivalence**).

Let X and Y be Banach spaces and let f be a function defined on an open subset U of X into Y. Then f is said to be **Gâteaux differentiable** at $a \in U$ if there is a continuous linear map $T: X \to Y$ such that for each $h \in X$, $\lim_{t\to 0} t^{-1}(f(a+th) - f(a)) = T(h)$. If this limit converges uniformly for all $h \in S_X$, then f is **Fréchet differentiable** at a. The map T is called the **differential** of f at a and is denoted by $Df(a)$.

Now suppose that f is a Lipschitz equivalence of X onto a subset of Y. If f is Gâteaux differentiable at $a \in X$, then $Df(a)$ is an isomorphism (i.e., linear homeomorphism) of X to a subspace of Y. If X is separable and Y has the ***Radon-Nikodým property*** (RNP), then any Lipschitz map of X into Y is Gâteaux differentiable at some point (see [3, Theorem 6.42]). Hence in this case X is Lipschitz equivalent to a subset of Y if and only if X is isomorphic to a subspace of Y. Heinrich and Mankiewicz have shown (1980) that the above holds true without the RNP of Y if $Y = Z^*$ for some Banach space Z (see [3, Theorem 7.10]). Thus if a separable Banach space X is Lipschitz equivalent to a subset of a Banach space Y, then X is isomorphic to a subspace of Y^{**}. All reflexive Banach spaces and separable dual Banach spaces have the RNP. Among our examples, $C(K)$ does not have the RNP if K is infinite compact and Hausdorff. For an infinite Γ, $\ell^p(\Gamma)$ has the RNP if $1 \leqslant p < \infty$, and $\ell^\infty(\Gamma)$, $c_0(\Gamma)$ do not have the RNP. Without the RNP and outside dual Banach spaces the situation is quite different: any separable metric space is Lipschitz homeomorphic to a subset of c_0, Aharoni 1974 [3, Theorem 7.11]. The Lipschitz or uniform classification of Banach spaces is known only for a limited classes of Banach spaces. For instance Deville, Godefroy and Zizler have proved (1990) that, for K compact, $C(K)$ is Lipschitz equivalent to some $c_0(\Gamma)$ if and only if $K^{(\omega_0)} = \emptyset$ (see [3, Theorem 7.13]). Marciszewski has recently shown that there is an Eberlein compact space K (see Section 3(A)) such that $K^{(3)} = \emptyset$ (hence, $C(K)$ is Lipschitz equivalent to some $c_0(\Gamma)$) but $C(K)$ is not isomorphic to any space of the form $c_0(\Gamma)$ (to be published).

3. Weak and weak* topologies

Let X be a normed space. The **weak topology** for X (respectively **weak* topology**) is the weakest topology for X (respectively X^*) that makes $x \mapsto f(x)$ (respectively $f \mapsto f(x)$) continuous for each $f \in X^*$ (respectively $x \in X$). Note that the dual space X^* also has the weak topology since it is a Banach space. The space X with the weak topology is denoted by (X, w), and similarly (X^*, w^*) is the dual X^* with the weak* topology. If Y is a linear subspace of a normed space X, then, by the Hahn–Banach extension theorem, the weak topology for Y is the restriction of the weak topology for X. Both (X, w) and (X^*, w^*) are Hausdorff locally convex spaces, i.e., 0 has a basis of convex neighbourhoods and the vector sum and the multiplication by members of $\mathbb{K}$ are continuous. Clearly on X the weak topology is weaker than the norm topology, and on X^* the weak* topology is weaker than the weak topology. However the Hahn–Banach extension theorem implies that if C is a (norm) closed convex subset of X, then C is weak-closed. A linear functional f on X is weakly continuous if and only if it is norm-continuous, i.e., $f \in X^*$. More generally, a linear map from a normed space into another is norm-norm continuous if and only if it is weak–weak continuous. A linear functional φ on X^* is weak*-continuous if and only if there exists an $x \in X$ such that $\varphi(f) = f(x)$ for each $f \in X^*$. When X is embedded in its bidual (cf. Section 1), X is weak*-dense in X^{**} and B_X is weak*-dense in $B_{X^{**}}$ (**Goldstine's theorem**).

One of the pleasant features of the weak and weak* topologies is that compact sets are relatively easy to come by. Whereas B_X is never norm-compact, unless X is finite dimensional, it is weak-compact if and only if X is reflexive.

Thus the unit balls of Hilbert spaces and of $\ell^p(\Gamma)$, with $1 < p < \infty$ and Γ arbitrary, are all weak-compact. Moreover, by the ***Tychonoff Product Theorem***, B_{X^*} is always weak*-compact (**Banach–Alaoglu's theorem**). Let X be a normed space and let K be (B_{X^*}, w^*). Then K is a compact Hausdorff space, and there is a natural linear map $\varphi: X \to C(K)$ given by $\varphi(x)(f) = f(x)$ for each $x \in X$, $f \in K = B_X^*$. Clearly φ is an isometry. If τ_p denotes the ***topology of pointwise convergence*** for $C(K)$, then φ maps (X, w) homeomorphically onto $(\varphi(X), \tau_p)$ and $\varphi(X)$ ***separates*** points of K. If X is a Banach space, then $\varphi(X)$ is τ_p-closed in $C(K)$. This shows that the study of the weak topology is very closely related to that of the pointwise topology for the spaces of the type $C(K)$ with K compact. Below we discuss a few topics related to weak and weak* topologies in more detail.

(A) Weak and weak-compact sets*

It had been observed by Šmulian, Eberlein and others that weak-compact subsets of Banach spaces possess properties similar to those of metrizable spaces. This can be summarized by saying that, for each Banach space X, (X, w) is angelic, where a regular Hausdorff space is said to be **angelic** if the closure of each relatively countably compact set A is compact and the closure consists of the limits of sequences in A. This, in turn, is a consequence of the fact that, for each compact Hausdorff space K, $(C(K), \tau_p)$ is angelic, [7]. Corson and Lindenstrauss (1966) conjectured that a weak-compact subset of a Banach space is homeomorphic to a weak-compact subset of $c_0(\Gamma)$ for a suitable set Γ. This conjecture was confirmed in the ground-breaking paper Amir–Lindenstrauss [1], in which the authors have suggested that a space homeomorphic to a weak-compact subset of a Banach space be called an ***Eberlein compact*** (EC). The confirmation is based on the main theorem of [1]. A Banach space X is said to be **weakly compactly generated** (WCG) if X is generated by a weak-compact subset $K \subset X$, i.e., the linear span of K is dense in X. The **Amir–Lindenstrauss theorem** states that if a Banach space X is WCG, then there exist a set Γ and a one-to-one continuous linear map on X into $c_0(\Gamma)$. For properties of Eberlein compacta, see the article on "Eberlein and Corson Compacta". Here we mention two properties of weak-compact sets related to the linear structure of Banach spaces. If K is a weak-compact subset of a Banach space, then the closed convex hull of K is weak-compact (Krein–Šmulian, 1940). This can be seen as a consequence of the remarkable, but much harder to prove, **James' theorem** (1972), [8]: if A is a subset of a real-Banach space X such that for each $f \in X^*$ there exists an $a \in A$ satisfying $f(a) = \sup\{f(x): x \in A\}$, then the weak-closure of A is weak-compact.

There is nothing remarkable about weak*-compact subsets of dual Banach spaces. In fact any compact Hausdorff space is homeomorphic to a weak*-compact subset of a dual Banach space. However, weak*-compact subsets of dual spaces of particular Banach spaces can be more special. A Banach space X is called an **Asplund space** if each real-valued convex continuous function defined on an open convex subset $U \subset X$ is Fréchet differentiable (cf. Section 2(E)) at each point of a dense G_δ subset of U. It is known that X is an Asplund space if and only if X^* has the ***Radon–Nikodým property*** (RNP). A compact space is said to be ***Radon–Nikodým compact*** (RN compact) if it is homeomorphic to a weak*-compact subset of the dual of an Asplund space. The properties of RN compact spaces are very similar to those of EC, and the class of EC is properly contained in the class of RN compact spaces. See the article on "Radon–Nikodým Compacta". Section 3(B) below describes another classes of Banach spaces for which their dual spaces have special w^*-compact subsets.

(B) Topological properties of (X, w)

The topological study of Banach spaces with the weak topology is a far more subtle matter than that for the norm topology (see Section 2(C)). Here the main problems arise in the non-separable case. In [6] Corson proved that if X is a Banach space, then (X, w) is ***paracompact*** if and only if (X, w) is ***Lindelöf***, and conjectured that (X, w) is Lindelöf if and only if X is WCG. More recently Rezničenko proved that (X, w) is normal if and only if it is Lindelöf. Talagrand proved that if a Banach space X is WCG, then (X, w) is ***K-analytic*** hence Lindelöf confirming a one-half of the Corson conjecture, see [12] for a detailed account about weakly K-analytic Banach spaces. The converse, however, is false since Rosenthal has constructed a WCG Banach space which has a non-WCG closed linear subspace.

It is proved in [1] that a compact Hausdorff space K is EC if and only if $(C(K), w)$ is WCG. Analogously, it is possible to classify K according to topological properties of $(C(K), w)$. Thus a compact Hausdorff space K is said to be **Talagrand compact** (respectively **Gul'ko compact**) if $(C(K), w)$ is ***K-analytic*** (respectively countably K-determined). Here a ***Tychonoff*** space T is said to be **countably K-determined** if there are a compactification Z of T and a sequence $\{K_n: n \in \mathbb{N}\}$ of compact sets in Z such that, for each $t \in T$, $\bigcap\{K_n: t \in K_n,\ n \in \mathbb{N}\} \subset T$. For X Banach space, (X, w) is K-analytic (respectively countably K-determined) if and only if (B_{X^*}, w^*) is Talagrand (respectively Gul'ko) compact space. We have the implications EC $\Rightarrow$ Talagrand compact $\Rightarrow$ Gul'ko compact $\Rightarrow$ ***Corson compact***, and none of the arrows can be reversed. (See the article by S. Negrepontis in [KV].)

Finally, we mention a striking result of Rosenthal [9]. By investigating thoroughly what it means for a uniformly bounded sequence of real-valued functions on a set *not* to have a convergent subsequence, he has proved the following: Let X be a separable real Banach space. Then X does not have any isomorphic copy of $\ell^1(\mathbb{N})$ if and only if each member of $B_{X^{**}}$ is the weak*-limit of a sequence in B_X. This means that $B_{X^{**}}$ is a pointwise compact set of ***first Baire class*** functions on the ***Polish space*** (B_{X^*}, w^*). A compact Hausdorff space is called **Rosenthal compact** if it is homeomorphic to a pointwise compact subset of the space $B_1(\Omega)$ of first Baire class functions on a Polish space Ω. A real breakthrough in this area was made by Bourgain, Fremlin and Talagrand [5] who proved that, for each Polish space Ω, $B_1(\Omega)$ is angelic with respect to the pointwise topology (compare 3(A)). See the article on “Rosenthal Compacta”.

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k-2 Banach Spaces and Topology (II)

This article is the continuation of "Banach spaces and topology (I)" (referred to as BT(I)). Whereas BT(I) mainly deals with properties of the norm topology and the weak topologies in Banach spaces by themselves, the present article will stress the interplay between properties of weak and norm topologies. To save space, some references are given by the author(s)'s names and Math. Reviews ID numbers. We shall also refer to articles cited in BT(I) as, e.g., [5,BT(I)]. In this article, Banach spaces are over the reals, unless otherwise indicated.

1. Properties related to weak, weak* and norm topologies

If a Banach space X is WCG or more generally if (X, w) is K-analytic, then (X, w) is a Lindelöf space. An important class of weak-Lindelöf Banach spaces are the so called **weakly Lindelöf determined** (**WLD**) Banach spaces, and they coincide with Banach spaces with weak*-***Corson compact*** dual unit ball (Mercourakis–Negrepontis survey [HvM]). They provide a framework where Amir–Lindenstrauss constructions for WCG Banach spaces [1,BT(I)] can be carried out with analogous consequences. More general than the notion of weak-Lindelöf space is the following: a Banach space X is said to have the **property** $(\mathcal{C})$ (of Corson) if each family of closed convex subsets with the countable intersection property has non-empty intersection. A Banach space X has property $(\mathcal{C})$ if and only if, whenever $A \subset B_{X^*}$, each element of $\overline{A}^{w^*}$ is in the weak*-closed convex hull of a countable subset of A (Pol, MR 82a:46022). Whereas property $(\mathcal{C})$ is stable under taking finite products, it is yet an open problem to decide if the product of a weak-Lindelöf Banach space by itself is again weak-Lindelöf.

Given a subset A of a Banach space X, a point $x \in A$ is said to be a **(weak to norm) continuity point** of A if the identity map $\mathrm{id}: (A, w) \to (A, \|\ \|)$ is continuous at x. Weak-compact subsets of Banach spaces have points of continuity. A Banach space X is said to have the **point of continuity property** (PCP) if each nonempty weakly closed and bounded subset A of X has a point of continuity. Edgar and Wheeler [3] proved that a Banach space X has the PCP if and only if (B_X, w) is ***hereditarily Baire***. If X is separable, then (B_X, w) is ***Polish*** if and only if X^* is separable and X has the PCP. A Banach space X is isomorphic to the direct sum of a reflexive subspace and a separable subspace with weak-Polish unit ball if and only if (B_X, w) is ***Čech-complete***. If X is a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of (X^{**}, w^*), then X has the PCP, and the converse is true if the Banach space X is assumed to be separable, see [3]. Ghoussoub and Maurey proved that if X is a separable Banach space, then X has the PCP if and only if there is a separable subspace Y of X^* such that $(B_X, \sigma(X, Y))$ is Polish, where $\sigma(X, Y)$ is the weakest topology for X that makes $x \mapsto f(x)$ continuous for each $f \in Y$ (Ghoussoub and Maurey, MR 88i:46022).

If C is a convex subset of a vector space, a point $a \in C$ is said to be an **extreme point** of C if it is not the midpoint of any proper segment in C; the set of extreme points of C will be denoted by $\mathrm{ext}(C)$. The Krein–Milman theorem states that each compact convex subset K of a locally convex space is the closed convex hull of $\mathrm{ext}(K)$, or equivalently, for each point $x \in K$, there exists a regular Borel measure μ on K supported by the closure of $\mathrm{ext}(K)$ such that $f(x) = \int_K f\, d\mu$ for each continuous real-valued affine function f on K. When K is metrizable it is possible to choose μ supported by $\mathrm{ext}(K)$ (**Choquet's theorem**). Among its consequences, we have the **Rainwater theorem**: a bounded sequence $\{x_n\}$ in a Banach space X weak-converges to $a \in X$ if $\{f(x_n)\}$ converges to $f(a)$ for each $f \in \mathrm{ext}(B_{X^*})$. More generally, it follows from Simons' inequality (Simons, MR 47#755) that Rainwater theorem remains true when $\mathrm{ext}(B_{X^*})$ is replaced by any subset $B \subset S_{X^*}$ with the property that for every $x \in X$ there is $x^* \in B$ such that $\|x\| = x^*(x)$.

Without compactness of C, $\mathrm{ext}(C)$ may be empty. A closed convex subset C of a Banach space is said to have the **Krein–Milman property** (KMP) if each nonempty bounded closed convex subset of C has an extreme point. If this is the case, C is the closed convex hull of $\mathrm{ext}(C)$. Given a nonempty bounded subset A of a Banach space X, a **slice of** A is a set of the form $\{a \in A\colon f(a) > \sup\{f(x)\colon x \in A\} - r\}$ with $f \in X^*$, $r > 0$. A point x of a closed bounded convex subset C is called a **denting point of** C if there are slices of C containing x of arbitrarily small $\|\ \|$-diameter. Note that each denting point of C is both a continuity and extreme point of C. The converse is true and non-trivial (Lin–Lin–Troyanski, MR 91g:46016). A non-empty bounded subset A of a Banach space is said to be **dentable** if A has non-empty slices of arbitrarily small diameter. It is known that a Banach space X has the ***Radon–Nikodým property*** (RNP) if and only if each nonempty bounded closed convex subset of X is dentable (cf. [1]). The RNP implies both the KMP and the PCP (Lindenstrauss, cf. [1]). Conversely the PCP and KMP implies the RNP (Schachermayer, MR 89c:46030). It is an open problem if the KMP implies the RNP. This is known to be true for some special cases. For instance this is true for dual Banach spaces (Huff and Morris, MR 50#14220) and for a Banach space which is isomorphic to its square (Schachermayer, MR 87e:46032).

Let c be a point of a bounded closed convex set C in a Banach space X and let $f \in X^*$. If $f(x) \leqslant f(c)$ for all $x \in C$, then we say that c is a **support point** of C and f a **support functional** of C. If, for some $g \in X^*$, $g(x) < g(c)$ for all $x \in C$, $x \neq c$, then we say that c is an **exposed point** of C. The point c is a **strongly exposed point** of C if, for some $g \in X^*$, the $\| \ \|$-diameter of the sets $\{x \in C\colon g(c) - r < g(x)\}$ tends to 0 as $r \downarrow 0$, and, if this is the case, the functional g is said to **expose** c **strongly**. A strongly exposed point is both a denting point and an exposed point, and an exposed point is both a support point and an extreme point. Bishop–Phelps theorem states that for any nonempty closed convex bounded subset C of a Banach space X, the support functionals of C are norm dense in X^* (cf. [2]). Let C be a bounded closed convex subset of a Banach space X with the RNP. Then C is the closed convex hull of its strongly exposed points, and the set of all continuous functionals that strongly expose points of C is a dense $\mathcal{G}_\delta$ subset of $(X^*, \| \ \|)$. The last conclusion has a strong converse: Let X be a Banach space. If, for each bounded closed convex subset C, the set of all support functionals of C is of the second category in $(X^*, \| \ \|)$, then X has the RNP (Bourgain–Stegall (ca. 1975), see [1]).

If in a dual Banach space X^* we replace the weak by the weak* topology in the definition of the PCP, we arrive at the notion of weak* point of continuity property (w^*-PCP for short). The w^*-PCP, the RNP, and the KMP are equivalent for a dual Banach space X^* (see [1]). If this is the case, X^* satisfies a stronger dentability condition: every nonempty convex bounded subset of X^* has nonempty relatively w^*-open slices of arbitrarily small diameter. There are examples of dual Banach spaces having the PCP but not the RNP: therefore PCP and w^*-PCP are not equivalent in dual Banach spaces (see [3]).

2. Smoothness and renorming

In a Banach space, one can change the norm to an equivalent one without affecting the norm, weak and weak* topologies. Therefore, in many instances, topological questions reduce to that of replacing the given norm with one with better geometric properties.

(A) Classical renorming results

The norm in any Hilbert space X satisfies the **Parallelogram Law**: $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in X$, and this law characterizes Hilbert spaces amongst complex Banach spaces (von Neumann 1935). Notice that, when $x, y \in S_X$, the distance $\|x - y\|$ depends only on how close $\|2^{-1}(x + y)\|$ is to 1. In particular, the midpoint of two distinct points of S_X is never on S_X. The norm $\| \ \|$ of a Banach space is said to be **rotund** or **strictly convex** if the unit sphere does not contain a non-trivial segment. The norm of a Hilbert space also enjoys the **smoothness** property of being Fréchet differentiable away from 0. Properties of rotundity and smoothness are in *"duality"* through Šmulyan criterion type results, see [2, Theorem I.1.4]. For instance, the norm of X is Gâteaux differentiable away from 0 if the dual norm on X^* is rotund.

Given $\varepsilon > 0$, a dyadic ε-tree with root $x \in X$ of length $N \in \mathbb{N} \cup \{\infty\}$ is a family $\{x(s)\}$ of elements of X indexed by $s \in \{-1, 1\}^{<N+1}$ such that $x = x(\emptyset)$, $x(s) = 2^{-1}(x(s, -1) + x(s, 1))$ and $\|x(s, -1) - x(s, 1)\| \geqslant \varepsilon$ for each $s \in \{-1, 1\}^{<N}$. An infinite dyadic ε-tree is not dentable. Thus a Banach space with the RNP does not contain an infinite bounded dyadic ε-tree for any $\varepsilon > 0$ but the converse is not true in general (Bourgain and Rosenthal, MR 82g:46044). A Banach space X is said to be **superreflexive** if for each $\varepsilon > 0$ there is $N(\varepsilon) \in \mathbb{N}$ such that each dyadic ε-tree contained in the unit ball B_X has length $N \leqslant N(\varepsilon)$. Superreflexive Banach spaces are reflexive. If X is superreflexive, then X^* is also superreflexive.

The **modulus of convexity** of a norm $\| \ \|$ is defined for $\varepsilon \in [0, 2]$ as follows: $\delta(\varepsilon) = \inf\{1 - 2^{-1}\|x + y\|\colon x, y \in S_X, \|x - y\| \geqslant \varepsilon\}$. The norm is said to be **uniformly rotund** if $\delta(\varepsilon) > 0$ for $\varepsilon \in (0, 2]$. The norm of X is uniformly rotund if and only if the dual norm on X^* is **uniformly smooth**, i.e., $\lim_{t \to 0} t^{-1}(\|x + th\| - \|x\|)$ exists uniformly in $(x, h) \in S_X \times S_X$. A Banach space with a uniformly rotund norm is necessarily reflexive. The spaces L^p with $1 < p < \infty$ are both uniformly rotund and uniformly smooth. A celebrated theorem of Enflo (MR 49#1073) states that X is superreflexive if and only if X admits an equivalent uniformly rotund norm (equivalently, a uniformly smooth norm). By a probabilistic method, Pisier (MR 52#14940) has improved Enflo's result by showing that the modulus of convexity of an equivalent uniformly rotund norm can be made to satisfy $\delta(\varepsilon) \geqslant C\varepsilon^p$ with $C > 0$, $p \geqslant 2$.

Given a Banach space X, $C \subset X$ a non-empty bounded set and $\varepsilon > 0$ define $D_\varepsilon(C) = \{x \in C\colon \| \cdot \|\text{-diam}(S) > \varepsilon$ for each slice S of C containing $x\}$. By induction we can define a transfinite sequence of sets by letting: $B_{\varepsilon,0} = B_X$, $B_{\varepsilon,\alpha+1} = D_\varepsilon(B_{\varepsilon,\alpha})$ and, for a limit ordinal α, $B_{\varepsilon,\alpha} = \bigcap\{B_{\varepsilon,\beta}\colon \beta < \alpha\}$. Then X is superreflexive if and only if for every $\varepsilon > 0$ there is $n(\varepsilon) < \omega$ such that $B_{\varepsilon,n(\varepsilon)} = \emptyset$. (Lancien used this fact in (MR 96e:46009) to give a non-probabilistic proof of Pisier's result.) The Banach space X has the RNP if and only if for each $\varepsilon > 0$ there is an ordinal $\alpha(\varepsilon)$ such that $B_{\varepsilon,\alpha(\varepsilon)} = \emptyset$. If $\alpha(\varepsilon)$ is a countable ordinal for each $\varepsilon > 0$, then X has an equivalent locally uniformly rotund norm (Lancien, MR 94h:46026). The norm $\| \ \|$ of a Banach space X is said to be **locally uniformly rotund** (LUR) if for each $x \in S_X$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x - y\| < \varepsilon$ whenever $y \in S_X$ and $\|2^{-1}(x + y)\| > 1 - \delta$. Certain topological properties of X ensure the existence of equivalent LUR norms, for instance X being WLD. The norm of X is Fréchet differentiable away from 0 if the dual norm on X^* is LUR. This fact provides a standard technique for finding an equivalent Fréchet differentiable norm as, for instance, for Banach spaces X such that X^* is weakly countably determined. See subsection (C) for further discussion on LUR-renorming.

(B) Asplund spaces

A Banach space X is said to be an **Asplund space** (respectively **weak Asplund space**) if each continuous convex real-valued function defined on a convex open subset of X is Fréchet (respectively Gâteaux) differentiable at all points of a dense $\mathcal{G}_\delta$ subset of its domain (cf. Section 3(A), BT(I)). Each separable Banach space is weak Asplund (Mazur 1933), and if X^* is separable, then X is indeed an Asplund space (Asplund, MR 37#6754).

The Banach space X is Asplund if and only if X^* has the RNP, and this is the case if and only if each separable subspace of X has the separable dual (Namioka–Phelps and Stegall, see [1]). In particular, for a compact Hausdorff space K the space $C(K)$ is Asplund if and only if K is scattered. Preiss (MR 91g:46051) established that real-valued Lipschitz functions on Asplund spaces are Fréchet differentiable on a dense subset. In the case of a separable space X, there is a very tight connection between renormability of the space and the Asplund property. Indeed, for separable X the following are all equivalent: separability of the dual space X^*; the existence on X of an equivalent norm with LUR dual norm; the existence on X of an equivalent norm which is Fréchet differentiable in $X \setminus \{0\}$. Day asked if the Fréchet renormability is necessary or sufficient condition for a non-separable Banach space to be Asplund. In one direction we have that X is an Asplund space (respectively weak Asplund) whenever it has an equivalent Fréchet (respectively Gâteaux) differentiable norm, Ekeland and Lebourg (MR 55#4254) (respectively Preiss–Phelps–Namioka, MR 92h:46021). The converse is not true (Haydon, MR 91h:46045). The result by Ekeland and Lebourg cited above needs only the existence of a non-trivial Fréchet differentiable function of bounded support. This type of functions are called **bump functions**. The question whether the Asplund property is characterized by the existence of a bump function remains open.

A Banach space X admits **C^k-smooth partitions of unity** ($k \in \mathbb{N}$ or $k = +\infty$) when each norm-open cover of X has a ***partition of unity*** subordinate to it consisting of Fréchet C^k-smooth functions. Toruńczyk (MR 49#4016) showed that a Banach space admits C^k-smooth partitions of unity if and only if there is a set Γ and a homeomorphic embedding φ from X into $c_0(\Gamma)$ such that the γ-th coordinate function $x \mapsto \varphi(x)(\gamma)$ is C^k-smooth on X, for each $\gamma \in \Gamma$. As a consequence he obtained that any Hilbert space admits C^∞-smooth partitions of unity, proved the existence of high order smooth partitions of unity on L^p-spaces and showed that any reflexive space admits C^1 -smooth partitions of unity. When X is WCG and X admits a C^k-bump function then X admits a C^k-smooth partition of unity (Godefroy–Troyanski–Whitfield–Zizler, MR 85d:46020). The spaces $C_0(\Upsilon)$ on trees Υ considered in [8] also admit C^∞-smooth partitions of unity. It is an open problem if the existence of a C^k-bump function in X implies that the space admits C^k-smooth partitions of unity. Hajek and Haydon have shown recently that this is the case when X is a space $C(K)$, to be published.

(C) Fragmentability conditions

The notion of fragmentability provides a tool to measure how far apart the weak and the norm topologies of a Banach space are. This term was introduced by Jayne and Rogers (MR 87a:28011) in their work on Borel selectors for functions taking values among subsets of Banach spaces. Let (T, τ) be a topological space, d a metric on T and $\varepsilon > 0$; the space T is said to be **ε-fragmented by** d if for each nonempty subset C of T there exists a τ-open subset V of T with $C \cap V \neq \emptyset$ and d-diam$(C \cap V) < \varepsilon$. When T is ε-fragmented by d for each $\varepsilon > 0$, we say that T is **fragmented by** d. Each bounded subset of a Banach space with the RNP is fragmented by the norm, but the converse is not true. However this is the case in dual Banach spaces: the dual ball (B_{X^*}, w^*) is fragmented by the norm if and only if X is an Asplund space (Namioka–Phelps, cf. [1]), i.e., X^* has the RNP. For Asplund spaces X, Jayne-Rogers' selection theorem provides a ***first Baire class*** map $f : X \to S_{X^*}$ with $\langle f(x), x\rangle = \|x\|$ for every $x \in X$; this selector for the duality map is a main tool to deal with the Amir–Lindenstrauss type construction in X^*. For further discussion, see [2] and [5].

There is not much known about the permanence of the weak Asplund property under the standard operations of Banach spaces. It is not even known if $X \times \mathbb{R}$ is weak Asplund when X is. However there is a much better subclass of weak Asplund spaces due to Stegall defined below (cf. [5] and references therein). Let T and S be topological spaces. A map $F : S \to 2^T$ is said to be **usco** if F is ***upper semi-continuous*** (i.e., whenever $U \subset T$ is open $\{s \in S\colon F(s) \subset U\}$ is open in S) and for each $s \in S$, $F(s)$ is compact and non-empty. A ***Tychonoff space*** T is said to belong to **class** $\mathcal{S}$ if, whenever B is a Baire space and $F : B \to 2^T$ is usco, there is a selector f of F (i.e., a map $f : B \to T$ such that $f(b) \in F(b)$ for each $b \in B$) which is continuous at each point of a dense $\mathcal{G}_\delta$ subset of B. It is shown that, if T is fragmented by a metric, then $T \in \mathcal{S}$. A Banach space X is said to belong to class $\widetilde{\mathcal{S}}$ (**Stegall's class**) if $(B_{X^*}, w^*) \in \mathcal{S}$. Each Banach space in $\widetilde{\mathcal{S}}$ is weak Asplund and this class has very good permanence properties. When a compact space K is fragmented, the Banach space $C(K)$ belongs to $\widetilde{\mathcal{S}}$ (Ribarska, MR 89e:54063). Compact spaces K such that $C(K)$ is weak Asplund are sequentially compact and contain dense $\mathcal{G}_\delta$ completely metrizable subset (Čoban–Kenderov, MR 91c:90119). A Banach space with a Gâteaux differentiable norm belongs to $\widetilde{\mathcal{S}}$ (Preiss–Phelps–Namioka, op. cit.).

A topological space (X, τ) is said to be **σ-fragmentable by a metric** d if for each $\varepsilon > 0$, $X = \bigcup_{n=1}^\infty X_{n,\varepsilon}$ where each $X_{n,\varepsilon}$ is ε-fragmented by d. The class of Banach spaces X such that (X, w) is σ-fragmentable by the norm metric has been extensively studied following the work of Jayne, Namioka and Rogers (MR 93i:46027, 94c:46028 and 94c:4601). Such Banach spaces are said to be **σ-fragmentable**. They have established the connection of this notion with descriptive set-theory, renormings and property $\mathcal{N}^*$. A norm in a Banach space X is called a **Kadec norm** if the norm and the weak topologies coincide on the unit sphere S_X. Each LUR norm is a Kadec norm because each point of the unit sphere of a LUR norm is a strongly exposed point of the unit ball. If a Banach space admits an equivalent Kadec norm, X is a Borel subset of (X^{**}, w^*)

and the Borel sets for the weak and the norm topology of X coincide (Edgar, MR 81d:28016). A Banach space X which is obtained through the ***Souslin operation*** applied to Borel subsets of (X^{**}, w^{*}) is called **weakly Čech-analytic**. Each weakly Čech-analytic Banach space is σ-fragmentable (Jayne–Namioka–Rogers, op. cit.). Therefore for a Banach space we have

$$\begin{aligned}\text{LUR renormable} &\Longrightarrow \text{Kadec renormable}\\ &\Longrightarrow \text{weakly Čech-analytic}\\ &\Longrightarrow \sigma\text{-fragmentable.}\end{aligned}$$

No example of σ-fragmentable Banach space without an equivalent Kadec norm is known. The first example of a Banach space with Kadec norm and without equivalent LUR norm is given by Haydon [8]. Kenderov and Moors have proved in (MR 2001f:46026) that a Banach space X is σ-fragmentable if and only if (X, w) is fragmented by a metric whose topology is stronger than the weak topology. Their argument is game theoretic.

A modification of σ-fragmentability characterizes LUR renormability of Banach spaces: X is LUR renormable if and only if for each $\varepsilon > 0$, $X = \bigcup_{n=1}^{\infty} X_{n,\varepsilon}$ where each $X_{n,\varepsilon}$ is the union of its slices of diameter less than ε (Molto–Orihuela–Troyanski, MR 98e:46011 and Raja [Mathematika **46** (1999), 343–358]). In particular, if every point of the unit sphere is a denting point then the space is LUR renormable (Troyanski, MR 86g:46030). When the previous sets $X_{n,\varepsilon}$ are simply the union of relatively weak open sets of diameter less than ε then X is said to have the **JNR property**. If a Banach space admits an equivalent Kadec norm , then it also has the JNR property. Conversely, the JNR property in X implies the existence of a symmetric homogeneous and weakly lower semi-continuous real-valued function F defined on X with $\| \ \| \leqslant F(\cdot) \leqslant 3 \| \ \|$ and such that the norm and the weak topologies coincide on $\{x \in X\colon F(x) = 1\}$, (Raja, MR 2000i:46003). A Banach space X has the JNR property if and only if there is a sequence $\{A_n\}$ of subsets of X such that the family $\{A_n \cap W\colon W \text{ weak-open}, n \in \mathbb{N}\}$ is a ***network*** for the norm topology. A Banach space admits an equivalent Kadec norm if and only if the sequence $\{A_n\}$ above can be chosen to be convex (Raja, MR 2000i:46003). The JNR property is equivalent to the fact that the weak topology has a ***σ-relatively-discrete*** network (cf. Hansell [Serdica Math. J. **27** (2001), 1–66] and Molto–Orihuela–Troyanski–Valdivia, MR 2000b:46031).

3. Heritage of S. Banach and the structural theory of Banach spaces

A sequence of vectors $\{x_i\colon i \in \mathbb{N}\}$ is called **basis of a Banach space** X if every $x \in X$ has a unique representation as $x = \sum a_i x_i$ with scalars a_i. If the convergence of the series is unconditional the basis is called an **unconditional basis**. In that case every infinite subset M of integers gives a continuous linear projection $P_M(\sum a_i x_i) = \sum_{i \in M} a_i x_i$. Each infinite dimensional Banach space contains an infinite dimensional subspace with a basis and Banach asked if each separable Banach space has a basis. A famous counterexample of Enflo [4] solved even a stronger version of the problem dealing with the approximation property of Grothendieck. After Enflo's counterexample, and for a long time, it was conjectured that each infinite dimensional Banach space contains copies of c_0 or ℓ^p or, at least, an infinite dimensional subspace with an unconditional basis. This is the case for Banach spaces with a C^{∞}-smooth bump function, (Deville, MR 90m:46023) and for the class of Orlicz spaces (Lindenstrauss and Tzafriri [9]). Nevertheless Tsirelson (MR 50#2871) constructed a reflexive Banach space not containing ℓ^p for $1 < p < +\infty$. A Banach space X is called stable if for any two bounded sequences $\{x_n\}$ and $\{y_n\}$ in X and for any two free ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$

$$\lim_{n \in \mathcal{U}} \lim_{m \in \mathcal{V}} \|x_n + y_m\| = \lim_{m \in \mathcal{V}} \lim_{n \in \mathcal{U}} \|x_n + y_m\|.$$

Krivine and Maurey (MR 83a:46030) have proved that stable Banach spaces contain, for every $\varepsilon > 0$, a subspace which is $(1 + \varepsilon)$-isometric to ℓ_p for some p. Tsirelson's construction has been modified by Schlumprecht (MR 93h:46023) opening the door for the construction by Gowers and Maurey [7] of a separable reflexive Banach space X that does not have any infinite-dimensional subspace with an unconditional basis (see also Gowers, MR 94j:46024). Gowers-Maurey's example X has the property that, for each subspace Z, any continuous linear projection P of X onto Z is trivial, i.e.,, either $\dim PZ < \infty$ or $\dim Z/PZ < \infty$. A Banach space with this property is said to be **hereditarily indecomposable**. A hereditarily indecomposable Banach space is not isomorphic to any of its proper subspaces and it provides an answer to Banach's *"hyperplane problem"* asking whether each infinite-dimensional Banach space is isomorphic to its hyperplanes, [6]. Recently Argyros has been able to construct non separable hereditarily indecomposable Banach spaces.

A dichotomy result by Gowers (MR 97m:46017) makes clear that hereditarily indecomposable spaces are not just pathological counterexamples but they are essential in the structural theory of general Banach spaces: Each infinite dimensional Banach space has a hereditarily indecomposable subspace or a subspace with an unconditional basis. The proof is combinatorial and it uses infinite Ramsey theory which turns out to be an important tool in the infinite-dimensional setting. As a consequence Gowers solved the classical *"homogeneous space problem"* by showing that ℓ^2 is the only Banach space which is isomorphic to each infinite dimensional subspace. Another well-known open problem was the following: Assume that X and Y are Banach spaces each of them isomorphic to a complemented subspace (i.e., the image of a continuous linear projection) of the other. Must X be isomorphic to Y? Again Gowers gave a counterexample: a Banach space X isomorphic to $X \oplus X \oplus X$ and not to $X \oplus X$ (MR 97d:46009).

4. Final comment

We refer the interested reader to the references here and in BT(I) with special mention to the book by Benyamini and Lindenstrauss [3,BT(I)] where many of the topics we have commented are expanded and the non-linear functional analysis theory is presented. We also refer to the *Handbook of the Geometry of Banach Spaces* (*Two volumes*) *W.B. Johnson and J. Lindenstrauss, eds, Elsevier, Amsterdam* (*2001 and 2003*).

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k-3 Measure Theory, I

The purpose of this article is, of course, to survey some of the interactions between measure theory and general topology. We should mention at the outset that Fremlin's grand treatise on measure theory [2–4] is used extensively as a reference, even though it is often not cited explicitly. All material not covered in his treatise has been given explicit references.

1. Measure spaces and measure algebras

The two primary objects of study in measure theory are measure spaces and measure algebras. Recall that a **measure space** is a triple $(X, \mathcal{E}, \mu)$ where $\mathcal{E}$ is a σ-algebra of subsets of X, and $\mu : \mathcal{E} \to [0, \infty]$ is a **measure** on X, i.e., $\mu(\emptyset) = 0$ and for every sequence $(E_n \colon n \in \mathbb{N})$ of pairwise disjoint members of $\mathcal{E}$,

$$\mu\left(\bigcup_{n=0}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \mu(E_n).$$

A **measure algebra** is a pair $(\mathcal{B}, \nu)$ where $\mathcal{B}$ is a σ-complete ***Boolean algebra*** and $\nu : \mathcal{B} \to [0, \infty]$ where $\nu(a) = 0$ iff $a = 0$, and for every sequence $(a_n \colon n \in \mathbb{N})$ of pairwise incompatible members of $\mathcal{B}$ (i.e., $a_m \cdot a_n = 0$ whenever $m \neq n$),

$$\nu\left(\sum_{n=0}^{\infty} a_n\right) = \sum_{n=0}^{\infty} \nu(a_n).$$

There are natural correspondences between measure spaces and measure algebras. Much of the theory of one object can be translated to the other, but not all of the results correspond and we cannot simply restrict our attention to only one of them without losing some valuable aspect of the measure theory. Given a measure space $(X, \mathcal{E}, \mu)$, let $\mathcal{N}_\mu$ denote the ideal of **null set**s (or **negligible subset**s) of X, i.e., N is null if there is an $E \in \mathcal{E}$ such that $E \supseteq N$ and $\mu(E) = 0$. It is almost a first principle of measure theory that negligible sets can be ignored. Thus in most cases (but not always) two sets which differ by a null set can be treated as the same set, and identifying sets in this way one arrives at the **measure algebra of a measure space** $(X, \mathcal{E}, \mu)$, the measure algebra $(\mathcal{B}, \bar{\mu})$ where $\mathcal{B} = \mathcal{E} / (\mathcal{N}_\mu \cap \mathcal{E})$ is the quotient algebra and $\bar{\mu}([E]_{\mathcal{N}_\mu \cap \mathcal{E}}) = \mu(E)$ for all $E \in \mathcal{E}$, where $[E]_{\mathcal{N}_\mu \cap \mathcal{E}}$ denotes the equivalence class of E.

Conversely, we have the following theorem: Every measure algebra is isomorphic, as a measure algebra, to the measure algebra of some measure space. Indeed, let $(\mathcal{B}, \nu)$ be a measure algebra and let $X = \mathrm{Ult}(\mathcal{B})$ be the ***Stone space*** of $\mathcal{B}$, so $\mathcal{B}$ is isomorphic to $\mathrm{Clop}(X)$ the Boolean algebra of ***clopen*** subsets of X – in what follows we identify $\mathcal{B}$ and $\mathrm{Clop}(X)$. We turn X into a measure space, as follows: let $\mathcal{M} = \mathcal{M}_X$ be the ideal of ***meager set***s and $\mathcal{A} = \{C \triangle M \colon C \in \mathrm{Clop}(X), M \in \mathcal{M}\}$. Then $\mathcal{A}$ is a σ-algebra on X – this is true whenever $\mathcal{B}$ is σ-complete; and since every compact Hausdorff space is a ***Baire space***, i.e., a space with no nonempty open meager sets, $C \mapsto [C]_\mathcal{M}$ defines an isomorphism between $\mathrm{Clop}(X)$ and $\mathcal{A} / \mathcal{M}$, and thus $\mu(C \triangle M) = \nu(C)$ is a well-defined measure on $\mathcal{A}$. It follows that $\mathcal{N}_\mu = \mathcal{M}$ and that $(\mathcal{A} / \mathcal{M}, \bar{\mu})$, where $\bar{\mu}$ is the natural quotient measure, is the measure algebra of the measure space $(X, \mathcal{A}, \mu)$, and that $C \mapsto [C]_\mathcal{M}$ is a measure algebra isomorphism between $(\mathcal{B}, \nu)$ and $(\mathcal{A} / \mathcal{M}, \bar{\mu})$.

Taxonomy of measures

We describe some classifications of measure spaces and measure algebras. The list here is not complete, but describes some of the primary properties which will be relevant later in our discussion. A measure algebra $(\mathcal{B}, \nu)$ is a **probability algebra** (and the measure a **probability measure**) if $\nu(1) = 1$; it is **totally finite** if $\nu(1) < \infty$; it is σ**-finite** if there is a sequence $(a_n \colon n \in \mathbb{N})$ in $\mathcal{B}$ such that $\sum_{n=0}^{\infty} a_n = 1$ and $\nu(a_n) < \infty$ for all n; it is **semifinite** if for all $a \in \mathcal{B}$ where $\nu(a) = \infty$, there is a nonzero $b \leqslant a$ such that $\nu(b) < \infty$; and it is **localizable** if it is semifinite and $\mathcal{B}$ is a complete Boolean algebra. Note that the strength of these properties is linearly ordered: probability algebra $\to$ totally finite $\to$ σ-finite $\to$ localizable $\to$ semifinite.

A **probability space** is a measure space $(X, \mathcal{E}, \mu)$ such that $\mu(X) = 1$; equivalently its measure algebra is a probability algebra. And a measure space is totally finite, σ-finite, localizable or semifinite, respectively, if its measure algebra is. Measure theory has very little to say about non-semifinite measure spaces and algebras.

There are two technical properties of measure spaces which are irrelevant for measure algebras. A **complete measure space** is one in which every null set is measurable (i.e., $E \in \mathcal{E}$ whenever $E \subseteq F \in \mathcal{E}$ and $\mu(F) = 0$), e.g., $(X, \mathcal{A}, \mu)$ above. A measure space is **locally determined** if for every $E \in \mathcal{P}(X) \setminus \mathcal{E}$ there is an $F \in \mathcal{E}$ with $\mu(F) < \infty$ such that $E \cap F \notin \mathcal{E}$. Another property not relevant to measure algebras: $(X, \mathcal{E}, \mu)$ is **strictly localizable** if there is a disjoint partition $(X_i \colon i \in I)$ of X into measurable sets of finite measure such that $\mathcal{E} = \{E \subseteq X \colon E \cap X_i \in \mathcal{E} \text{ for all } i \in I\}$, and $\mu(E) = \sum_{i \in I} \mu(E \cap X_i)$ for all $E \in \mathcal{E}$. Note that we have the implications: σ-finite $\to$ strictly localizable $\to$ localizable and locally determined. Since elementary measure theory only deals with σ-finite measures, one can completely ignore the notions strictly localizable, localizable and locally determined in the elementary case.

For the more important classes of measure algebras in terms of these classifications, more can be said of its description as the measure algebra of its Stone space; for example, if $(\mathcal{B}, \nu)$ is a semifinite measure algebra, then every meager subset of its Stone space is nowhere dense. Recall that for any Boolean algebra $\mathcal{B}$, the regular open algebra $\mathrm{RO}(\mathrm{Ult}(\mathcal{B}))$ of the Stone space of $\mathcal{B}$ is the usual completion of $\mathcal{B}$, and that $\mathcal{B}$ is a dense (with respect to the Boolean algebra ordering) subalgebra of $\mathrm{RO}(\mathrm{Ult}(\mathcal{B}))$ via the identification between $\mathcal{B}$ and $\mathrm{Clop}(\mathrm{Ult}(\mathcal{B}))$. It follows that if a measure algebra $(\mathcal{B}, \nu)$ is localizable, then $\mathrm{RO}(\mathrm{Ult}(\mathcal{B}))$ is equal to the algebra of clopen subsets of the Stone space $\mathrm{Ult}(\mathcal{B})$; and since for any Baire space X, $\mathrm{RO}(X)$ is isomorphic to the **category algebra** $\mathcal{C}_X / \mathcal{M}_X$ of X where $\mathcal{C}_X$ is the family of subsets $E \subseteq X$ with the ***Baire property***, i.e., $E = U \triangle M$ where U is open and M is meager, $(\mathcal{B}, \nu)$ is isomorphic to the measure algebra of the measure space $(\mathrm{Ult}(\mathcal{B}), \mathcal{C}_{\mathrm{Ult}(\mathcal{B})}, \mu)$ where $\mu(E) = \nu(a_E)$ and $a_E \in \mathcal{B}$ is the element that corresponds to $[E]_{\mathcal{M}_{\mathrm{Ult}(\mathcal{B})}}$ under the identifications $\mathcal{B} \cong \mathrm{RO}(\mathrm{Ult}(\mathcal{B})) \cong \mathcal{C}_{\mathrm{Ult}(\mathcal{B})} / \mathcal{M}_{\mathrm{Ult}(\mathcal{B})}$. Furthermore, if $(\mathcal{B}, \nu)$ is semifinite, then ν has a unique extension $\hat{\nu}$ to $\mathrm{RO}(\mathrm{Ult}(\mathcal{B}))$ such that $(\mathrm{RO}(\mathrm{Ult}(\mathcal{B})), \hat{\nu})$ is a localizable measure algebra.

Topology of a measure algebra

For a measure algebra $(\mathcal{B}, \nu)$, let $\mathcal{B}_{\mathrm{f}} = \mathcal{B}_{\mathrm{f}}(\nu) = \{a \in \mathcal{B}: \nu(a) < \infty\}$. Then each $a \in \mathcal{B}_{\mathrm{f}}$ defines a pseudometric by

$$\rho_a(b, c) = \nu\big(a \cdot (b \triangle c)\big).$$

Then the **measure algebra topology** on $\mathcal{B}$ is the topology generated by the family ρ_a $(a \in \mathcal{B}_{\mathrm{f}})$ of pseudometrics. Note that if $(\mathcal{B}, \nu)$ is totally finite, then ρ_1 is a complete metric and the measure algebra topology coincides with this metric topology. Moreover, measure algebras can be classified according to the topological properties of the corresponding measure algebra topology: The measure algebra topology is metrizable iff $(\mathcal{B}, \nu)$ is a σ-finite measure algebra; the topology is Hausdorff and complete iff $(\mathcal{B}, \nu)$ is localizable; and the topology is Hausdorff iff $(\mathcal{B}, \nu)$ is semifinite.

A measure algebra is a **separable measure algebra** if its topology is separable, and a measure space is *separable* if its measure algebra is. Notice that every separable semifinite measure is σ-finite.

Examples

The preeminent measure space is of course $(\mathbb{R}, \mathcal{E}, \mu)$ where μ is the Lebesgue measure and $\mathcal{E}$ is the algebra of Lebesgue measurable subsets of the real line. The Lebesgue measure space is complete, σ-finite and separable. An example of a noncomplete measure space is $(\mathbb{R}, \mathcal{B}, \mu \restriction \mathcal{B})$ where $\mathcal{B} \subseteq \mathcal{E}$ is the algebra of Borel subsets of $\mathbb{R}$.

A basic example is the measure space $(S, \mathcal{P}(S), \mu)$ where μ is the **counting measure** on an arbitrary set S: $\mu(E)$ is the cardinality of E when E is finite, and $\mu(E) = \infty$ otherwise. If S is uncountable then μ is strictly localizable and complete, but is not σ-finite. In this atypical case, $\mathcal{N}_\mu = \{\emptyset\}$, and thus the corresponding measure algebra is equivalent to the original counting measure space.

Liftings

Let $\mathcal{B} / \mathcal{I}$ be a quotient algebra. Then a **lifting** of $\mathcal{B} / \mathcal{I}$ is a Boolean homomorphism $f : \mathcal{B} / \mathcal{I} \to \mathcal{B}$ such that $[f(s)]_{\mathcal{I}} = s$ for all $s \in \mathcal{B} / \mathcal{I}$. In other words a lifting is a homomorphism which is a right inverse of the quotient map.

There is a natural correspondence between liftings f of $\mathcal{B} / \mathcal{I}$, and Boolean homomorphisms $g : \mathcal{B} \to \mathcal{B}$ such that $g(a) = 0$ for all $a \in \mathcal{I}$ and $a \triangle g(a) \in \mathcal{I}$ for all $a \in \mathcal{B}$. Given a lifting f, $a \mapsto f([a]_{\mathcal{I}})$ defines such a function g, and conversely, given such a function $g : \mathcal{B} \to \mathcal{B}$, $[a]_{\mathcal{I}} \mapsto g(a)$ is a well-defined lifting. Thus if $\mathcal{I}$ is nontrivial, one cannot obtain a lifting from the identity map on $\mathcal{B}$, and in general quotient algebras do not have liftings.

One of the fundamental theorems of measure theory is that the measure algebra of any 'interesting' complete measure space does have a lifting. This is known as the von Neumann–Maharam lifting theorem (see [11]).

THEOREM 1 (von Neumann–Maharam). *The measure algebra of a complete strictly localizable measure space has a lifting.*

The proof of the lifting theorem relies on repeated use of the axiom of choice, and in general there is no choice of lifting that distinguishes itself as a natural one. For example, none of the liftings of Lebesgue measure on the real line stands out as canonical. The one exceptional case is with Stone spaces. Indeed if $(\mathcal{B}, \bar{\nu})$ is a measure algebra and $(\mathrm{Ult}(\mathcal{B}), \mathcal{A}, \mu)$ is the measure space obtained from the Stone space of $\mathcal{B}$, then $f : \mathcal{A} / \mathcal{M} \to \mathcal{A}$ defined by $[C]_{\mathcal{M}} \mapsto C$ is a lifting of the measure algebra $(\mathcal{A} / \mathcal{M}, \bar{\mu}) \cong (\mathcal{B}, \bar{\nu})$ into $(\mathrm{Ult}(\mathcal{B}), \mathcal{A}, \mu)$.

In general, liftings associated with measure spaces can be described using Stone spaces. Suppose that $(\mathcal{B}, \bar{\nu})$ above is the measure algebra of some measure space $(X, \mathcal{E}, \nu)$. Then there is a natural correspondence between liftings f of $\mathcal{B} = \mathcal{E} / (\mathcal{N}_\nu \cap \mathcal{E})$, and functions $g : X \to \mathrm{Ult}(\mathcal{B})$ such that $g^{-1}[C] \in \mathcal{E}$ and $[g^{-1}[C]]_{\mathcal{N}_\nu} = C$ for all $C \in \mathcal{B}$, via $f(C) = g^{-1}[C]$ for each $C \in \mathcal{B}$. Furthermore, in cases where the lifting theorem applies, e.g., $(X, \mathcal{E}, \nu)$ is complete and locally determined, the conditions on g are equivalent to: g is inverse measure preserving and the homomorphism between $\mathcal{A} / \mathcal{M}$ and $\mathcal{B}$ (the measure algebras of μ and ν, respectively) induced by g^{-1} is, up to identification, the identity.

2. Topological measure theory

A **topological measure space** is a quadruple $(X, \tau, \mathcal{E}, \mu)$ where $(X, \mathcal{E}, \mu)$ is a measure space and (X, τ) is a topological space such that $\tau \subseteq \mathcal{E}$. Two properties relevant to topological measure spaces are of particular importance. One is inner regularity for compact sets, i.e., a measure space $(X, \mathcal{E}, \mu)$ is **inner regular** for some subfamily $\mathcal{F} \subseteq \mathcal{E}$ if

$$\mu(E) = \sup\{\mu(F)\colon E \supseteq F \in \mathcal{F}\}$$

for all $E \in \mathcal{E}$. The other property is τ-additivity, where a topological measure space $(X, \tau, \mathcal{E}, \mu)$ is **τ-additive** if for every family $\mathcal{G}$ of open sets which is upwards directed – i.e., for all $U, V \in \mathcal{G}$ there exists $W \in \mathcal{G}$ such that $U \cup V \subseteq W$ – $\mu(\bigcup \mathcal{G}) = \sup_{U \in \mathcal{G}} \mu(U)$. It is straightforward to prove that every topological measure space which is inner regular with respect to the compact sets is τ-additive. Also a **locally finite topological measure space** is one in which if every point has an open neighbourhood of finite measure.

The central concept in the interactions between measures and topologies is that of Radon measures.

DEFINITION 2. A topological measure space $(X, \tau, \mathcal{E}, \mu)$ is a **Radon measure space** if $(X, \mathcal{E}, \mu)$ is complete and locally determined, (X, τ) is Hausdorff, and μ is locally finite and inner regular with respect to the compact sets.

Abstract topological measure theory is a relatively new subject, and the terminology is still not fixed. The definition of **Radon measure** is commonly simplified by omitting the first clause, i.e., it is a locally finite Hausdorff topological measure space which is inner regular for compact sets. However, this is not much of a disagreement because for every Hausdorff topological measure space $(X, \tau, \mathcal{E}, \mu)$, μ is locally finite and inner regular for compact sets iff μ has an extension to what we have defined as a Radon measure. Furthermore, in this case the extension is unique and is the **complete locally determined (cld) version** of $(X, \mathcal{E}, \mu)$: Given any measure space $(X, \mathcal{E}, \mu)$, let $\mathcal{E}_{\mathrm{f}} = \mathcal{E}_{\mathrm{f}}(\mu) = \{E \in \mathcal{E}\colon \mu(E) < \infty\}$. Then the cld version of the measure space $(X, \mathcal{E}, \mu)$ is the complete locally determined measure space $(X, \widetilde{\mathcal{E}}, \tilde{\mu})$ where $\widetilde{\mathcal{E}}$ is the set of all subsets $F \subseteq X$ such that for every $E \in \mathcal{E}_{\mathrm{f}}$, there is a μ-null set N_E for which $(E \cap F) \triangle N_E \in \mathcal{E}$, and $\tilde{\mu}(F) = \sup_{E \in \mathcal{E}_{\mathrm{f}}} \mu((E \cap F) \triangle N_E)$. Note that $\tilde{\mu}$ extends μ iff μ is semifinite, and if μ is locally finite and inner regular for compact sets then μ is semifinite.

With regards to our previous classification scheme for measure spaces, we remark that it is a theorem that Radon measures are strictly localizable. And clearly every Radon measure on a compact space is totally finite.

As indicated in [4, Section 435], it is reasonable to say that the fundamental question of topological measure theory is: What kinds of measures can exist on what kinds of topological spaces?

We have mentioned that two of the most important properties of a topological measure space are inner regularity for compact sets and τ-additivity. However, it is a theorem that: Every totally finite τ-additive topological measure on a regular topological space is inner regular for closed sets. Thus inner regularity and τ-additivity coincide on a major proportion of the classes of topological measure spaces with a compact topology. And this can be generalized much further. A Hausdorff space X is a **K-analytic space** if it is the image $R[\mathbb{N}^{\mathbb{N}}]$ of an **usco-compact** relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$, i.e., $R[\{h\}]$ is compact for every $h \in \mathbb{N}^{\mathbb{N}}$, and $R^{-1}[F]$ is closed for every closed $F \subseteq X$. Note that every compact Hausdorff space is K-analytic. And: On a K-analytic space, every semifinite topological measure which is inner regular for closed sets is inner regular for compact sets. Thus for regular K-analytic spaces the two most important properties coincide for totally finite measures. We digress to state a very interesting fact: In a complete locally determined Hausdorff topological measure space $(X, \tau, \mathcal{E}, \mu)$ every K-analytic subspace of X is measurable (i.e., a member of $\mathcal{E}$).

This still leaves a large gap between the 'pathological' topological measure spaces which are not inner regular for compact sets, and the 'well behaved' ones which are. Obviously it is helpful to restrict the domain of a measure for obtaining inner regularity. A **Borel measure** μ on a topological space X is a topological measure with a minimum of measurable sets, i.e., the domain of μ is the algebra of Borel subsets of X. Recall that an **analytic space** is a space that is the continuous image of a ***Polish space***; or equivalently a continuous image of $\mathbb{N}^{\mathbb{N}}$, and thus analytic spaces are K-analytic. It is a theorem that: In an analytic Hausdorff space, every semifinite Borel measure is inner regular for compact sets. Moreover, as we have seen, any such Borel measure that is also locally finite can be extended to a Radon measure by taking the cld version.

While analytic spaces form an important class, one often wants to look further for spaces with no pathological measures. This leads to the following definition: a Hausdorff space X is a **Radon space** if every totally finite Borel measure on X is inner regular with respect to the compact sets.

Examples

Every Hausdorff analytic space is a Radon space. In particular, it follows that Lebesgue measure on $\mathbb{R}$, and more generally the Lebesgue measure on $\mathbb{R}^n$ for a positive integer n, is a Radon measure, because the restriction of Lebesgue measure to the Borel subsets of the unit cube $[0, 1]^n$ is a Borel probability measure whose cld version is the Lebesgue measure on $[0, 1]^n$.

In the order topology on ω_1, a set $E \subseteq \omega_1$ is Borel iff either there is a club contained in E, or a club contained in $\omega_1 \setminus E$. **Dieudonné's measure** μ is defined on the family of Borel sets by letting $\mu(E) = 1$ if E is in the club filter and $\mu(E) = 0$ otherwise. It is a complete topological probability measure and is inner regular for closed sets. Clearly μ is not τ-additive, and thus is not inner regular for compact sets. Thus ω_1 is not a Radon space, and one can use this to show that $\omega_1 + 1$ is also not a Radon space. For an index set I, the ***Cantor cube*** $\{0, 1\}^I$ with its product topology is a Radon space iff $[0, 1]^I$ is a Radon space iff I is countable. This can be proved using Dieudonné's measure, because $\omega_1 + 1$ embeds as a closed subset of $\{0, 1\}^I$ when I is uncountable.

Every hereditarily Lindelöf K-analytic space is Radon. In particular, Alexandroff's ***split interval*** $[0, 1] \times \{0, 1\}$ is a Radon space.

3. Topological groups

A **topological group** is a triple $(G, \tau, \cdot)$ where $(G, \cdot)$ is a group and τ is a topology on G such that the group operations $(a, b) \mapsto a \cdot b$ and $a \mapsto a^{-1}$ are both continuous. Note that, in this case, each of the maps $x \mapsto a \cdot x$ is a homeomorphism.

A **left Haar measure** on a topological group G is a nonzero Radon measure μ which is invariant for the left action of G on itself, i.e., $\mu(a \cdot E) = \mu(E)$. Similarly we define a **right Haar measure**, and a **Haar measure** refers to a measure which both a left and right Haar measure. Note that all three notions coincide for Abelian groups.

Every locally compact topological group has both left and right Haar measures; see [4, §441] or [5, Chapter XI]. If G is compact then every Haar measure on G must be totally finite (since they are Radon). In fact, a locally compact topological group has a totally finite left Haar measure iff it is compact; and it has a σ-finite left Haar measure iff it is σ-compact.

Furthermore, it is a theorem that Haar measures are unique up to a constant multiple: For every topological group G, every two left Haar measures on G are multiples of each other; and it is thus apparent why Haar measures dominate the theory of locally compact groups. Thus it is also a theorem, due to Haar, that: Compact groups have unique left Haar probability measures.

Examples
For any index set I, $\{0, 1\}^I$ endowed with the product topology is a compact topological group under coordinatewise addition modulo 2. $\mathbb{R}^n$ is a locally compact and σ-compact topological group with coordinatewise addition, and the Lebesgue measure is a Haar measure. The group of p-adic integers with the usual metric topology is an example of a σ-compact non-compactly generated topological group.

4. Maharam's Theorem

While the Stone space construction gives a canonical representation of a measure algebra as a measure space, surely the most important representation of a measure algebra is given by Maharam's Theorem [10]. A **simple product** of a family $\{(\mathcal{B}_i, \nu_i): i \in I\}$ of measure algebras is the product Boolean algebra $\prod_{i \in I} \mathcal{B}_i$ which is a measure algebra with the functional ν given by

$$\nu(a_i: i \in I) = \sum_{i \in I} \nu_i(a_i).$$

In a **simple weighted product** the measures ν_i are each multiplied by some nonnegative constant.

THEOREM 3 (Maharam). *A measure algebra $(\mathcal{B}, \mu)$ is localizable iff it is isomorphic as a measure algebra to a simple weighted product of a family of measure algebras $\{(\mathcal{B}_i, \mu_i): i \in I\}$ for some index set I such that each $(\mathcal{B}_i, \mu_i)$ is the measure algebra of the Haar probability measure on $\{0, 1\}^{\kappa_i}$ (with the usual topological group structure) where κ_i is either 0 or an infinite cardinal.*

REMARK 4. In the case of σ-finite measure algebras one obtains a sharper result: $(\mathcal{B}, \mu)$ is isomorphic to a simple weighted product $\mathcal{P}(S) \times \prod_{i \in I} \mathcal{B}_{\kappa_i}$, where S is countable with the counting measure, and the κ_i's are distinct infinite cardinals.

Maharam's Theorem is one of the fundamental results of measure theory. What makes it so important is that it gives an explicit description of an arbitrary measure algebra, allowing a wide variety of problems to be resolved.

A subalgebra $\mathcal{C}$ of a Boolean algebra $\mathcal{B}$ is **order closed** in $\mathcal{B}$ if $\sum \mathcal{D} \in \mathcal{C}$ whenever $\mathcal{D} \subseteq \mathcal{C}$ and $\sum \mathcal{D} \in \mathcal{B}$ exists. The **Maharam type** of a Boolean algebra $\mathcal{B}$ is the smallest cardinality of a subset $\mathcal{D} \subseteq \mathcal{B}$ such that the smallest order closed subalgebra containing $\mathcal{D}$ is $\mathcal{B}$ itself. For a measure algebra $(\mathcal{B}, \nu)$, the Maharam type of $\mathcal{B}$ is closely related to the topological density of $\mathcal{B}$ with its measure algebra topology. Indeed, if $(\mathcal{B}, \nu)$ is localizable and of infinite Maharam type then they are equal.

A Boolean algebra $\mathcal{B}$ is **Maharam homogeneous** if $\mathcal{B}_a = \{b \in \mathcal{B}: b \leqslant a\}$ has the same Maharam type as $\mathcal{B}$ for every nonzero $a \in \mathcal{B}$. Note that the Maharam type of a Maharam homogeneous Boolean algebra is either 0 or infinite. We have chosen to state Maharam's theorem as a characterization of localizability; however, one should also consider the theorem of Maharam which states that: Every Maharam homogeneous probability algebra $(\mathcal{B}, \mu)$ is isomorphic as a measure algebra to the measure algebra of the Haar probability measure on $\{0, 1\}^\kappa$ where κ is the Maharam type of $\mathcal{B}$. Thus, for example, the measure algebras of Lebesgue measure on the unit interval and the Haar probability measure on $\{0, 1\}^{\mathbb{N}}$ are isomorphic.

Continuous maps onto $[0, 1]^\kappa$
Having in mind Maharam's Theorem, given a Hausdorff space X one can define $\Gamma(X)$ to be the set of all cardinals κ for which there is a Radon probability measure on X whose measure algebra is isomorphic to the measure algebra of the Haar measure on $\{0, 1\}^\kappa$. For then from $\Gamma(X)$, we can determine exactly which measure algebras appear as the measure algebra of some localizable measure on X; namely, the infinite κ's in Maharam's Theorem are always from the set $\Gamma(X)$; moreover, for σ-finite measures we get an even sharper result according to remark 4. Since Radon measures are determined by their behaviour on compact sets, it makes sense to look at $\Gamma(X)$ for compact Hausdorff X; in fact, for any Hausdorff space X, $\Gamma(X) = \bigcup\{\Gamma(K): K \subseteq X$ is compact$\}$. It was shown some time ago by Haydon [6] that if there is a continuous map of X onto $[0, 1]^\kappa$ then $\kappa \in \Gamma(X)$, and he asked whether the converse is true. The question was answered positively by Fremlin in [1].

THEOREM 5 (Fremlin). *Assume that Martin's Axiom holds for families of κ many dense sets. Then for every compact*

Hausdorff space X, $\kappa \in \Gamma(X)$ iff X maps continuously onto $[0,1]^\kappa$.

In particular, assuming MA_{ω_1}, X has a nonseparable Radon measure iff X maps continuously onto $[0,1]^{\omega_1}$. Moreover, it is known that some additional set theoretic assumptions are necessary, since Haydon [7] showed that if the Continuum Hypothesis holds then there is a compact Hausdorff space not map continuously onto $[0,1]^{\omega_1}$, which also carries a nonseparable Radon probability measure; this result was adapted by Kunen [8] who used it to construct a compact ***L-space*** from CH, and it was strengthened by Kunen and van Mill [9] and Plebanek [12].

One should also note that if X is second-countable then every Radon measure on X is separable, because the family of equivalence classes of finite unions of members of some base for the topology on X is dense in the measure algebra topology.

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k-4 Measure Theory, II

In this part we consider supporting a strictly positive measure as a topological chain condition, and the role of topological measures in functional analysis. We deal with the study of compact subsets of topological vector spaces and necessary and/or sufficient conditions for the separability of every Radon measure on these subsets.

1. Chain conditions

The chain condition method is a very important tool for studying the role of countability in topology. A topological space has the **countable chain condition** (ccc) if every pairwise disjoint collection of opens sets is countable. Since it originated from the famous Souslin problem, it is also called **Souslin's condition**. A much stronger chain condition is ***separability*** (i.e., the existence of a countable dense set), and the strongest chain condition of all is ***second-countability*** (i.e., the existence of a countable base). Two other major chain conditions are **Shanin's condition** that every uncountable collection of open sets contains an uncountable subcollection with a nonempty intersection, and **property** K (also called **Knaster's condition**) that every uncountable collection of opens sets has an uncountable subcollection in which no two sets are disjoint. Note that these five properties are linearly ordered in strength: second-countable $\to$ separable $\to$ Shanin's condition $\to$ property K $\to$ ccc.

Examples
The ***Cantor set*** $\{0,1\}^{\mathbb{N}}$ is second-countable, and the ***Cantor cube*** $\{0,1\}^{\omega_1}$ is separable but not second-countable, while $\{0,1\}^{\mathfrak{c}^+}$ satisfies Shanin's condition but is nonseparable. The ***Souslin line*** has the ccc but does not satisfy property K. Let us take note here that the Stone space of a semifinite measure algebra has the ccc (not to be confused with the measure algebra topology) iff it is σ-finite, and thus the index set in Maharam's Theorem is countable iff the measure is σ-finite.

Here we are interested in one of the motivating forces behind the study of the chain condition method, namely the existence of **strictly positive measures**, topological measures for which every nonempty open set is of nonzero measure. More generally, the **support** of a topological measure space $(X,\tau,\mathcal{E},\mu)$ is a closed subset $F\subseteq X$ such that $\mu(F\cap U)>0$ whenever U is open and $F\cap U\neq\emptyset$, and $\mu(X\setminus F)=0$. (Thus μ is strictly positive iff X is a support.) Note that there is at most one support, and if μ is τ-additive then it has a support. More generally still, a topological space X **supports a strictly positive measure** if there is a totally finite measure $(X,\mathcal{E},\mu)$ such that $\mathcal{E}$ includes a π***-base*** $\mathcal{U}$ for X (i.e., a collection $\mathcal{U}$ of open sets such that for every nonempty open $U\subseteq X$ there is a nonempty $V\in\mathcal{U}$ included in U) with $\mu(V)>0$ for every nonempty $V\in\mathcal{U}$.

Supporting a strictly positive measure is in fact a chain condition whose strength falls naturally between separability and the property K. This can be seen by considering Kelly's **intersection numbers** [5]. The intersection number $I(\mathcal{G})$ of a family of sets $\mathcal{G}$ is given by

$$I(\mathcal{G})=\inf_s\frac{\max\{|F|:\ F\subseteq|s|,\ \bigcap_{i\in F}s(i)\neq\emptyset\}}{|s|}$$

where the infimum is taken over all sequences $s:n\to\mathcal{G}$ for some $n\geqslant 1$. For example, in a probability space $(X,\mathcal{E},\mu)$ one has $I(\{E\in\mathcal{E}\colon \mu(E)\geqslant\varepsilon\})\geqslant\varepsilon$ for all $0<\varepsilon\leqslant 1$; moreover, μ only needs to be finitely additive. Kelley's theorem states that a topological space X supports a strictly positive measure iff there is a countable decomposition of the family of nonempty open sets into pieces with positive intersection numbers.

The terminology "supports a strictly positive measure" is not standardized; for example, in some of the literature the measure is required to be Radon. However, for locally compact Hausdorff spaces these definitions coincide. Indeed in this case the proof of Kelley's theorem in [5] yields a strictly positive totally finite Radon measure.

The separability of a space (X,τ) is equivalent to the existence of a countable decomposition $\mathcal{G}_n$ $(n\in\mathbb{N})$ of $\tau\setminus\{\emptyset\}$ where $\bigcap\mathcal{G}_n\neq\emptyset$ for all n; hence, Kelly's theorem demonstrates why separability implies supporting a strictly positive measure. Also by Kelly's theorem, if X supports a strictly positive measure then it follows from the Dushnik–Miller relation $\omega\to(\omega_1,\omega)^2$ that X has property K. In fact, supporting a strictly positive measure entails finer intersection properties than property K. There are a number of results on this, including the following very general one [3] of Fremlin: If $(X,\mathcal{E},\mu)$ is a probability space, $\delta>0$ and $\mathcal{F}$ an uncountable subfamily of $\mathcal{E}$ consisting of sets of measure at least δ, then for every $\varepsilon>0$ and $k\geqslant 2$ there is an uncountable $\mathcal{G}\subseteq\mathcal{F}$ such that $\mu(E_0\cap\cdots\cap E_{k-1})\geqslant\delta^k-\varepsilon$ for all $E_0,\dots,E_{k-1}\in\mathcal{G}$.

Observe that a Haar measure on any topological group must be strictly positive: If U were a nonempty open set with $\mu(U)=0$ then $\mathcal{U}=\{\bigcup_{a\in F}a\cdot U\colon F\in[G]^{<\omega}\}$ is upward directed and hence by τ-additivity, $\mu(G)=\sup_{V\in\mathcal{U}}\mu(V)=0$. Thus we obtain the following theorem.

THEOREM 1. *Locally compact groups support strictly positive measures.*

It was proved by Tkachenko [10] that all σ-compact groups satisfy the property K. However, this was not proved by constructing a strictly positive measure, and could not have been as follows from the following more recent result of Todorčević [11]. The ***free Abelian topological group*** over the one-point compactification of a discrete space of size continuum does not support a strictly positive measure. Recall that the free Abelian group $G(X)$ over some set X consists of all irreducible formal sums $\sum_{i=0}^{n-1} z_i \cdot x_i$ where $n \in \mathbb{N}$, $z_i \in \mathbb{Z}$ and $x_i \in X$; and $G(X)$ has the minimal topology such that every continuous function from X into some Abelian topological group H extends to a continuous function from $G(X)$ into H (where X naturally identifies with a subspace of $G(X)$). Since X is a compact subspace of $G(X)$ whenever X is compact, the group in Todorčević's theorem is compactly generated, and in particular σ-compact.

2. Functional analysis

A considerable number of concepts and results in topological measure theory are either linked with or inspired by functional analysis. Adding more structure to topological groups, one arrives at the notion of a *topological vector space*; that is a pair (V, τ) where V is a vector space over some field $\mathbb{F}$ endowed with a topology ρ, and τ is a topology on V such that the vector operations $(u, v) \mapsto u + v : V \times V \to V$ and $(\lambda, u) \mapsto \lambda u : \mathbb{F} \times V \to V$ are continuous. Typically topological vector spaces are taken over the fields $\mathbb{R}$ or $\mathbb{C}$, and throughout this article the field $\mathbb{R}$ is always assumed.

Recall that a *linear functional* on a vector space V over the field $\mathbb{R}$ is a function $f : V \to \mathbb{R}$ satisfying linearity: $f(u + \lambda v) = f(u) + \lambda f(v)$. If $\mathcal{F}$ is a family of linear functionals on a vector space V (over $\mathbb{R}$), then for each $f \in \mathcal{F}$, $\|u\|_f = |f(u)|$ defines a seminorm on V, and this seminorm induces a pseudometric $d_f(u, v) = \|u - v\|_f$ on V. The family of pseudometrics $\{d_f\colon f \in \mathcal{F}\}$ defines a topology on V which renders V a topological vector space. We call this the *topology generated by* $\mathcal{F}$.

Let V be a topological vector space (over the reals). We let V^* denote the *dual* of V, the vector space under pointwise addition and multiplication by real constants of all continuous linear functionals from V into $\mathbb{R}$. The *weak topology* on V is the topology generated by V^*. The *weak* topology* on V^* is generated by the family $\{\hat{u}\colon u \in V\}$ of linear functionals on V^*, where $\hat{u}(f) = f(u)$ for all $f \in V^*$. The weak* topology on V^* is precisely the *topology of pointwise convergence*, i.e., the topology V^* inherits from the product topology on $\mathbb{R}^V$.

The most common topological vector spaces are *normed vector spaces*, i.e., a vector space X (over $\mathbb{R}$) with a norm $x \mapsto \|x\|$ on X. A normed vector space is a topological vector space with the metric topology induced by the norm: $d(x, y) = \|x - y\|$. And a normed vector space is called a *Banach space* if it is complete with respect to the norm metric. A linear functional f on X is *bounded* if

$$\|f\| = \sup\{|f(x)|\colon \|x\| \leqslant 1\} < \infty. \tag{1}$$

Recall that a linear functional on a normed vector space is bounded iff it is continuous. The weak topology on X coarser than the norm topology on X, and is strictly coarser whenever X is infinite dimensional. The operator defined in 1 is in fact a norm on X^*, and if X is a Banach space then so is X^* with this norm. Note that $\{\hat{x}\colon x \in X\}$ is a dense vector subspace of X^{**}, i.e., $X^{**} = (X^*)^*$ where X^* is given its norm topology. In particular, the weak* topology on X^* is coarser than the weak topology on X^* (the one generated by X^{**}), and is equal if X is *reflexive*, i.e., $\{\hat{x}\colon x \in X\} = X^{**}$.

Our first examples of topological vector spaces are the $L^0(\mu)$ spaces. For a given measure space $(X, \mathcal{E}, \mu)$ we let $\mathcal{L}^0(\mu)$ denote the family of all real-valued **$\mathcal{E}$-measurable function**s f on some conegligible subset of X, i.e., $f : C \to \mathbb{R}$ where $X \setminus C$ is null, and there is a conegligible $D \subseteq X$ such that for every open $U \subseteq \mathbb{R}$, $f^{-1}[U] \cap D \in \mathcal{E}$. Then $L^0(\mu)$ is the collection of equivalence classes of $\mathcal{L}^0(\mu)$ over the equivalence relation $f \sim g$ if $f = g$ *almost everywhere*, i.e., $f(x) = g(x)$ for conegligibly many $x \in X$. Note that $L^0(\mu)$ is a vector space over $\mathbb{R}$ where $[f] + [g] = [f + g]$ (pointwise addition) for all $f, g \in \mathcal{L}^0(\mu)$, and $c \cdot [f] = [c \cdot f]$ for all $c \in \mathbb{R}$.

For a measure space $(X, \mathcal{E}, \mu)$, a **μ-simple function** is a function of the form $\sum_{i=0}^{n-1} \lambda_i \chi_{E_i}$ where $\lambda_i \in \mathbb{R}$ and $E_i \in \mathcal{E}_{\mathrm{f}}(\mu)$ for all i; and an **$\mathcal{E}$-simple function** is of the form $\sum_{i=0}^{n-1} \lambda_i \chi_{E_i}$ where $\lambda_i \in \mathbb{R}$ and $E_i \in \mathcal{E}$ – simple functions are also called **step function**s. Recall that $\int f \,\mathrm{d}\mu$ is defined on the nonnegative $\mathcal{E}$-simple functions in the obvious manner (taking $0 \cdot \infty = 0$); for each $f \in \mathcal{L}^0(\mu)$ with $f \geqslant 0$, $\int f \,\mathrm{d}\mu = \sup\{\int g \,\mathrm{d}\mu\colon g$ is a nonnegative $\mathcal{E}$-simple function, $g \leqslant f$ a.e.$\}$; and $\int f \,\mathrm{d}\mu$ is determined on $\mathcal{L}^0(\mu)$ by integrating the positive and negative parts of f separately as in $\int f \,\mathrm{d}\mu = \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\mu$ (when both integrals are infinite $\int f \,\mathrm{d}\mu$ is undefined).

Every $F \in \mathcal{M}$ of finite measure defines a seminorm on $L^0(\mu)$ by

$$\|[f]\|_F = \int_F |f| \,\mathrm{d}\mu$$

for all $f \in \mathcal{L}^0(\mu)$. The standard topology on $L^0(\mu)$ is the **topology of convergence in measure** given by the associated pseudometrics $d_F([f], [g]) = \|[f] - [g]\|_F$ $(F \in \mathcal{E}_{\mathrm{f}})$, and with this topology $L^0(\mu)$ is a topological vector space. One also speaks of the topology of convergence in measure on $\mathcal{L}^0(\mu)$, given by the pseudometrics $d'_F(f, g) = \int_F |f - g| \,\mathrm{d}\mu$. Thus the topology on $L^0(\mu)$ is the quotient of the topology on $\mathcal{L}^0(\mu)$. This topology on $L^0(\mu)$ is very closely related to the measure algebra topology of μ. For example, the topology of convergence in measure is Hausdorff iff μ is semifinite iff the measure algebra of μ is semifinite iff the topology of the measure algebra is Hausdorff; the topology of convergence in measure is metrizable iff the topology of the measure algebra of μ is metrizable; and it is Hausdorff and complete iff the measure algebra topology is. Moreover, $[E] \mapsto [\chi_E]$ $(E \in \mathcal{E})$ is a well-defined embedding of the measure algebra of μ into $L^0(\mu)$.

There is an important relationship between the topology of convergence in measure and pointwise convergence. If a sequence $(f_n\colon n \in \mathbb{N})$ in $\mathcal{L}^0(\mu)$ converges pointwise to $f \in \mathcal{L}^0(\mu)$ almost everywhere then $([f_n]\colon n \in \mathbb{N})$ converges to $[f]$ in measure. Moreover, in the case where μ is σ-finite the topology of convergence in measure is metrizable and thus can be described in terms of sequences. In this case, $([f_n]\colon n \in \mathbb{N}) \to [f]$ in measure iff every subsequence $([f_{n_k}]\colon k)$ has a sub-subsequence $([f_{n_{k_i}}]\colon i)$ such that $f_{n_{k_i}} \to f$ almost everywhere.

Perhaps the most important spaces in functional analysis are the $L^p(\mu)$'s for $1 \leqslant p \leqslant \infty$. For a given measure space $(X, \mathcal{E}, \mu)$, for $1 \leqslant p < \infty$, $L^p(\mu) \subseteq L^0(\mu)$ is the subset of all equivalence classes $[f]$ such that

$$\big\|[f]\big\|_p = \left(\int |f|^p \,\mathrm{d}\mu\right)^{1/p} < \infty. \tag{2}$$

This defines a norm on $L^p(\mu)$ for which it is a Banach space. For $1 < p < \infty$, setting $q = p\,/\,(p-1)$, the map $\Psi : L^p(\mu) \to L^q(\mu)^*$ defined by

$$\Psi\big([f]\big)\big([g]\big) = \int f \cdot g \,\mathrm{d}\mu \tag{3}$$

is an ***isometric*** (i.e., norm preserving) ***isomorphism*** (i.e., continuous linear bijection). In particular, by considering the isomorphisms between $L^p(\mu)$ and $L^q(\mu)^*$ and between $L^q(\mu)$ and $L^p(\mu)^*$ one concludes that $L^p(\mu)$ is reflexive. Now for the case $p = \infty$. For $[f] \in L^0(\mu)$,

$$\big\|[f]\big\|_\infty = \inf\big\{\alpha \in [0,\infty]\colon f^{-1}\big[[\alpha,\infty)\big] \text{ is null}\big\} \tag{4}$$

is well-defined, $L^\infty(\mu) \subseteq L^0(\mu)$ is the set of all $[f]$ such that $\|[f]\|_\infty < \infty$, and $\|\cdot\|_\infty$ defines a norm on $L^\infty(\mu)$ which makes it a Banach space. The map $\Psi : L^\infty(\mu) \to L^1(\mu)^*$ defined as in 3 is a isometric isomorphism iff $(X, \mathcal{E}, \mu)$ is localizable. The duality is of great importance in functional analysis, demonstrating the value of isolating the property of localizability. Let us also point out that $L^1(\mu)$ is not reflexive except in the trivial case where $L^1(\mu)$ is finite-dimensional.

It is clear that the family of all equivalence classes of μ-simple functions is a dense subspace of the Banach space $L^p(\mu)$ for $1 \leqslant p < \infty$, and the family of all equivalence classes of $\mathcal{E}$-simple functions is dense in $L^\infty(\mu)$. An important consequence is that, for localizable measures, Maharam's Theorem gives a complete description up to Banach space isomorphisms of the spaces $L^p(\mu)$ for $1 \leqslant p < \infty$, and also for $p = \infty$ for totally finite measures. In particular, a measure space is separable iff $L^p(\mu)$ is a separable Banach space.

There is one more example which should be introduced here. For a topological space X, let $C(X)$ be the family of all continuous real-valued functions on X. Then $C(X)$ is a vector space (with the usual operations). We let $C^*(X)$ (not to be confused with $C(X)^*$) denote the vector subspace of bounded continuous functions on X. There is a natural norm on $C^*(X)$, the **uniform norm** given by

$$\|f\|_u = \sup_{x \in X}\big|f(x)\big| \tag{5}$$

for all $f \in C^*(X)$, for which $C^*(X)$ is a Banach space. If $(X, \mathcal{E}, \mu)$ is a measure space where $\mathcal{E}$ includes the Baire algebra of X (e.g., if μ is a topological measure), then $C(X) \subseteq \mathcal{L}^0(\mu)$ – recall that the **Baire σ-algebra** of X is the smallest σ-algebra for which every continuous real-valued function on X is measurable and its elements are the **Baire sets**. If moreover μ is strictly positive, then it is clear that the map $f \mapsto [f]$ is injective, and thus $C(X)$ identifies with a subspace of $L^0(\mu)$. Also in this case, $C^*(X)$ identifies with a complete subspace of $L^\infty(\mu)$ and the uniform norm agrees with the L^∞ norm, i.e., $\|f\|_u = \|[f]\|_\infty$.

We also consider $C_0(X) \subseteq C^*(X)$, the set of all **continuous functions vanishing at infinity**, i.e., for all $\varepsilon > 0$ there exists a compact $K \subseteq X$ such that $|f(x)| < \varepsilon$ when $x \notin K$; it is a Banach subspace of $C^*(X)$. And $C_{00}(X) \subseteq C_0(X)$ denotes the vector subspace of all **functions with compact support**, i.e., there is a compact $K \subseteq X$ such that $f(x) = 0$ for all $x \notin K$. In this context we are most interested in locally compact spaces X, because there are many continuous functions with compact support; for example, the characteristic function of any compact subset of X extends to a member of $C_{00}(X)$. In general, $C_{00}(X)$ is not complete; indeed, (for any X) its closure in the uniform norm metric is $C_0(X)$.

Representations of linear functionals

One of the most important subjects in topological measure theory is the representation of linear functionals by Radon measures. It is an important link between measure theory and functional analysis, not to mention an important tool for constructing measures.

The majority of introductions to measure theory begin with the construction of Lebesgue measure on the real line from an outer measure on the real line, and then the theory of integration with respect to a measure (e.g., the Lebesgue integral) is developed from the simple functions through various convergence theorems. All of this can be done with no mention of topology. However, equally important to the construction of integrals from measures, is the construction of measures from integrals or more generally the construction of measures from linear functionals. And clearly integration with respect to some measure space $(X, \mathcal{E}, \mu)$ gives a linear functional on the vector space $L^1(\mu)$.

A linear functional F on $C(X)$ is *positive* if $F(f) \geqslant 0$ whenever $f \geqslant 0$. Considering a linear functional F on the subspace $C_0(X)$, note that if F is positive then it follows from the completeness of $C_0(X)$ that F must be bounded and thus continuous. The main theorem for identifying measures with positive linear functionals is the following.

THEOREM 2 (Riesz Representation Theorem). *Let X be a locally compact Hausdorff space. If $F : C_0(X) \to \mathbb{R}$ is a pos-*

itive linear functional, then there is a unique Radon measure μ on X such that $F(f) = \int f \, \mathrm{d}\mu$ for every $f \in C_0(X)$; moreover, μ is totally finite.

The measure in the theorem is given by

$$\mu(E) = \sup\{F(f)\colon f \in C_{00}(X),\ 0 \leqslant f \leqslant \chi_E\} \qquad (6)$$

for every $E \in \mathrm{dom}(\mu)$. Note that $\mu(X) \leqslant \|F\| < \infty$. Thus we obtain a complete description of the positive linear functionals on $C_0(X)$ for a locally compact space X: Every positive linear functional on $C_0(X)$ is given by integrating against some totally finite Radon measure.

A positive linear functional on $C_{00}(X)$ need not be continuous as $C_{00}(X)$ is not in general complete. There is another version of the Riesz Representation Theorem for positive linear functionals on $C_{00}(X)$, which is the same as theorem 2 except that μ need not be totally finite.

Let $M(X)$ denote the set of all totally finite Radon measures on X. By the Riesz Representation Theorem, the map $\mu \mapsto \int \cdot \, \mathrm{d}\mu$ is a bijection between $M(X)$ and the family of positive linear functionals on $C_0(X)$. This identification is used to give $M(X)$ the topology it inherits from $C_0(X)^*$ with the weak* topology, sometimes called the **vague topology** on $M(X)$. Furthermore, note that $P(X) \subseteq M(X)$, the space of Radon probability measures on X, is a compact subspace of $C_0(X)^*$. One can see this by noting that $P(X)$ is closed, and applying Alaoglu's theorem that for any normed vector space X, the closed unit ball $B_{X^*} = \{f \in X^*\colon \|f\| \leqslant 1\}$ is weak* compact in X^*. Since $P_\alpha(X) = \{\mu \in M(X)\colon \mu(X) \leqslant \alpha\}$ is similarly compact for all $0 < \alpha < \infty$, we see that the space of positive linear functionals on a locally compact Hausdorff space X is a σ-compact subspace of $C_0(X)^*$.

Center of mass

Recall that a (finite) **convex combination** of a subset C of some vector space (over $\mathbb{R}$) is a vector of the form $\lambda_0 u_0 + \cdots + \lambda_{n-1} u_{n-1}$ where each $u_i \in C$, each $\lambda_i \in [0, 1]$ and $\sum_{i=0}^{n-1} \lambda_i = 1$. A subset C of a vector space is a **convex set** if $\lambda u + (1 - \lambda)v \in C$ for all $u, v \in C$ and $\lambda \in [0, 1]$; equivalently if C is closed under convex combinations. The set of all convex combinations of elements of a set is its **convex hull**. A vector u is an **extreme point** of C if it is not a convex combination of two members of $C \setminus \{u\}$. A topological vector space is **locally convex** if its topology has a base consisting of convex sets. It is a theorem that a topological vector space is locally convex iff its topology is generated by a family of seminorms. Thus, for example, $L^0(\mu)$ with the topology of convergence in measure is locally convex.

Observe that every locally compact Hausdorff space X is homeomorphic via the **Dirac measure**s $x \mapsto \delta_x$, to the set $\mathrm{ext}(P(X))$ of all extreme points of $P(X)$ (i.e., $\delta_x(E)$ equals 1 if $x \in E$ and equals 0 otherwise). For any Hausdorff space, the Dirac measures are Radon measures. And δ_x is an extreme point of $P(X)$ because its support has cardinality one, and conversely, since X is Hausdorff, any topological measure with support of cardinality at least two cannot be an extreme point of $P(X)$. Finally, the injection is continuous, because for all $x \in X$, $f \in C(X)$ and $\varepsilon > 0$, there is an open subset $U \ni x$ of X such that $f[U] \subseteq (f(x) - \varepsilon / 2, f(x) + \varepsilon / 2)$, so that $\|\delta_x - \delta_y\|_{\hat{f}} < \varepsilon$ for all $y \in U$.

Measures on locally convex topological vector spaces are a central theme in functional analysis, and one of the central concepts is that of the 'center of mass' of a measure:

DEFINITION 3. Let X be a locally convex Hausdorff topological vector space. For a probability measure μ on a subset $A \subseteq X$ such that $X^* \subseteq \mathcal{L}^0(\mu)$ a point $x \in X$ is the **barycenter** of μ if $\int_A f \, \mathrm{d}\mu = f(x)$ for all $f \in X^*$. Note that μ has at most one barycenter because X is Hausdorff and locally convex, and by the Hahn–Banach Theorem for topological vector spaces.

EXAMPLE. Every member of $(\mathbb{R}^2)^*$ is of the form $(x, y) \mapsto cx + dy$. For a strictly decreasing function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$, let $R(f) \subseteq \mathbb{R}^2$ denote the region bounded by $x = 1$, $y = 0$ and f. When $R(f)$ has finite area $A = \int_1^\infty f(x)\, \mathrm{d}x$, the probability measure $\nu = A^{-1} \cdot \mu$ on $R(f)$ has barycenter $A^{-1}\big(\int_1^\infty x f(x)\, \mathrm{d}x, \frac{1}{2}\int_1^\infty f(x)^2\, \mathrm{d}x\big)$ when it exists. Thus $R(x^{-\alpha})$ has a barycenter iff $\alpha > 2$.

The following two results describe the existence of a barycenter for a given measure, and the existence of a measure with a given barycenter, respectively, for any locally convex Hausdorff topological vector space X (see, e.g., [6]).

(1) For every compact convex set $K \subseteq X$, every topological probability measure on K has a barycenter in X belonging to K.
(2) For every compact $K \subseteq X$ the closure of the convex hull of K is equal to the set of barycenters of Radon probability measures on K.

3. Compact subsets of function spaces

Another major theme in functional analysis is compact sets of measurable functions. The **Eberlein compact**a form a broad class of compacta, consisting of spaces which are homeomorphic to a weakly compact (i.e., compact in the weak topology) subset of some Banach space. The Eberlein compacta possess many nice properties. For example, it is a theorem of Rosenthal that all of the chain conditions are equivalent (see [8, 9]): Every Eberlein compactum satisfying the ccc is second countable.

The requirement that a topological measure space $(X, \tau, \mathcal{E}, \mu)$ has separable support, is an important variation of the restrictive hypothesis that $(X, \mathcal{E}, \mu)$ is separable. Indeed if X is completely regular (e.g., if X is Hausdorff and locally compact) and μ is totally finite and has a support, then the separability of the support is entailed by the separability of the measure space. This is because when X is completely regular, if X nonseparable then so is $C^*(X)$. And if

$(X, \mathcal{E}, \mu)$ is separable, then so is the strictly positive measure $\nu = \mu \upharpoonright (\mathcal{E} \cap \mathcal{P}(\mathrm{supp}(\mu)))$ on $\mathrm{supp}(\mu)$. Thus, the Banach space $L^\infty(\nu)$ is separable and therefore, being a metric space, second-countable; hence, so is $C^*(\mathrm{supp}(\mu))$ since it embeds into $L^\infty(\nu)$.

A more general class of compacta still possessing many pleasant properties is the **Corson compacta**, spaces which are homeomorphic to a compact subset of some ***Σ-product*** $\Sigma(\mathbb{R}^I)$ of the real line. Note that on a Corson compactum, and thus on an Eberlein compactum, the two properties considered in the preceding paragraph coincide for Radon measures because $\mathbb{R}^\mathbb{N}$ is second-countable. By Rosenthal's theorem, every Radon measure on an Eberlein compactum has separable support, and thus every Radon measure on an Eberlein compactum is separable. For the Corson compacta additional hypotheses are required: Every Radon measure on a closed convex subset of some $\Sigma([0,1]^I)$ is separable.

It is instructive to see how this is proved: Assume that $K \subseteq \Sigma([0,1]^I)$ is convex and closed. If μ is a Radon measure on K, then we may assume that μ is a probability measure since it is totally finite. Then μ has a barycenter $x_\mu \in K$ (since K is in fact compact). Since x_μ is in the Σ-product, we finish by showing that $\mathrm{supp}(\mu) \subseteq \{x\colon x(i) = 0$ whenever $x_\mu(i) = 0\}$. If this were not the case, then for some $j \in I$ and $\varepsilon > 0$, $x_\mu(j) = 0$ and $\mu(U \cap K) > 0$ where $U = \{x\colon x(j) > \varepsilon\}$. But the projection map π_j is a linear functional on $\mathbb{R}^I$, giving the contradiction $0 = x_\mu(j) = \pi_j(x_\mu) = \int_K \pi_j \,\mathrm{d}\mu \geqslant \varepsilon \cdot \mu(U \cap K) > 0$.

This theorem cannot be strengthened to arbitrary convex Corson compacta – spaces homeomorphic to a compact convex subset of some $\Sigma(\mathbb{R}^I)$. While this is consistent with ZFC, it was proved very recently [7] that assuming the Continuum Hypothesis, there is a compact convex subset of $\Sigma(\mathbb{R}^{\omega_1})$ with a nonseparable Radon measure.

Notice that the converse of this theorem is also true, i.e., for every compact $K \subseteq \Sigma(\mathbb{R}^I)$, if every Radon probability measure on K has separable support then the closure of the convex hull of K is contained in the Σ-product. For if $x \in \overline{\mathrm{conv}}(K)$, then there is a $\mu \in P(K)$ with barycenter x. Assuming $\mathrm{supp}(\mu)$ is separable, there is a countable $A \subseteq I$ such that $\mathrm{supp}(\mu) \subseteq \{y\colon y(i) = 0$ for all $i \notin A\}$. Thus for all $i \notin A$, $x(i) = \int_K \pi_i \,\mathrm{d}\mu = 0$.

In fact we have the following characterization from [2]:

THEOREM 4 (Argyros–Mercourakis–Negrepontis). *The following are equivalent for every Corson compactum K.*

(1) *Every Radon measure on K is separable.*
(2) *The Banach space C(K) is* ***weakly Lindelöf***.

To prove the theorem we need a general fact about normed vector spaces X: It clearly follows from the Hahn–Banach Theorem for normed vector spaces that $u \mapsto \hat{u} \upharpoonright B_{X^*}$ is an isometric isomorphism between X with its weak topology and its image in $C(B_{X^*})$ with its pointwise topology. Moreover, if X is a Banach space then its image is closed. In our case $X = C(K)$, $P(K)$ is the 'essential part' of $B_{C(K)^*}$ – indeed $B_{C(K)^*}$ is the image of $(P(K) \times [0,1])^2$ under the map $((\mu,\alpha),(\nu,\beta)) \mapsto \alpha\cdot\mu - \beta\cdot\nu$ – and the preceding holds true replacing $B_{C(K)^*}$ with $P(K)$. Hence $C(K)$ with its weak topology identifies with a closed subspace of $C(P(K))$ with its pointwise topology.

The implication from (a) to (b) in theorem 4 now follows from the well-known result of Alster and Pol [1] and Gul'ko [4] asserting that $C(K)$ is Lindelöf in its pointwise topology for every Corson compactum K, and the following fact. If K is a Corson compactum and every Radon probability measure on K has separable support, then $P(K)$ is a Corson compactum.

To sketch a proof assume that $K \subseteq \Sigma(\mathbb{R}^I)$, and that every Radon probability measure on K is separable. Let $\mathcal{Q}$ be the set of all rational open intervals which do not contain 0, and put $J = \bigcup_{z \in [I]^{<\omega}} \mathcal{Q}^z$. For each $s \in J$, choose a sequence $(f_n^s\colon n \in \mathbb{N})$ of functions in $C(K)^*$ with $0 \leqslant f_n^s \leqslant \chi_{\{x\colon x(i) \in s(i) \text{ for all } i \in \mathrm{dom}(s)\}}$ such that $\sup_{n\in\mathbb{N}} \int f_n^s \,\mathrm{d}\mu$ is equal to $\mu(\{x\colon x(i) \in s(i)$ for all $i \in \mathrm{dom}(s)\})$ (see (6)). Put $H = \bigcup_{s \in J}\{f_n^s\colon n \in \mathbb{N}\}$. Define $h\colon P(K) \to \mathbb{R}^H$ by

$$h(\mu)(f) = \int f \,\mathrm{d}\mu \quad (f \in H).$$

Then clearly h is continuous, and it is injective because μ is completely determined by the information in $h(\mu)$. Excluding 0 from the intervals in $\mathcal{Q}$ ensures that the image of h is in the Σ-product.

The implication from (b) to (a) in theorem 4 is in fact valid for any compact Hausdorff K: For suppose μ is a Radon measure on K. Then applying weak Lindelöf property of $C(K)$ to the collection of sets

$$C_x^n = \left\{f \in C(K)\colon f(x) = 0,\ \int f \,\mathrm{d}\mu \geqslant (n+1)^{-1}\right\}$$

(where $x \in \mathrm{supp}(\mu)$ and $n \in \mathbb{N}$), we obtain for each n, a countable $D_n \subseteq \mathrm{supp}(K)$ such that $\bigcap_{x \in D_n} C_x^n = \emptyset$. And then by Urysohn's Lemma, $\bigcup_{n=0}^\infty D_n$ is a dense subset of $\mathrm{supp}(K)$.

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k-5 Polyhedra and Complexes

Polyhedra are special topological spaces determined by geometric objects called complexes. Complexes are formed by simplices, which are certain subsets of real vector spaces V. To define simplices, one first defines **geometrically independent** finite sets of points as sets $\{v_0, \ldots, v_n\}$ for which the vectors $v_1 - v_0, \ldots, v_n - v_0 \in V$ are linearly independent or, equivalently, the equations

$$\sum_{i=0}^{i=n} \lambda_i = 0, \quad \sum_{i=0}^{i=n} \lambda_i v_i = 0, \quad \lambda_i \in \mathbb{R}, \tag{1}$$

imply $\lambda_0 = 0, \ldots, \lambda_n = 0$. An n**-dimensional geometric simplex** (or simply an n**-simplex**), spanned by a geometrically independent set of points $\{v_0, \ldots, v_n\}$ from V and denoted by $\sigma = [v_0, \ldots, v_n]$, is the ***convex hull*** of that set. The points $\{v_0, \ldots, v_n\}$ are completely determined by σ and are the **vertices** of σ. Clearly, $[v_0, \ldots, v_n]$ consists of all points $x \in V$ of the form

$$x = \sum_{i=0}^{i=n} \lambda_i v_i, \tag{2}$$

where $\lambda_i \geqslant 0$ and $\sum_{i=0}^{i=n} \lambda_i = 1$. Because of (1), the real numbers $\lambda_i \in [0, 1]$ in (2) are completely determined by x and are called **barycentric coordinate**s of x. The convex hull of a (proper) subset of the set of vertices of σ is again a simplex, called a (proper) **face** of σ.

A **geometric simplicial complex** is a set K (finite or infinite) of simplices, contained in a real vector space V, such that every face τ of a simplex $\sigma \in K$ also belongs to K and the intersection $\sigma_1 \cap \sigma_2$ of two simplices of K is a face τ of both of them. All proper faces of a simplex σ form a complex $\partial\sigma$, called the **boundary** of σ. For vertices $\partial v = \emptyset$. The union of σ and $\partial\sigma$ is also a complex, denoted again by σ. A **subcomplex** L of a complex K is a subset $L \subseteq K$, which is itself a complex. Unions and intersections of arbitrary collections of subcomplexes of a complex K are again subcomplexes of K. The subcomplex $K^n \subseteq K$, $n \geqslant 0$, formed by all simplices $\sigma \in K$ of dimension $\dim\sigma \leqslant n$, is the n**-skeleton** (or n**-dimensional skeleton**) of K. In particular, K^0 is the set of **vertices** of K. Clearly, $K^0 \subseteq K^1 \subseteq \cdots$. The **carrier** $|K|$ of a complex K is the union of all simplices from K and thus, $|K| = \bigcup_{n\geqslant 0} |K^n|$. If L is a subcomplex of K, then $|L| \subseteq |K|$. If $\sigma \in K$, then the set $\sigma^\circ = \sigma \backslash \partial\sigma \subseteq |K|$ is the **interior** of the simplex σ. For vertices $v^\circ = v$. If $v \in K^0$ and $x \in |K|$ belongs to a simplex $\sigma = [v_0, \ldots, v_n]$ such that v is one of its vertices v_i, then the **barycentric coordinate** $\lambda_v(x) = \lambda_i$ is defined by (2). One extends this definition to all remaining vertices $v \in K^0$ by putting $\lambda_v(x) = 0$. In this way, for every vertex v of K, $\lambda_v : |K| \to [0, 1]$ is a well-defined function. A **subdivision** of a complex K is a geometric simplicial complex M, which has the property that every simplex $\tau \in M$ is contained in some simplex $\sigma \in K$ and every $\sigma \in K$ is the union of a finite collection of simplices $\tau \in M$. Clearly, $|K| = |M|$. An important example is the **barycentric subdivision** K' of K. Its vertices are the barycenters $b_\sigma = \frac{1}{n}(v_0 + \cdots + v_n)$ of all simplices $\sigma = [v_0, \ldots, v_n]$ of K. Barycenters $b_{\sigma_0}, \ldots, b_{\sigma_k}$ span a k-simplex of K' provided they can be labeled in such a way that $\sigma_0 \subseteq \cdots \subseteq \sigma_k$.

Let K and L be geometric simplicial complexes and let $f : K^0 \to L^0$ be a map having the property that, whenever the vertices $v_0, \ldots, v_n$ span a simplex of K, then the vertices $f(v_0), \ldots, f(v_n)$ span a simplex of L. The **geometric simplicial map** induced by f is the map $|f| : |K| \to |L|$ defined by the formula

$$f(x) = \sum_{i=0}^{i=n} \lambda_i f(v_i), \tag{3}$$

where $x \in [v_0, \ldots, v_n] \subseteq |K|$ is given by (2).

On the carrier $|K|$ of a geometric simplicial complex K one can define a ***metric*** $d = d_K$ using the formula

$$d(x, y) = \sum_{v\in K^0} \left|\lambda_v(x) - \lambda_v(y)\right| \tag{4}$$

[8, Chapter III.9]. The ***topology*** induced by this metric is called **strong topology** (or **metric topology**) and the corresponding space is denoted by $|K|_m$. Barycentric coordinates $\lambda_v : |K|_m \to [0, 1]$ are ***continuous*** functions. Consequently, the **stars** $\mathrm{St}(v, K) = (\lambda_v)^{-1}(0, 1]$ are open subsets of $|K|_m$. More general, for every simplex $\sigma \in K$, the **star** $\mathrm{St}(\sigma, K)$, defined as the union of the interiors of all simplices having σ for a face, is also an open set in $|K|_m$. For a subcomplex $L \subseteq K$, $|L|$ is always a closed subset of $|K|_m$. If $V = \mathbb{R}^m$ is a Euclidean space, $|K| \subseteq V$ and K^0 is ***discrete*** in V, then the strong topology of $|K|$ is just the Euclidean topology inherited from V. In particular, the metric topology of a simplex always coincides with its Euclidean topology. A map $f : X \to |K|_m$, defined on a topological space X, is continuous if and only if, for every vertex v of K, the composition $\lambda_v f : X \to [0, 1]$ is continuous. Geometric simplicial maps $|f| : |K| \to |L|$ are continuous with respect to the metric topologies of $|K|$ and $|L|$.

If M is a subdivision of K, the strong topologies on $|M| = |K|$, induced by the metrics d_M and d_K, in general differ. E.g., if K is the cone with basis $\{v_1, v_2, \ldots\}$ and vertex v_0 and $w_i = (1 - t_i)v_0 + t_i v_i$, $i = 1, 2, \ldots,$ where

$0 < t_i \leqslant 1$, $t_i \to 0$, then in $|K|_m$ the sequence w_i converges to v_0. If M is the subdivision obtained from K by adding the points w_i as new vertices, the distance $d_M(w_i, v_0) = 2$ and thus, w_i cannot converge towards v_0 in $|M|_m$. However, in the case of the barycentric subdivision, $|K'|_m = |K|_m$ (see [10, Theorem 9.10], [7, Corollary V.6] or [12, Appendix 1.1.3, Theorem 13]). Using the second barycentric subdivision K'' of K and the fact that also $|K''|_m = |K|_m$, it is easy to prove that, for an arbitrary subcomplex L of K, $|L|$ is a **retract** of some neighbourhood U of $|L|$ in $|K|_m$. This means that there exists a **retraction** $r : U \to |L|$, i.e., a map such that the restriction of r to $|L|$ is the identity map on $|L|$. In fact, $U = \mathrm{St}(|L|, K'') = \bigcup_{\tau \in L} \mathrm{St}(\tau, K'')$ has the desired property. D. W. Henderson proved that, for every complex K and open cover $\mathcal{U}$ of $|K|_m$, there exists a subdivision M of K such that $|M|_m = |K|_m$ and the simplices of M refine $\mathcal{U}$, i.e., every simplex $\tau \in M$ is contained in some $U \in \mathcal{U}$ (see [7, Lemma V.7]).

A **simplicial approximation** of a map $g : |K| \to |L|$ is a geometric simplicial map $|f| : |K| \to |L|$, which is a **modification** of g, i.e., has the property that, whenever for a point $x \in |K|$, $g(x)$ belongs to a simplex $\tau \in L$, then also $f(x) \in \tau$. If $g : |K|_m \to |L|_m$ is continuous, then $|f| : |K|_m \to |L|_m$ is homotopic to g, because the map $H : |K| \times [0, 1] \to |L|$, defined by $H(x, t) = (1 - t)g(x) + t|f|(x)$, is a homotopy which connects g to $|f|$. Using Henderson's subdivision theorem one can show that every continuous map $g : |K|_m \to |L|_m$ admits a subdivision M of K and a simplicial approximation $|f| : |M| \to |L|$ of g.

For a geometric simplicial complex K, $|K|_m$ is ***compact*** if and only if K is finite. $|K|_m$ is ***locally compact*** if and only if K is a **locally finite complex**, i.e., every vertex $v \in K^0$ belongs only to a finite number of simplices of K. The space $|K|_m$ has a countable ***base*** if and only if K is countable. For an arbitrary complex K, $|K|_m$ is ***locally contractible***. In fact, $|K|_m$ is always an **absolute neighborhood retract** (ANR) (for ***metrizable space***s), i.e., whenever $|K|_m$ is embedded as a closed set in a metrizable space X, then there exists a neighbourhood of $|K|_m$ in X which retracts to $|K|_m$ (see [12, Appendix 1.1.3, Theorem 11] or [6, Theorem 3.3.10]). This fact can be derived from a theorem of J. Dugundji, which asserts that every convex subset C of a normed vector space (or more generally, of a locally convex topological vector space) is an **absolute extensor** (AE) for metrizable spaces, i.e., for every closed subset A of a metrizable space X, every map $f : A \to C$ admits an extension $\tilde{f} : X \to C$ to all of X (see [12, I.3.1., Theorem 3]).

The carrier $|K|$ of a simplicial complex K can also be endowed with the **weak topology** yielding thus a space denoted by $|K|_w$. By definition, in this topology every simplex $\sigma \in K$ has the Euclidean topology and a set $U \subseteq |K|$ is open provided, for every simplex $\sigma \in K$, the intersection $U \cap \sigma$ is open in σ. A map $f : |K|_w \to Y$ is continuous if and only if, for every $\sigma \in K$, the restriction $f|\sigma$ is continuous. In the weak topology too, the barycentric coordinates $\lambda_v : |K|_w \to [0, 1]$ are continuous maps and the stars $\mathrm{St}(v, K)$ are open sets. Moreover, the carrier $|L|$ of every subcomplex $L \subseteq K$ is closed in $|K|_w$ and $|L|$ is a neighbourhood retract of $|K|_w$. Furthermore, for an arbitrary subdivision M of K, the weak topology of $|M|$ is the topology inherited from $|K|_w$ and every open cover $\mathcal{U}$ of $|K|_w$ admits a subdivision M of K such that the simplices of M refine $\mathcal{U}$. Geometric simplicial maps $|f| : |K| \to |L|$ are continuous with respect to the weak topologies of $|K|$ and $|L|$. Every continuous map $g : |K|_w \to |L|_w$ admits a subdivision M of K and a simplicial approximation $|f| : |M| \to |L|$ of g (see [13, Theorem 16.5]).

The identity map $i : |K|_w \to |K|_m$ is always continuous. It is a homeomorphism if and only if K is locally finite, which means that in this case the strong and the weak topologies coincide. In general, there exists a (continuous) map $j : |K|_m \to |K|_w$ which is a modification of the identity map $|K| \to |K|$. The map $i : |K|_w \to |K|_m$ is always a homotopy equivalence (see [4, Theorem 1], or [12, Appendix 1.1.3, Theorem 10]). Using Henderson's subdivision theorem one can easily show that i is even a fine homotopy equivalence, i.e., for every open cover $\mathcal{V}$ of $|K|_m$ there exists a map $j : |K|_m \to |K|_w$ such that ij and ji are $\mathcal{V}$-homotopic to the identity maps.

An **abstract simplicial complex** $\mathcal{K}$ consists of a set E, called the set of **vertices** of $\mathcal{K}$, and of a set of finite nonempty subsets $s = \{e_0, \dots, e_n\} \subseteq E$ such that $s \in \mathcal{K}$ and $\emptyset \neq t \subseteq s$ implies $t \in \mathcal{K}$ and every singleton $\{e_0\}$ from E belongs to $\mathcal{K}$. Elements $s \in \mathcal{K}$ are called **abstract n-dimensional simplices** of $\mathcal{K}$. 0-dimensional simplices $\{e_0\}$ are identified with vertices e_0 of $\mathcal{K}$. An **abstract simplicial map** $f : \mathcal{K} \to \mathcal{L}$ is a map which sends vertices of $\mathcal{K}$ into vertices of $\mathcal{L}$ and $f(s) = \{f(e_0), \dots, f(e_n)\} \in \mathcal{L}$ whenever $s = \{e_0, \dots, e_n\} \in \mathcal{K}$. An **isomorphism** of abstract simplicial complexes $f : \mathcal{K} \to \mathcal{L}$ is a simplicial map, which induces a bijection between the sets of vertices and has the property that $s \in \mathcal{K}$ if and only if $f(s) \in \mathcal{L}$. With every geometric simplicial complex K is associated an abstract simplicial complex $\mathcal{K}$. It consists of the vertices $v \in K$ and of the sets $\{v_0, \dots, v_n\}$, where $[v_0, \dots, v_n] \in K$. Conversely, every abstract simplicial complex $\mathcal{K}$ admits a **geometric realization**, i.e., admits a geometric simplicial complex K such that the associated abstract complex is isomorphic to $\mathcal{K}$. All geometric simplicial maps are induced by abstract simplicial maps. In some situations (e.g., in homology theory) one also considers **ordered simplicial complexes**. These are abstract simplicial complexes $\mathcal{K}$ endowed with an ***ordering***. The ordering need not be linear, but one requires that its restriction to every subcomplex formed by a simplex and its faces be a linear ordering. A simplicial map between ordered simplicial complexes is ordered if it is order-preserving.

A **polyhedron** is a topological space P, which admits a geometric simplicial complex K and a ***homeomorphism*** $\varphi : |K|_w \to P$. We refer to φ as to the **triangulation** of P. Consequently, polyhedra are **triangulable** spaces. In 1961 J.G. Ceder showed that polyhedra (more generally, CW-complexes) are ***stratifiable spaces***. As such they are (***Hausdorff***) ***paracompact*** and hereditarily ***paracompact***, ***perfectly normal***, ***monotonically normal σ-spaces*** (see [MN, Chapter 11] and [KV, Chapter 10.5]). Another important property of polyhedra is that they are **absolute neighborhood extensor**s (ANEs) for stratifiable spaces (but not for paracompact spaces), i.e., if X is a stratifiable space, A is a closed subset of X and $f : A \to P$ is a map into a polyhedron P, then there exists an ***extension*** $\tilde{f} : U \to P$ of f to some neighbourhood U of A in X (see [3, Theorems 1.4 and 2.1]). Since stratifiable spaces generalize ***metrizable spaces***, it follows that polyhedra are also ANEs for metrizable spaces. In 1934 S. Cairns proved that differentiable C^1-manifolds, hence also C^r-manifolds, $r \geqslant 2$, and C^∞-manifolds, are triangulable, i.e., are polyhedra. For topological manifolds triangulability was proved, for dimension $n = 2$ by T. Rado in 1925, and for dimension $n = 3$ by E. Moise in 1952. It follows from the work of A.J. Casson (1975) and M.H. Freedman (1982) that there exist topological 4-manifolds with boundary, which are not triangulable. Triangulability of closed n-dimensional topological manifolds, for $n \geqslant 4$, is still an open problem. Freedman proved the existence of a closed 1-connected topological 4-manifold with intersection matrix E_8 and showed that triangulability of that manifold implies failure of the 3-dimensional Poincaré conjecture [5, Corollary 1.6].

Spaces which are not polyhedra are often approximated by polyhedra. More precisely, with every ***open cover*** $\mathcal{U}$ of a space X one associates its **nerve**. This is the abstract simplicial complex $\mathcal{N}(\mathcal{U})$, whose vertices are members U of $\mathcal{U}$ and vertices $U_0, \ldots, U_n$ form a simplex provided $U_0 \cap \cdots \cap U_n \neq \emptyset$. The carrier of the geometric realization $N(\mathcal{U})$ of $\mathcal{N}(\mathcal{U})$, endowed with the weak topology, is a polyhedron $|N(\mathcal{U})|_w$. If a cover $\mathcal{V}$ refines $\mathcal{U}$, i.e., if every V from $\mathcal{V}$ is contained in some U from $\mathcal{U}$, then one can define an abstract simplicial map $p_{\mathcal{U}\mathcal{V}} : \mathcal{N}(\mathcal{V}) \to \mathcal{N}(\mathcal{U})$, by putting $p_{\mathcal{U}\mathcal{V}}(V) = U$. This map induces a geometric simplicial map $|p_{\mathcal{U}\mathcal{V}}| : |N(\mathcal{V})| \to |N(\mathcal{U})|$. Any two such maps are contiguous with respect to $N(\mathcal{U})$, i.e., for every point $x \in |N(\mathcal{V})|$, the images of x under both maps belong to some simplex of $N(\mathcal{U})$. It follows that in the weak topology the two maps are homotopic. A **partition of unity** subordinated to $\mathcal{U}$ is a collection of maps $\varphi_U : X \to [0, 1]$, indexed by $U \in \mathcal{U}$, such that the support $\mathrm{supp}(\varphi_U) = \{x \in X : \varphi_U(x) \neq 0\} \subseteq U$ and $\sum_{U \in \mathcal{U}} \varphi_U = 1$. Here the sum is interpreted as the least upper bound of all the sums $\sum_U \varphi_U$, where U ranges over finite subsets of $\mathcal{U}$. A **normal cover** of a topological space X is an open cover $\mathcal{U}$ of X which admits a subordinated partition of unity. One can always achieve that the supports of φ_U form a ***locally finite*** collection in which case, for every $x \in X$, the sum $\sum_U \varphi_U(x)$ reduces to a finite sum [12, Appendix 1.3.1, Theorem 3]. If X is paracompact, every open cover is normal. Interpreting $\varphi_U(x)$ as the barycentric coordinate, belonging to the vertex U of $N(\mathcal{U})$ of a point $\varphi_{\mathcal{U}}(x)$, one obtains a map $\varphi_{\mathcal{U}} : X \to |N(\mathcal{U})|_w$, called **canonical map**, characterized by the property that $\varphi^{-1}(\mathrm{St}(U, N(\mathcal{U})) \subseteq U$. If $\mathcal{V}$ refines $\mathcal{U}$ and $\varphi_{\mathcal{V}} : X \to |N(\mathcal{V})|_w$ is a canonical map subordinated to $\mathcal{V}$, then the maps $\varphi_{\mathcal{U}}$ and $p_{\mathcal{U}\mathcal{V}}\varphi_{\mathcal{V}}$ are contiguous with respect to $N(\mathcal{U})$ and therefore, are homotopic. It follows that polyhedra $|N(\mathcal{U})|_w$ and the homotopy classes $|p_{\mathcal{U}\mathcal{V}}|$ form an inverse system $\mathbf{X}$ in the homotopy category H(Top) of spaces and homotopy classes of maps, called the **Čech system** of X. Moreover, the homotopy classes $[\varphi_{\mathcal{U}}]$ form a morphism $X \to \mathbf{X}$ in the category of inverse systems pro-H(Top).

The most useful generalization of geometric simplicial complexes are the CW-complexes, introduced in 1949 by J.H.C. Whitehead [14] as the natural framework for homotopy theory. A **CW-complex** (or **CW-space**) is a ***Hausdorff space*** X endowed with a **CW-structure**, i.e., an increasing sequence of closed ***subspaces*** $X^0 \subseteq X^1 \subseteq \cdots$ of X which satisfy the following conditions:

(i) X^0 is a discrete space.
(ii) For every $n > 0$, X^n is obtained from X^{n-1} by attaching a collection of n-cells.
(iii) $X = \bigcup X^n$ is the **direct limit** of the inclusion sequence $X^0 \hookrightarrow X^1 \hookrightarrow \cdots$.

To attach a collection of n-cells to X^{n-1} means to form the ***adjunction space*** $X^n = Y \sqcup_\varphi X^{n-1}$, where Y is the ***topological sum*** (***coproduct***) $Y = \bigsqcup_\alpha B^n_\alpha$ of a collection of copies of the n-**ball** $B^n = \{x \in \mathbb{R}^n : \|x\| \leqslant 1\}$ and $\varphi : \bigcup_\alpha \partial B^n_\alpha \to X^{n-1}$ is a map of the form $\varphi = \bigsqcup_\alpha \varphi_\alpha$, where $\varphi_\alpha : \partial B^n_\alpha \to X^{n-1}$ are arbitrary maps. Condition (iii) means that a subset $U \subseteq X$ is open in X if and only if $U \cap X^n$ is open in X^n, for every $n \geqslant 0$. One refers to X^n as to the n-**skeleton** of the complex. For $n > 0$, the restriction of the natural ***quotient map*** $q : Y \sqcup X^{n-1} \to X^n \subseteq X$ to the interior $(B^n_\alpha)^\circ$ of the n-ball B^n_α is a homeomorphism and its image $e^n_\alpha \subseteq X$ is an **open** n-**cell** of the CW-structure of X (but not an open set of X). Points of K^0 are considered to be open 0-cells. The n-**cell**s of a CW-structure are its open n-cells, $n \geqslant 0$. Cells of all dimensions $n \geqslant 0$ form a decomposition of the space X. Every map $\chi_\alpha : B^n \to X$, whose restriction to $(B^n)^\circ$ is a homeomorphism $(B^n)^\circ \to e^n_\alpha$, is a **characteristic map** of the n-cell e^n_α. Its restriction to the boundary $\chi_\alpha|\partial B^n$ is an **attaching map** of the n-cell e^n_α. The image $\chi_\alpha(B^n) = \overline{e^n_\alpha}$ is the closure of e^n_α and is a **closed** n-**cell** of the CW-structure. Closed cells are compact. A **regular** n-**cell** is one whose closure is an n-ball. A **regular CW-complex** is one in which all cells of its CW-structure are regular.

An important property of CW-complexes is that every compact subset of X is contained in the union of a finite number of open cells. In particular, $\overline{e^n_\alpha}$ meets only finitely many open cells. This "closure-finiteness" property explain the letter C in the name of CW-complexes; W comes from "weak" topology. Every polyhedron is a regular CW-complex, but there exist CW-complexes which are not polyhedra [6, 3.4, Example]. In the theory of CW-complexes **cellular maps** play an important role. These are maps $f : X \to Y$ between CW-complexes which have the property that $f(X^n) \subseteq Y^n$, for all n. Many properties of CW-complexes can be derived from analogous properties of polyhedra using a theorem proved in 1972 by R. Cauty. The theorem asserts that every CW-complex embeds as a (closed) neighbourhood retract in a polyhedron $|K|_w$ [2, Theorem 8]. In particular, this implies that CW-complexes are ANEs for stratifiable spaces.

Many notions which refer to geometric simplicial complexes and their weak topology have their analogues in CW-structures and CW-complexes, e.g., the notions of subcomplex and subdivision. Similarly, many theorems have their analogues. An example is the **cellular approximation theorem**, which asserts that every map $g: X \to Y$ between CW-complexes is homotopic to a cellular map $f: X \to Y$ [11, Theorem 8.5]. However, there are also cases when analogy fails. E.g., every open set of a polyhedron is a polyhedron, but Cauty has shown in 1992 that an open subset of a CW-complex need not be a CW-complex.

An important generalization of ordered simplicial complexes are **simplicial sets** (in older literature called **semi-simplicial complexes**). A simplicial set S consists of a sequence of sets S^n, $n = 0, 1, \ldots,$ and two sequences of functions $d_j : S^n \to S^{n-1}$, $n \geqslant 1$, and $s_j : S^n \to S^{n+1}$, $n \geqslant 0$, $j = 0, \ldots, n$, which satisfy the following conditions (see [9, 1.3]):

$$d_j d_i = d_{i-1} d_j, \ j < i, \tag{5}$$

$$s_j s_i = s_i s_{j-1}, \ i < j, \tag{6}$$

$$d_j s_i = \begin{cases} s_{i-1} d_j, & j < i, \\ \text{id}, & j = i, i+1, \\ s_i d_{j-1}, & i+1 < j. \end{cases} \tag{7}$$

The elements of S^n are called n**-simplices**, while d_j and s_j are called the **face operators** and the **degeneracy operators**, respectively. A **simplicial map** $f: S \to T$ between simplicial sets is a sequence of functions $f^n : S^n \to T^n$, which commutes with the face and degeneracy operators,

$$d_j f^n = f^{n-1} d_j, \qquad s_j f^n = f^{n+1} s_j. \tag{8}$$

The example which in 1950 has motivated S. Eilenberg and J.A. Zielber to introduce simplicial sets is the **singular complex** $S(X)$ of a topological space X. Its n-simplices are singular n-simplices of X, i.e., maps $s : \Delta^n \to X$, where Δ^n denotes the standard geometric n-simplex $\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : t_i \geqslant 0, \sum t_i = 1\}$ in $\mathbb{R}^{n+1}$. The **geometric realization** $\tilde{S}$ of a simplicial set S is a space constructed in the following way. For every $n \geqslant 0$, one considers the direct product $S^n \times \Delta^n$, where S^n is endowed with the ***discrete topology***, i.e., $S^n \times \Delta^n$ is the disjoint sum (coproduct) of card(S^n) copies of Δ^n. Then one considers the disjoint sum

$$\overline{S} = (S^0 \times \Delta^0) \sqcup (S^1 \times \Delta^1) \sqcup \cdots \sqcup (S^n \times \Delta^n) \sqcup \cdots. \tag{9}$$

Finally, $\widetilde{S} = \overline{S}/\sim$, where $\sim$ is the equivalence relation generated by all relations of the form

$$(d_j s, t) \sim (s, \delta_j t), \qquad (s_j s, t) \sim (s, \sigma_j t); \tag{10}$$

here $\delta_j : \Delta^{n-1} \to \Delta^n$ and $\sigma_j : \Delta^{n+1} \to \Delta^n$ are the standard embeddings into faces and projections onto faces, respectively. The space $\widetilde{S}$ has a natural CW-structure and thus, is a CW-complex (see [6, Theorem 4.3.5]). However, $\widetilde{S}$ also admits a triangulation and therefore, is actually a polyhedron (see [6, Corollary 4.6.12]).

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k-6 Homology

The definition of **homology groups** of a ***simplicial complex*** is based on the notion of the **oriented boundary** of the oriented n-simplex $\Delta^n = [v_0, \dots, v_n]$, spanned by the vertices $v_0, \dots, v_n$. The boundary is defined by the formula $\partial[v_0, \dots, v_n] = \sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$, where $[v_0, \dots, \hat{v}_i, \dots, v_n]$ denotes the ith face of Δ^n, i.e., the face opposite to the vertex v_i. An **orientation** of a simplex is a linear ordering of its vertices. Two orientations are equivalent if they have the same parity as permutations. Thus, $-[v_0, \dots, v_n]$ means the opposite orientation. Let K be a simplicial complex and let G be an Abelian group. The nth **chain group** $C_n(K; G)$ is defined as a group of finite linear combinations with coefficients in G of oriented simplices $\sum g_i \Delta_i^n$, where $g(-\Delta^n) = -g\Delta^n$. The boundary formula defines the **boundary homomorphism** $\partial_n : C_n(K; G) \to C_{n-1}(K; G)$. The kernel $\operatorname{Ker} \partial_n = Z_n(K; G)$ is called the group of n-**cycle**s and the image $\operatorname{Im} \partial_{n+1} = B_n(K; G)$ is called the **group of** n-**boundaries**. Since $\partial^2 = \partial\partial = 0$, we have $B_n(K; G) \subset Z_n(K; G)$. The **simplicial homology groups** of a complex K are defined as $H_n(K; G) = Z_n(K; G)/B_n(K; G)$. If the group $H_n(K; \mathbb{Z})$ is finitely generated, then the rank of the free part of it is called the n-th **Betti number** of K.

A collection of Abelian groups $\{C_n, \partial_m\}$ with homomorphisms satisfying $\partial^2 = 0$ is called a **chain complex** and the groups $H_n = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$ are **homology groups of a chain complex**. Thus, the simplicial homology groups $H_n(K; G)$ of a simplicial complex K are the homology groups of the simplicial chain complex $\{C_n(K; G), \partial_n\}$.

The group of n-**cochains** on a complex K is the group of homomorphisms $\operatorname{Hom}(C_n(K; \mathbb{Z}), G)$. It means that any n-cochain is determined by a map φ on the set of n-simplices in K with values in G. The **coboundary homomorphism** is defined by the formula $d_n f([v_0, \dots, v_{n+1}]) = \sum_i (-1)^i f([v_0, \dots, \hat{v}_i, \dots, v_{n+1}])$. The group of n-**cocycles** $Z^n(K; G)$ is defined as the kernel of d_n and the group of n-**coboundaries** $B^n(K; G)$ is defined as the image of d_{n-1}. Then the n-th **cohomology group** of a complex K is defined as $Z^n(K; G)/B^n(K; G)$.

The notion of chains, boundaries and coboundaries first appeared in the Calculus and in different areas of Physics. The homologies were introduced by H. Poincare on the eve of 20th century. The cohomologies were independently discovered by J.W. Alexander and A.N. Kolmogorov in the 30s.

The singular homology theory was developed in the works of O. Veblen, W. Hurewicz, S. Lefschetz, C. Dowker, J. Dugundji and S. Eilenberg as an extension of the simplicial homology to general spaces. It is topologically invariant by the definition and it coincides with the simplicial homology for simplicial complexes. An n-dimensional **singular simplex** σ^n on a topological space X is a continuous map $\sigma^n : \Delta^n \to X$ of the standard n-simplex $\Delta^n = \{(x_0, \dots, x_n) \in \mathbf{R^n} \mid \sum x_i = 1, x_i \geqslant 0\}$ to X. The **singular** n-**chain group** $S_n(X; G)$ of a topological space X with coefficients in G is the group of finite linear combinations $\sum g_i \sigma_i^n$. The oriented boundary of a singular simplex can be defined by means of the above boundary formula. The n-dimensional **singular homology group** of a topological space X with coefficients in G is defined as the n-homology group of the singular chain complex $\{S_n(X; G), \partial\}$. For every subset $A \subset X$ one can define a **relative singular homology group** $H_n(X, A; G)$ as the n-homology group of the chain complex $\{S_n(X; G)/S_n(A; G)\}$. Every continuous map $f : X \to Y$ induces homomorphisms of singular homologies $f_* : H_n(X; G) \to H_n(Y; G)$. The same holds for a map of pairs $f : (X, A) \to (Y, B)$, $f(A) \subset B$. So, for every n we have a ***covariant functor*** $H_n : \mathcal{TOP}_2 \to \mathcal{A}$ from the ***category*** of pairs of topological spaces to the category of Abelian groups. The category of topological spaces $\mathcal{TOP}$ is contained in $\mathcal{TOP}_2$ by means of the identification $X = (X, \emptyset)$. These functors satisfy certain properties which are known as the **Eilenberg–Steenrod axioms**.

Homotopy Axiom: If $f, g : (X, A) \to (A, B)$ are ***homotopic***, then $f_* = g_*$.

Exactness Axiom: There is a sequence of natural transformations $\partial_n : H_n \to H_{n-1} \circ R$, where $R : \mathcal{TOP}_2 \to \mathcal{TOP}$ is the restriction functor, i.e., R takes (X, A) to A, such that the following sequence is exact

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow \cdots.$$

Here i_* and j_* are induced by inclusions $i : A \to X$ and $j : (X, \emptyset) \to (X, A)$. The exactnes of a sequence $\cdots \xrightarrow{\varphi} H \xrightarrow{\psi} \cdots$ at H means the equality $\operatorname{Im} \varphi = \operatorname{Ker} \psi$. A sequence is called an **exact sequence** if it exact at each term.

Excision Axiom: For every pair (X, A) and every open subset $U \subset X$ such that the ***closure*** $\overline{U}$ lies in the ***interior*** $\operatorname{Int}(A)$ of A, the inclusion $(X \setminus U, A \setminus U) \subset (X, A)$ induces isomorphisms $H_n(X \setminus U, A \setminus U) \to H_n(X, A)$ for all n.

Dimension Axiom: For a one-point space there are equalities $H_n(pt) = 0$ for all $n > 0$. The group $H_0(pt)$ is the **coefficient group**.

S. Eilenberg and N. Steenrod proved that any two homology theories with isomorphic coefficient groups on the category of compact polyhedral pairs are isomorphic. Hu extended this to the category of finite CW-complexes. Eilenberg–Steenrod theorem introduces a possibility of axiomatic approach in algebraic topology. Thus, if the Dimension Axiom is dropped one obtains the notion of **extraordinary homology theory**. The Eilenberg–Steenrod theorem holds for cohomologies as well. In that case we consider a ***contravariant functors*** and reverse all arrows in the axioms. The Excision Axiom can be replaced by the following [16].

Mayer–Vietoris Exact Sequence: If A and B are subspaces of X with $X = Int(A) \cup Int(B)$, then the following sequence is exact

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{(i_*^A, i_*^B)} H_n(A) \oplus H_n(B)$$
$$\xrightarrow{j_*^A - j_*^B} H_n(X) \xrightarrow{\alpha} H_{n-1}(A \cap B) \longrightarrow \cdots .$$

Here i^A, i^B, j^A and j^B are corresponding inclusions. The homomorphism α is defined by means of ∂_n from the Exactness Axiom and the Excision Axiom.

Here is the list of facts about homology and cohomology of compact polyhedra which can be derived from the axioms.

(1) There is the **suspension isomorphism** $H_i(X) = H_{i+1}(\Sigma X)$, $i > 0$. The **suspension** ΣX of a space X is the quotient space of $X \times [0, 1]$ in which $X \times \{0\}$ is identified to a point and $X \times \{1\}$ is identified to another point. The same is true for cohomology. The **reduced homology** is defined as $\widetilde{H}_n(X) = H_n(X, x_0)$, where $x_0 \in X$. For the reduced homology the suspension homomorphism holds for all i. The reduced homology are natural in the category of pointed topological spaces.

Everywhere below $H_n(X)$ and $H^n(X)$ denote homology and cohomology with integer coefficients.

(2) **Künneth Formula**: There are formulas

$$H_n(X \times Y) = \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \oplus \bigoplus_{k+l=n-1} \operatorname{Tor}\big(H_k(X), H_l(Y)\big)$$

and

$$H^n(X \times Y) = \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \oplus \bigoplus_{k+l=n+1} \operatorname{Tor}\big(H^k(X), H^l(Y)\big).$$

There are analogues of it for coefficients in any principal ideal domain R.

(3) **Universal Coefficient Theorem**: For every space X and every Abelian group G there are isomorphisms between $H^n(X; G)$ and $\operatorname{Hom}(H_n(X), G) \oplus \operatorname{Ext}(H_{n-1}(X), G)$ for all n. For every finite polyhedron X and every Abelian group there are isomorphisms

$$H_n(X; G) = \operatorname{Hom}\big(H^n(X), G\big) \oplus \operatorname{Ext}\big(H^{n+1}(X), G\big)$$

for all n. For the group $G = S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$ this turns into the **Pontryagin Duality** $H_n(X; \operatorname{Char}\mathbb{Z}) = \operatorname{Char}(H^n(X; \mathbb{Z}))$. Here $\operatorname{Char}(G) = \operatorname{Hom}(G, S^1)$. Note that $\mathbb{Z}$ in the Pontryagin Duality can be replaced by any discrete group and the isomorphism holds on the level of topological groups.

(4) **Universal Coefficient Formula**: For every space X and every Abelian group G there are isomorphisms $H_n(X; G) = H_n(X) \otimes G \bigoplus \operatorname{Tor}(H_{n-1}(X), G)$. If X is a compact polyhedron, there are the formulas $H^n(X; G) = H^n(X) \otimes G \bigoplus \operatorname{Tor}(H^{n+1}(X), G)$.

Here $A \otimes B$ means ***tensor product*** over the integers. Let $0 \to R \xrightarrow{i} F \to A \to 0$ be a short exact sequence, where F is free Abelian. Then the **Tor** is defined as $\operatorname{Tor}(A, B) = \operatorname{Ker}(i \otimes 1_B)$ and the **Ext** is defined as $\operatorname{Ext}(A, B) = \operatorname{Hom}(R, B)/i^* \operatorname{Hom}(F, B) = \operatorname{Coker}(i^*)$.

(5) **Coefficients Long Exact Sequence**: For a short exact sequence of Abelian groups $0 \to G' \to G \to G'' \to 0$ there are **Bockstein homomorphisms** $H_n(X; G'') \to H_{n-1}(X, G')$ and $H^n(X; G'') \to H^{n+1}(X, G')$ such that the sequences

$$\cdots \to H_n(X; G') \to H_n(X; G)$$
$$\to H_n(X, G'') \to H_{n-1}(X; G') \to \cdots$$

and

$$\cdots \to H^n(X; G') \to H^n(X; G)$$
$$\to H^n(X, G'') \to H^{n+1}(X; G') \to \cdots$$

are exact.

There are **cup product** and **cap product**: $\cup : H^p(X) \otimes H^q(X) \to H^{p+q}(X)$ and $\cap : H^p(X) \otimes H_n(X) \to H_{n-p}(X)$. The former defines a ***ring*** structure on cohomologies $H^*(X)$. The latter participates in the formulation of different duality theorems, in particular, the **Poincaré Duality** Theorem: For an ***orientable closed n-manifold M*** there are isomorphisms $H^p(M) = H_{n-p}(M)$. Both product can be defined for arbitrary coefficients: $H^p(X, T_1) \otimes H^q(X; T_2) \to H^{p+q}(X, T_1 \otimes T_2)$ and $H^p(X, T_1) \otimes H_n(X; T_2) \to H_{n-p}(X; T_1 \otimes T_2)$.

There is the Hurewicz homomorphism $h_n : \pi_n(X) \to H_n(X)$, where $\pi_n(X)$ is the n-dimensional **homotopy group** of X. By the definition $\pi_n(X)$ is the set of the homotopy classes of $[S^n, X]$ of maps $\varphi : (S^n, s_0) \to (X, x_0)$ in the pointed category. The group operation comes from the quotient map of S^n collapsing the equator to a point. The 1-dimensional homotopy group is called the **fundamental group**. The homomorphism h_1 is the ***Abelianization*** homomorphism. The Hurewicz theorem states that h_n is an isomorphism, provided the space X is ***$(n-1)$-connected***. The axiomatic approach gives the following isomorphism $H_n(X) = \pi_n(FA(X)) = [S^n, FA(X)]$ for compact polyhedra, where $FA(X)$ is the ***free Abelian topological group*** generated by X [3]. The cohomology groups can be defined as $H^n(X) = [X, FA(S^n)]$. The latter holds because of the representation theorem: $H^n(X; G) = [X, K(G, n)]$, since $FA(S^n)$ is an Eilenberg–MacLane space of the type $K(\mathbb{Z}, n)$.

An ANE-space K is an **Eilenberg–MacLane space** of type $K(G,n)$ if $\pi_i(K)=0$ for $i\neq n$ and $\pi_n(K)=G$.

Formally the **singular homology** and **cohomology** are defined for arbitrary topological spaces but the natural area of their applications is the category of finite complexes. It can be automatically enlarged to spaces homotopy equivalent to finite complexes such as compact ANR. More general spaces require different construction of homologies. Luckily, all natural definitions of cohomologies on general spaces agree with each other. For all topological spaces X the n-dimensional cohomology group $H^n(X;G)$ equals $[X,K(G,n)]$. The structure of an Abelian group on the set of homotopy classes $[X,K(G,n)]$ comes from the fact that $K(G,n)$ is the free Abelian group $FA(M(G,n))$ generated by a Moore complex $M(G,n)$. Usually this cohomology group is called the **Čech cohomology** (= Alexander–Spanier cohomology) and often it is denoted as $\check{H}^n(X;G)$ whenever X is not a finite complex. We consider here with more details homologies and cohomologies for compact metric spaces. Every compact metric space X can be presented as the limit space of an ***inverse sequence*** of finite polyhedra $X=\varprojlim\{K_i,p_i^{i+1}\}$. This leads to the natural definition of homologies and cohomologies: $\check{H}_n(X;G)=\varinjlim H_n(K_i;G)$ and $\check{H}^n(X;G)=\varinjlim H^n(K_i;G)$.

The homology theory $\check{H}_*$ is called the **Čech homology**. It was constructed independently and using different approaches by P.S. Alexandroff, S. Lefschetz and L. Vietoris [2] in late 20s. This theory is also called the Vietoris homology or the Alexandroff–Čech homology. Čech gave a construction which works for general spaces. The Čech homologies do not satisfy the Exactness Axiom.

The cohomologies $\check{H}^*$ are called Čech cohomologies. They coincides with the cohomology defined by maps to an Eilenberg–MacLane space and satisfy all Eilenberg–Steenrod axioms. For general spaces the Excision Axiom is formulated as follows: For every closed subset $A\subset X$ the quotient map $q:(X,A)\to(X/A,pt)$ induces isomorphisms of cohomology groups. They also satisfy the following.

Additivity Axiom: If a compactum X is presented as an infinite ***wedge*** $X=\vee_i X_i$ with a common point x. Then

$$H^n\big(X,\{x\}\big)=\bigoplus_i H^n\big(X_i,\{x\}\big),$$

and

$$H_n\big(X,\{x\}\big)=\prod_i H_n\big(X_i,\{x\}\big)$$

for homologies.

Milnor proved that there is a unique cohomology theory on compact metric spaces satisfying the Eilenberg–Steenrod axioms plus the Additivity Axiom. The Mayer–Vietoris sequence, the Coefficient Long Exact sequence, The Universal Coefficient Formula, the Künneth Formula hold for cohomologies of compacta without changes. Additionally the Čech cohomologies of compacta are **continuous**: $H^n(X;G)\varinjlim H^n(X_i;G)$ for every inverse system $\{X_i\}$ with X as the limit space. For a compact subset $X\subset S^n$ of an n-dimensional sphere there is the **Alexander Duality** $H^k(X;G)=H_{n-k-1}(S^n\setminus X;G)$ for all k. It holds for any orientable closed n-manifold M for k with $H^k(M;G)=H^{k+1}(M;G)=0$. The Alexander duality implies the Jordan–Brouwer theorem which states that if S is a subset of $\mathbb{R}^n$ homeomorphic to the $(n-1)$-sphere S^{n-1}, then $\mathbb{R}^n\setminus S$ has exactly two components. A corollary of this is the **Invariance of Domain** theorem: The image $f(U)$ of an open subset of $\mathbb{R}^n$ under a continuous injective map $f:U\to\mathbb{R}^n$ is an open set.

One of the areas of application of cohomology in general topology is dimension theory. It seems that only by means of cohomology it can be proven that the multiplication of a topological space by an interval raises the dimension by one (K. Morita). The cohomological dimension theory allows to compute the dimension of product of two compacta. The **cohomological dimension** of a space X with respect to a coefficient group G is defined as $\dim_G X=\max\{n\mid H^n(X,A;G)\neq 0,\ A\subset X \text{ is closed}\}$. Since the relative Čech cohomologies vanish in dimensions greater than the covering dimension of a space, it follows that $\dim_{\mathbb{Z}} X\leqslant \dim X$. Alexandroff proved that for a compact metric space $\dim X=\dim_{\mathbb{Z}} X$, provided $\dim X<\infty$. There exist infinite dimensional compacta with finite cohomological dimension [4, 8]. These compacta lead to examples of homology manifolds with infinite covering dimension. S. Ferry discovered that such manifolds can appear as limit points in some moduli spaces. Using these compacta and Jensen's principle $\diamondsuit$ V.V. Fedorchuk constructed a smooth separable (not paracompact) 4-manifold M^4 with $\dim_{\mathbb{Z}} M^4=4$, and $\dim M^4=\infty$. The Universal Coefficient Formula implies that $\dim_G X\leqslant\dim_{\mathbb{Z}} X$ for all Abelian groups G. Let $\mathcal{P}$ be the set of all prime numbers. The localization $\mathbb{Z}_{(p)}$ of the integers at a prime p is the group of rationals with denominators not divisible by p. M.F. Bockstein proved that $\dim_{\mathbb{Z}} X=\max\{\dim_{\mathbb{Z}_{(p)}} X\mid p\in\mathcal{P}\}$ for all compact metric spaces X. For compacta there is an alternative: either

(1) $\dim_{\mathbb{Z}_{(p)}} X=\dim_{\mathbf{Q}} X=\dim_{\mathbb{Z}_p} X=\dim_{\mathbb{Z}_{p^\infty}} X$, or
(2) $\dim_{\mathbb{Z}_{(p)}} X=\max\{\dim_{\mathbf{Q}} X,\dim_{\mathbb{Z}_{p^\infty}} X+1\}$ [6].

Here $\mathbb{Z}_m=\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}_{p^\infty}=\varinjlim\mathbb{Z}_{p^k}$. In the first case a compactum X is called **p-regular**, in the second case it is called **p-singular**. All cohomology dimensional information about a compactum X is completely determined by the following data: the number $q_X=\dim_{\mathbf{Q}} X$, the sets of primes $S\subset\mathcal{P}$ for which X is p-singular, its subset $D_X=\{p\mid\dim_{\mathbb{Z}_{p^\infty}} X\neq\dim_{\mathbf{Z}_p} X\}$ and a function $d_X:S\to\mathbf{N}\cup\infty$ defined as $d_X(p)=\dim_{\mathbb{Z}_p} X$. Note that if $\dim_{\mathbb{Z}_{p^\infty}} X\neq\dim_{\mathbb{Z}_p} X$, then $\dim_{\mathbb{Z}_{p^\infty}} X=\dim_{\mathbb{Z}_p} X-1$. This family $\{q_X,D_X,S_X,d_X\}$ is called the **cohomological dimension type** of X and is denoted as $\mathrm{DIM}(X)$. The Bockstein formula for the cohomological dimension of the product of two compacta turns into the following $\mathrm{DIM}(X\times Y)$

$= \{q_X + q_Y, D_X \cup D_Y, S_X \cup S_Y, d_X + d_Y\}$ [6]. Then the formula for dimension $\dim(X \times Y)$ can be recovered from this in view of Alexandroff's and Bockstein's theorems. It can be derived from this formula that for a compactum X either $\dim X^n = n \dim X$ or $\dim X^n = n \dim X - n + 1$. The first occurs if and only if there is a ***field*** F such that $\dim_F X = \dim X$. A compactum X is called **dimensionally full-valued** if $\dim(X \times Z) = \dim X + \dim Z$ for all compacta Z. A compactum is dimensionally full-valued if and only if $\dim_G X = \dim X$ for all Abelian groups G (V.G. Boltyanskij). The cohomological dimension type of a dimensionally full-valued compactum is $\{n, \emptyset, \emptyset, n\}$. An abstract cohomological dimension type $\mathcal{F} = \{q, D, S, d\}$ consists of a number $q \in \mathbf{N} \cup \infty$, two subsets of primes $D \subset S$ and a function $d : S \to \mathbf{N} \cup \infty$. For every abstract cohomological dimension type $\mathcal{F}$ there is a compactum X such that $\mathcal{F} = \mathrm{DIM}(X)$ [4, 6]. This implies that for compacta of dimensions $\dim X = n$ and $\dim Y = m$ the dimension of the product $\dim(X \times Y)$ can be any number between $\max\{n, m\} + 1$ and $n + m$ and each possibility can be realized. First examples of compacta with $\dim(X \times Y) < \dim X + \dim Y$ were constructed by Pontryagin (Pontryagin surfaces). First example of a compactum with $\dim X^2 \neq 2 \dim X$ was constructed by Boltyanskij. Cohomological dimension theory is used to show that the dimension of the intersection $f(X) \cap g(Y)$ of the images of compacta does not exceed $\dim(X \times Y) - n$ for a dense G_δ set $\{(f, g)\} \subset (\mathbb{R}^n)^X \times (\mathbb{R}^n)^Y$ in the map space [5]. The computation performed in [5] were used by S.A. Bogatyi to prove the equality between algebraic and geometric defects of finite dimensional compacta and by E.V. Shchepin to obtain the formula for the dimension of the multiple intersection. Shchepin rearranged the Bockstein algebra of cohomological dimension types in an elegant arithmetic of triples which was further improved by J. Dydak. Cohomological dimension theory is useful in the theory of compact group actions. It is well-known that the possibility of an effective action of p-adic integers A_p on a manifold constitute the Hilbert–Smith Conjecture. If the group A_p effectively acts on a closed n-manifold, then the orbit space has the following cohomological dimensions: $\dim_{\mathbf{Q}} M/A_p = \dim_{\mathbb{Z}_q} M/A_p = \dim_{\mathbb{Z}_{(q)}} M/A_p = n$, where q is relatively prime to p; $\dim_{\mathbb{Z}_p} M/A_p = n + 1$ and $\dim_{\mathbb{Z}_{(p)}} M/A_p = n + 2$ (C.T. Yang, G. Bredon, F. Raymond, R.F. Willams). Thus, the covering dimension $\dim M/A_p$ equals either $n + 2$ or ∞. First example of a compact group action on a comact space whose projection to the orbit space raise the dimension was constructed by Kolmogorov. Further examples are due to Raymond and Williams. An action where the dimension is raised to infinity was constructed in [7].

A visual duality between homology and cohomology is a part of general duality between covariant and contravariant approaches to Topology. According to that, the dual to compact metric spaces is the class of countable CW-complexes. Every countable CW-complex K is the direct limit of finite subcomplexes $K = \varinjlim\{K_i\}$. Then $H_n(K; G) \varinjlim H_n(K_i; G)$ for the singular homology. In the case of cohomology there exists the **Milnor exact sequence**:

$$0 \to \varprojlim{}^1 \{H^{n-1}(K_i; G)\} \to H^n(K; G) \to \varprojlim\{H^n(K_i; G)\} \to 0.$$

Let $\{A_i, f_i^{i+1}\}$ be an inverse sequence of Abelian groups and let $\varphi : \prod A_i \to \prod A_i$ be a homomorphism defined by the formula $\varphi(a_1, a_2, a_3 \ldots) = (a_1 - f_1^2(a_2), a_2 - f_2^3(a_3), \ldots)$. Then $\varprojlim\{A_i\} = \mathrm{Ker}(\varphi)$ and $\varprojlim{}^1 A_i = \mathrm{Coker}\,\varphi$. Dually, in the case of compacta the right homology theory must satisfy the Milnor Formula

$$0 \to \varprojlim{}^1 \{H_{n+1}(K_i; G)\} \to H_n(X; G) \to \check{H}_n(X; G) \to 0,$$

where $X = \varprojlim\{K_i\}$. The corresponding homology theory is called the **Steenrod homology**. It satisfies all Eilenberg–Steenrod axioms and the Additivity Axiom. Milnor proved that these axioms determine the theory on pairs of compacta. Before defining the Steenrod homology we give a brief description of (co)homologies defined by infinite chains.

For a locally finite simplicial complex K one can consider infinite chains and the notion of the boundary still will make sense. This leads to the definition of **simplicial locally finite homology** $H_*^{lf}(K; G)$. Dually, one can consider cochains on infinite chains. To make things working one should take only cochains with compact supports. This defines the **simplicial cohomology with compact supports** $H_c^*(K)$. An infinite singular chain on a topological space X is locally finite if for every compact set $C \subset X$ there are finitely many singular simplexes σ from the chain having nonempty intersection $\mathrm{Im}(\sigma) \cap C \neq \emptyset$. Locally finite singular chains define **singular locally finite homology**. A cochain ξ on a space X has a compact support if there is compact $C \subset X$ such that $\xi(\sigma) = 0$ for all singular simplexes σ with $\mathrm{Im}(\sigma) \cap C = \emptyset$. Cochains with compact supports define **singular cohomology with compact supports**. These (co)homology also can be characterized axiomatically. To do that one has to transform the Excision Axiom into the following: $H_*^{lf}(X, Y) = H_*^{lf}(X \setminus Y)$ and $H_c^*(X, Y) = H_c^*(X \setminus Y)$ for every closed subset $Y \subset X$. S. Petkova proved [15] the uniqueness theorem for these (co)homology in the category of locally compact second-countable spaces with proper maps using the above Eilenberg–Steenrod axioms and the dual to Milnor's additivity axiom.

Dual Additivity Axiom: Let K_i be components of K. Then $H_n^{lf}(K) \prod_i H_n^{lf}(K_i)$ and $H_c^n(K) = \bigoplus H_c^n(K_i)$. For a locally compact space X cohomologies with compact support coincide with the Čech cohomology of the one-point compactification of X.

Assume that a compactum X is embedded in a face $I^\omega \times \{0\}$ of the Hilbert cube $Q = I^\omega \times [0, 1]$. Then the Steenrod homology groups can be defined as $H_n(X; G) = H_{n+1}^{lf}(Q \setminus X; G)$ and for pairs $H_n(X, A; G) = H_n(X/A; G)$.

The Universal Coefficient Theorem which expresses the homology in terms of cohomologies holds for the Steenrod homology and the Čech cohomology. There is the **Steenrod–Sitnikov Duality**: $H_k(X; G) = H^{n-k-1}(S^n \setminus X; G)$ [12]. Borel and Moore constructed the homology theory which agrees with the Steenrod homology for finitely generated coefficients. Generally the Borel–Moore homologies do not satisfy the Additivity Axiom.

For a nonsimply connected spaces (co)homologies naturally appear with **twisted coefficients**. A twisted coefficient system on X is a family of Abelian groups $\{G_x \mid x \in X\}$ and isomorphisms $\{\xi_s : G_{x_0} \to G_{x_1} \mid s : [0,1] \to X, s(0) = x_0, s(1) = x_1\}$ such that $\xi_s = \xi_{s'}$ for every homotopic relatively endpoints paths s and s'. Thus, the fundamental group $\pi_1(X, x_0)$ acts on the group $G = G_{x_0}$. This turns the group G into a module over the group ring $\mathbb{Z}[\pi_1(X)]$. Every action of $\pi_1(X)$ on G defines twisted coefficients on X. There is the natural action of the fundamental group on the singular (co)chains on the universal cover $\overline{X}$ of X which turns the chain groups $S_n(\overline{X})$ and the cochain group $S^n(\overline{X})$ into $\mathbb{Z}[\pi_1(X)]$-modules. To define (co)homology groups $H_*(X; \underline{G})$ and $H^*(X; \underline{G})$ with twisted coefficients (the other name is **local (co)homology**) we consider the following chain and cochain complexes $S_n(\overline{X}) \otimes G$ and $\mathrm{Hom}(S_n(\overline{X}), G)$, where the tensor product and Hom are taken over the ring $\mathbb{Z}[\pi_1(X)]$. Thus, the first is the quotient of the singular chain complex $S_n(\overline{X}; G)$ on the universal cover $\overline{X}$ with the kernel generated by elements of the type $s \otimes \alpha g - \alpha^{-1} c \otimes g$, the second is a subgroup of $S^n(\overline{X}; G) = \mathrm{Hom}(S_n(\overline{X}), G)$ which consists of homomorphisms φ such that $\varphi(\alpha s) = \alpha\varphi(s)$, where $s \in S_n(\overline{X})$, $g \in G$, and $\alpha \in \pi_1(X)$. The simplest example of local coefficients is the **orientation sheaf** on an n-manifold X which is defined by the formula $G_x = H_n(X, X \setminus x)$ and ξ_s is defined by means of isotopy on X containing a path s. The local cohomology is used to define cohomological dimensions for discrete groups Γ: $\mathrm{cd}_{\mathbf{L}}\,\Gamma = \max\{n \mid H^n(K(\Gamma, 1); \underline{\mathbf{L}[\Gamma]})$, where $\mathbf{L}$ is a principle ideal domain. Bestvina and Mess proved the equality $\mathrm{cd}_{\mathbf{L}}\,\Gamma = \dim_{\mathbf{L}} \partial\Gamma$ for the groups Γ that admit Bestvina's Z-boundary $\partial\Gamma$, in particular for all hyperbolic groups.

The local cohomology is a partial case of the **sheaf cohomology**. A **sheaf** on X is a topological space S together with a surjective ***local homeomorphism*** $\pi : S \to X$ such that every fiber $S_x = \pi^{-1}(x)$ is an Abelian group, called a **stalk** at $x \in X$ and group operation is continuous. The latter means that the function $\{(\alpha, \beta) \in S \times S \mid \pi(\alpha) = \pi(\beta)\} \to S$ taking (α, β) to $\alpha - \beta$ is continuous. For every open subset $U \subset X$ the set of sections $s : U \to S$ forms a group $\Gamma_U(S)$. There are different ways to define sheaf cohomologies. Here is an approach in spirit of Alexandroff–Čech. For an open cover $\mathcal{U} = \{U_i\}$ of X we define an n-cochain as a skew symmetric function $\xi : \mathcal{U}^n \to \Gamma_\varphi(S)$, where $\varphi : \mathcal{U}^n \to \mathcal{U}$ is the operation of taking the intersection. This defines a cochain complex and the cohomology $H^n(\mathcal{U}; S)$. Then the sheaf cohomologies of X are defined as the direct limit of the sheaf cohomologies of open covers. The sheaf cohomology theory is useful if one works with general topological spaces. Thus analogs of Hurewicz finite-to-one map theorems for general spaces first were proved using sheaf cohomologies (E.G. Skljarenko and A.V. Zarelua). The cohomological dimension theory of paracompact spaces is based on the sheaf cohomologies [10, 1]. For every map between compacta $f : X \to Y$ there is the **Leray sheaf** L on Y whose stalks are $L_x = \check{H}^*(f^{-1}(x); G)$. There is the Leray spectral sequence connecting $H^*(Y; L)$ with $H^*(X; G)$. The Leray spectral sequence implies in particular the Vietoris-Begle theorem: A map between compacta $f : X \to Y$ induces isomorphisms of cohomologies provided $\check{H}^*(f^{-1}(y)) = 0$ for all $y \in Y$. It implies that a G-**acyclic map** (i.e., a map $f : X \to Y$ with $\check{H}^*(f^{-1}(y); G) = 0$) cannot raise the cohomological dimension with respect to G. For a locally compact space X there is the **homology sheaf** $\mathcal{H}_*(X; G)$ whose stalks are the groups $\varinjlim\{H_*(X, X \setminus U; G) \mid x \in U\}$. Here H_* is the Steenrod homology and U runs over all open neighbourhoods of x. A locally compact space X is called an n-dimensional **homology manifold** over G ($n - hm_G$) if $\mathcal{H}_i(X; G)$, $i \neq n$, is the zero sheaf and $\mathcal{H}_n(X; G)$ is a locally constant sheaf with stalks isomorphic to G. In this case $\mathcal{H}_n(X; G)$ is called the **orientation sheaf** of a homology manifold X. A theorem of G. Bredon states that if for a connected and locally connected locally compact space X and a field F the sheaf $\mathcal{H}_*(X; F)$ is locally constant and $\dim_F X = n < \infty$, then X is an $n - hm_F$ [1].

As it was mentioned, there is a unique cohomology theory for all kind of topological spaces. The situation is different with homologies. Even for compact metric spaces the best homology theory, the Steenrod homology, has a flaw: it is not continuous. The Čech homology has this property but it is not exact. This moves it to the rank of an auxiliary theory. The most natural homologies for metric spaces are Steenrod-Sitnikov homologies [13] which are defined as $H_n(X; G) = \varinjlim\{H_n(C; G) \mid C$ is compact in $X\}$. The same theory for more general spaces is defined in [12]. In some literature it is called the homologies with compact supports. There are other theories which appeared naturally in different areas of Topology but all of them have some deficiency. The Borel–Moore homology theory with general coefficients does not coincides with the Steenrod homology for compacta. The strong homology defined by Yu. Lisica and S. Mardesic in the Strong Shape Theory [11] are nontrivial in negative dimensions under the ***Continuum Hypothesis*** (S. Mardešič, A. Prasolov). They coincide with the Steenrod homologies for all compact Hausdorff spaces. Axiomatic approach to the homology for compact Hausdorff spaces was studied by B.I. Botvinnik and V.I. Kuzminov, N.A. Berikashvili, H.N. Inasaridze and L.D. Mdzinarishvili [15]. Extraordinary (co)homology (generalized (co)homology) theory can be also defined for general spaces. A request for such generalizations appeared in the Controlled Topology [9].

J. Roe brought to life the notion of (co)homology to the coarse category. The **coarse category** consists of metric spaces and coarsely uniform metrically proper maps. A map between metric spaces $f : X \to Y$ is called **coarsely uniform** if there is a tending to infinity function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$

such that $d_Y(f(x), f(x')) \leqslant \rho(d_X(x, x'))$. It is called **metrically proper** if the preimage of every bounded set is bounded. Let $\mathcal{U}_\alpha$ be a directed ***cofinite system*** of uniformly bounded open covers of a metric space Z. This defines a direct system of nerves N_α of these covers. Such a system is called an **anti-Čech approximation** of Z. Then the **coarse homology** $HX_*(Z; G)$ is defined as $\varinjlim H_*^{lf}(N_\alpha; G)$. It is (anti) analogous to the Čech cohomology. The definition of coarse cohomologies is more complicated. The approach based on approximations by polyhedra meets with the same difficulties as in the case of the construction of ordinary homologies [14].

The idea of (co)homology is very fruitful in Mathematics. It came from Topology and appeared in many different areas. The applications of (co)homologies of various kind are very deep and impressive. They are so numerous, e.g., cyclic homology, elliptic (co)homology, Floer homology, intersection homology, quantum homology, etc, that there is no chances to list them all in a small note like this.

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k-7 Homotopy, I

In this and the following article we shall give a bird's-eye view of basic facts in homotopy theory. *Spaces* in these sections are *topological spaces* unless otherwise stated. *Maps* are *continuous functions*. H_* and H^* are *singular homology* and *cohomology theories*, respectively. When we consider integral coefficient (co)homology groups, we will not specify the group $\mathbb{Z}$ of integers.

1. Basic definitions

The concept of homotopy may be found in Lagrange's method in the calculus of variations. Even in the 19th century many mathematicians used the idea of deformation. In 1911 L.E.J. Brouwer, first, gave the general definition of homotopy between two maps. Thus, let X and Y be spaces. Two maps f_0 and f_1 from X to Y are **homotopic**, denoted $f_0 \simeq f_1$, if there exists a map $F: X \times \mathbb{I} \to Y$ with

$$F(x,0) = f_0(x) \quad \text{and}$$
$$F(x,1) = f_1(x) \quad \text{for all } x \in X,$$

where $\mathbb{I}$ is the unit interval $[0,1]$. Such a map F is called a **homotopy** joining f_0 and f_1. The existence of homotopy clearly induces an equivalence relation in the set $C(X,Y)$ of maps from X to Y. Then the equivalence class of a map $f: X \to Y$, denoted by $[f]$, is called the **homotopy class** of f. We use $[X;Y]$ to denote the set of homotopy classes. If all maps from X to Y are homotopic to each other, we shall write $[X;Y] = 0$. If X is the singleton, $[X;Y] = 0$ means that Y is **pathwise connected**, i.e., any two points of Y can be connected by a path. Moreover, suppose that $f_0 \restriction A = f_1 \restriction A$ for a subset A of X. Then f_0 is **homotopic to** f_1 **relative to** A, denoted by $f_0 \simeq f_1$ rel A, if there exists a homotopy $H: X \times \mathbb{I} \to Y$ joining f_0 and f_1 such that $H(a,t) = f_0(a)$ for all $a \in A$ and $t \in \mathbb{I}$. H is called a **relative homotopy** joining f_0 and f_1 with respect to A or a **homotopy relative to** A joining f_0 and f_1 and is denoted by $H: f_0 \simeq f_1$ rel A. We use the symbols $[f]_A$ and $[X;Y]_A$ to denote the relative homotopy class of f and the set of relative homotopy classes, respectively.

The concept is generalized to maps between n-tuples of spaces. Here an n-tuple $(X, A_1, \ldots, A_n)$ consists of a space X and its subspaces $A_1, \ldots, A_n$. A map $f: (X, A_1, \ldots, A_n) \to (Y, B_1, \ldots, B_n)$ means a map $f: X \to Y$ with $f(A_i) \subset B_i$ for all $i = 1, \ldots, n$. Two maps $f_0, f_1: (X, A_1, \ldots, A_n) \to (Y, B_1, \ldots, B_n)$ are homotopic as maps of n-tuples if there exists a homotopy $F: X \times \mathbb{I} \to Y$ joining f_0, f_1 such that $F(A_i \times \mathbb{I}) \subset B_i$ for all $i = 1, \ldots, n$, and we denote by $f_0 \simeq f_1: (X, A_1, \ldots, A_n) \to (Y, B_1, \ldots, B_n)$. As the same as in absolute case we use symbols $[f]$ and $[X, A_1, \ldots, A_n; Y, B_1, \ldots, B_n]$ to denote the homotopy class of a map f and the set of homotopy classes.

For maps $f, f': (X, A_1, \ldots, A_n) \to (Y, B_1, \ldots, B_n)$ and $g, g': (Y, B_1, \ldots, B_n) \to (Z, C_1, \ldots, C_n)$, if $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$ as maps from $(X, A_1, \ldots, A_n)$ to $(Z, C_1, \ldots, C_n)$. Hence we can define the composition of homotopy classes $[f]$ and $[g]$ as follows:

$$[g] \circ [f] = [g \circ f] \in [X, A_1, \ldots, A_n; Z, C_1, \ldots, C_n].$$

Moreover the composition $g_*([f]) = [g \circ f] = f^*([g])$ induces the following two functions

$$g_*: [X, A_1, \ldots, A_n; Y, B_1, \ldots, B_n] \to [X, A_1, \ldots, A_n; Z, C_1, \ldots, C_n]$$

and

$$f^*: [Y, B_1, \ldots, B_n; Z, C_1, \ldots, C_n] \to [X, A_1, \ldots, A_n; Z, C_1, \ldots, C_n].$$

If $f \simeq f'$ then $f^* = f'^*$ and if $g \simeq g'$, then $g_* = g'_*$.

Let X be a space with a distinguished point x_0. We denote the pair $(X, \{x_0\})$ by (X, x_0) and call it a **pointed space** (with **base point** x_0). Also a pointed pair (X, A, x_0) is the triple $(X, A, \{x_0\})$ such that $x_0 \in A$. In these cases maps and homotopies are ones for pairs and triples. We call them **pointed maps** and **pointed homotopies**. If we do not need to specify base points, we write "a pointed space X" and "a pointed map $f: X \to Y$" …, etc. We denote the set of homotopy classes of pointed maps by $[X;Y]_0$. If a map $f: X \to Y$ is homotopic to the constant map $c_{y_0}: X \to \{y_0\} \subset Y$, we say that it is **null-homotopic** or **inessential** and denote this by $f \simeq 0$. $[X;Y]_0 = 0$ means that all maps are null-homotopic. Let $S^0 = \{\pm 1\}$ be the 0-sphere. Then $[X;S^0]_0 = 0$ if and only if X is ***connected***. $[S^0;Y]_0 = 0$ if and only if Y is ***arcwise connected***.

Note that if K is an infinite ***CW-complex***, there may exist a map $f: K \to Y$ such that $f \restriction K^{(n)} \simeq 0$ for all n, but f is not null-homotopic, where $K^{(n)}$, $n = 0, 1, \ldots$, are the ***n-skeletons*** of K. We call such a map a **phantom map**. In fact, there exist uncountable many phantom maps from the **infinite-dimensional complex projective space** $\mathbb{CP}^\infty$ to S^3 [1]. Phantom maps are important subjects in homotopy theory [2].

To study geometric topology we often make a classification of homeomorphisms and embeddings. Thus, homeomorphisms f and g of a space X onto itself are **isotopic** if there exists a homotopy $H : X \times \mathbb{I} \to X$ joining f and g such that for every $t \in \mathbb{I}$ the map $H_t : X \to X$ given by $H_t(x) = H(x,t)$, $x \in X$, is a homeomorphism. Such an H is called an **isotopy** joining f and g. If a homeomorphism $h : X \to X$ is isotopic to the identity map id_X, then we call an isotopy joining h and id_X an **ambient isotopy**. Embeddings f and g of a space Y into a space X is **isotopically equivalent** if there exists an ambient isotopy $h : X \to X$ such that $h \circ f = g$.

2. Homotopy type of spaces

For spaces X and Y, if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$, we say that X and Y have the same **homotopy type** or that they are **homotopy equivalent**. Moreover these f and g are called **homotopy equivalence**s and g is a **homotopy inverse** of f. We denote the equivalence by f: $X \simeq Y$ or simply $X \simeq Y$. For a homotopy equivalence $f : X \to Y$ and a space Z the maps $f_* : [Z, X] \to [Z, Y]$ and $f^* : [Y, Z] \to [X, Z]$ are bijections. Therefore in homotopy theory spaces having the same homotopy type may be identified. If we have maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$, then we say that X is **homotopy dominated** by Y or Y **homotopy dominates** X. We use the same notations when we consider n-tuples of spaces and pointed spaces (by considering suitable maps and homotopies).

A compact space has the homotopy type of a CW-complex (not necessarily compact) if and only if it is homotopy dominated by a finite CW-complex. For a space X which is homotopy dominated by a finite CW-complex, Wall [3] defined an obstruction, called **finiteness obstruction**, $[X] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ such that $[X] = 0$ if and only if X is homotopy equivalent to a finite CW-complex. Moreover, for a finitely presented group G, every nontrivial element $\sigma \in \widetilde{K}_0(\mathbb{Z}[G])$ is a finiteness obstruction of a countable CW-complex X which is homotopy dominated by a finite CW-complex and $\pi_1(X) = G$ (see 3 of this section for the definition of fundamental groups).

Let A be a subspace of a space X and let $i : A \to X$ be the inclusion map. If there exists a map $r : X \to A$ such that $r \upharpoonright A = \mathrm{id}_A$, then A is a **retract** of X and we call such a map r a **retraction** of X to A. Moreover, if $i \circ r \simeq \mathrm{id}_X$ (or $i \circ r \simeq \mathrm{id}_X$ rel A), then A is a **deformation retract** (or **strong deformation retract**) of X. A space is said to be **contractible to a point** (or simply a **contractible space**) if some point is a deformation retract of it. Convex subsets and star-shaped subsets in a topological linear space are contractible. If a space is contractible to a point then it is arcwise connected and contractible to any point.

A metric space X is an **absolute neighborhood retract** for metrizable spaces, shortly ANR, if for every closed embedding $h : X \to Y$ to a metric space Y, there exist a neighbourhood U of $h(X)$ and a retraction $r : U \to h(X)$. Compact ***topological manifold***s and ***polyhedra*** are ANR. A compact ANR in the Hilbert cube has arbitrarily small neighbourhoods which have the form of the product of polyhedra and a copy of the Hilbert cube and admit retractions onto the original space. A long-standing problem posed by Borsuk if a compact ANR is homotopy equivalent to a finite CW-complex was affirmatively solved by West [4]. See C.10 for more information.

3. Fundamental groups

The idea of homotopy groups appeared in Poincaré's paper of 1895. He defined $\pi_1(X, x_0)$ of a pointed space X with a base point x_0 as the set $[\mathbb{I}, \dot{\mathbb{I}}; X, x_0]$. We call a map $\omega : \mathbb{I} \to X$ from the unit interval to a space X a **path** in X. $f(0)$ and $f(1)$ are called the **initial point** and **terminal point** of f, respectively. In particular, if $f(0) = f(1) = x_0$, we call such a path f a **loop** at x_0. For a path $f : \mathbb{I} \to X$, the path $\bar{f} : \mathbb{I} \to X$ defined by $\bar{f}(t) = f(1-t)$ is called the **inverse path** of f. For paths $f_1, f_2 : \mathbb{I} \to X$ with $f_1(1) = f_2(0)$ we define a path $f_1 * f_2 : \mathbb{I} \to X$, called the **product** of f_1 and f_2, by

$$(f_1 * f_2)(t) = \begin{cases} f_1(2t), & 0 \leqslant t \leqslant 1/2, \\ f_2(2t-1), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

For two loops f_1, f_2 at x_0, we can always define the product and $(\pi_1(X, x_0), *)$ is a group, which is called the **fundamental group** of the pointed space (X, x_0). Then the identity element of $\pi_1(X, x_0)$ is the homotopy class of the constant loop c_{x_0} at x_0, and the inverse element of $[f] \in \pi_1(X, x_0)$ is the homotopy class of the inverse loop $\bar{f}$. For a path $\omega : \mathbb{I} \to X$, $\omega(0) = x_0$, $\omega(1) = x_1$, and an element $\alpha = [f] \in \pi_1(X, x_0)$, the homotopy class $[\bar{\omega} * f * \omega] \in \pi_1(X, x_1)$ depends on only homotopy classes of ω and f. Thus, we have the function $h_{[\omega]} : \pi_1(X, x_0) \to \pi_1(X, x_1)$. In fact, $h_{[\omega]}$ is an isomorphism and if ω is a loop at x_0, then $h_{[\omega]}$ is conjugation by $[\omega]$. Hence, if X is arcwise connected, then for any two points $x_0, x_1 \in X$, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ and we can simply write $\pi_1(X)$. If X is arcwise connected and $\pi_1(X)$ is trivial, we say that X is **simply connected**. For example, All contractible spaces are simply connected. Every n-sphere S^n, $n \geqslant 2$, is simply connected. A **trivial group** is one with just one element, denoted 1 (or 0 in case of Abelian groups); a **trivial homomorphism** maps its domain to the unit element.

The following is one of the most important tools for calculating fundamental groups.

Seifert and van Kampen theorem
Assume that U and V are arcwise connected open subsets of a space X such that $X = U \cup V$ and $U \cap V$ is nonempty and arcwise connected. Choose a base point $x_0 \in U \cap V$. Let $i_{1*} : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $i_{2*} : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$ be the induced homomorphisms by the inclusion maps $i_1 : (U \cap V, x_0) \to (U, x_0)$ and $i_2 : (U \cap V, x_0) \to (V, x_0)$. Then $\pi_1(X, x_0)$ is isomorphic to the amalgamated product of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ over $\pi_1(U \cap V, x_0)$.

Here let G, T_1 and T_2 be groups, and suppose that we have homomorphisms $f_1 : G \to T_1$ and $f_2 : G \to T_2$. The

amalgamated product of T_1 and T_2 over G is essentially defined as the smallest group generated by T_1 and T_2 with the relations $f_1(x) = f_2(x)$ for $x \in G$. Indeed, let F be the free group generated by the set $T_1 \cup T_2$. We write $x \cdot y$ for the product in F. Thus every element of F is of the form $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $\varepsilon_i = \pm 1$ and $x_i \in T_1 \cup T_2$, $i = 1, \dots, n$. Consider the words $(xy)^1 \cdot y^{-1} \cdot x^{-1}$ defined if both x and y belong to either T_1 or T_2, and $f_1(g)^1 \cdot (f_2(g))^{-1}$ for $g \in G$. Let R be the normal subgroup generated by these words. Then the amalgamated product of T_1 and T_2 over G, written $T_1 *_G T_2$, is the quotient group F/R.

Let X be a pointed space. The **Hurewicz homomorphism** $\varphi_1 : \pi_1(X) \to H_1(X)$ is defined by

$$\varphi_1\big([f]\big) = f_*(\iota_1) \quad \text{for } [f] \in \pi_1(X),$$

where $\iota_1 \in H_1(S^1)$ is a generator. Then we have

Hurewicz isomorphism theorem
If X is arcwise connected, the Hurewicz homomorphism $\varphi_1 : \pi_1(X) \to H_1(X)$ is an epimorphism (surjective homomorphism) and the kernel is the commutator subgroup of $\pi_1(X)$. Thus, $H_1(X)$ is isomorphic to the Abelianization of $\pi_1(X)$.

Here for a group G the subgroup of G generated by the set $\{aba^{-1}b^{-1}\colon a, b \in G\}$ is called the **commutator subgroup** of G and denoted by G'. Then G' is a normal subgroup of G and the quotient group G/G' is Abelian. If H is a normal subgroup of G, then G/H is Abelian if and only if H contains G'. Hence we call the quotient group G/G' the **Abelianization** of G.

EXAMPLE. The fundamental group $\pi_1(S^1)$ is $\mathbb{Z}$. For pointed spaces (X, x_0) and (Y, y_0) one has $\pi_1((X, x_0) \times (Y, y_0)) \cong \pi_1(X, x_0)) \oplus \pi_1(Y, y_0)$. Therefore $\pi_1((S^1)^n) \cong \bigoplus_n \mathbb{Z}$. For any group G there exists a 2-dimensional CW-complex K such that $\pi_1(K) = G$. For the orientable compact closed surface M of genus n, $\pi_1(M)$ is the group generated by $2n$ elements $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ with one relation $\prod_{k=1}^n a_k b_k a_k^{-1} b_k^{-1} = 1$. Its Abelianization is isomorphic to $\bigoplus_{2n} \mathbb{Z}$.

4. Covering spaces

Let X be a space. A **covering space** of X is a pair consisting of an arcwise connected space $\widetilde{X}$ and a map $p : \widetilde{X} \to X$ such that the following condition holds: each point $x \in X$ has an open neighbourhood U such that $p^{-1}(U)$ is the disjoint union of open subsets of $\widetilde{X}$ each of which is mapped homeomorphically onto U by p. Then U is said to be **evenly covered** by p. X is called the **base space** of the covering space. The map p is often called a **covering projection**. For a map $f : Y \to X$ there may exist a map $\tilde{f} : Y \to \widetilde{X}$ such that $p \circ \tilde{f} = f$. We call such a map $\tilde{f}$ a **lifting** of f.

Let $(\widetilde{X}, p)$ be a covering space of X. Let $f : (\mathbb{I}, 0) \to (X, x_0)$ be a path. For a point $\tilde{x}_0 \in f^{-1}(x_0)$, there exists a unique path $\tilde{f} : (\mathbb{I}, 0) \to (\widetilde{X}, \tilde{x}_0)$ with $p \circ \tilde{f} = f$ (**path lifting property**). Let $f, g : \mathbb{I} \to X$ be paths in X from x_0 to x_1 such that $f \simeq g$ rel $\dot{\mathbb{I}}$. Let $\tilde{x}_0$ be in $p^{-1}(x_0)$ and let $\tilde{f}, \tilde{g} : \mathbb{I} \to \widetilde{X}$ be liftings of f, g, respectively, with $\tilde{f}(0) = \tilde{x}_0 = \tilde{g}(0)$. Then $\tilde{f} \simeq \tilde{g}$ rel $\dot{\mathbb{I}}$. In particular, $\tilde{f}(1) = \tilde{g}(1)$ (**monodromy theorem**). Hence $p_* : \pi_1(\widetilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is a ***monomorphism*** (injective homomorphism).

A standard example of covering projections is the map $p : \mathbb{R} \to S^1$ defined by $p(t) = (\cos(2\pi t), \sin(2\pi t))$. Applying above properties to this covering projection, we have that $\pi_1(S^1) = \mathbb{Z}$.

Let G be a topological group. Note that any group becomes a topological group when it is given the discrete topology. A **left action** of G on a space X is a map $\varphi : G \times X \to X$ such that (using $g \cdot x$ to denote $\varphi(g, x)$) the following are satisfied:

(1) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G, x \in X$,
(2) $1 \cdot x = x$ for all $x \in X$, where 1 denotes the identity of G.

If $g \cdot x = x$ for all $g \in G$ and $x \in X$, we say that G acts **trivially** on X.

For $x \in X$ the **orbit** of x is the set $ob(x) = \{g \cdot x\colon g \in G\}$. We say that G acts **transitively** on X if, for each $x, x' \in X$, there exists $g \in G$ such that $g \cdot x = x'$. The **isotropy subgroup** of $x \in X$ is the subgroup $G_x = \{g \in G\colon g \cdot x = x\}$. Then the cardinality of $ob(x)$ is equal to that of the set of distinct right cosets of G_x in G which is called the index of G_x in G. Note that G acts transitively on X if and only if $ob(x) = X$ for every $x \in X$. Hence, if G acts transitively on X, the cardinality of X is equal to the index of G_x in G. If $G_x = \{1\}$ for all $x \in X$, we say that G acts **freely** on X. If $\{g \in G\colon g \cdot x = x \text{ for all } x \in X\} = \{1\}$, we say that G acts **effectively** on X.

A space with an action of G on the space is called a **G-space**. Let X be a G-space. By X/G we denote the space of equivalence classes of X under $x \sim g \cdot x$, with the quotient topology and call it the **orbit space** of X by G. Suppose that X is a G-space such that for each $x \in X$ there exists an open neighbourhood V_x such that $V_x \cap (g \cdot V_x) = \emptyset$ for all $g \neq 1$ in G. We say that G acts **properly** on X. Then the quotient map $p : X \to X/G$ is a covering projection. In fact, p is regular (see below for the definition).

We shall give important examples of proper G-action. The action $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ given by $t \cdot x = t + x$ induces the covering projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ mentioned above. An important example is the action $\mathbb{Z}_2 \times S^n \to S^n$, where the nontrivial element of $\mathbb{Z}_2 = \{1, -1\}$ acts by $(-1) \cdot x = -x$. This induces the covering projection $S^n \to S^n/\mathbb{Z}_2$. We call the orbit space the **real projective n-space** and denote it by $\mathbb{RP}^n$. Next, let p and q be relatively prime integers. Regard S^3 as the set $\{(z_0, z_1) \in \mathbb{C}^2\colon |z_0|^2 + |z_1|^2 = 1\}$. Let ξ be the primitive pth root of the unity. We define the action of $\mathbb{Z}_p$ on S^3 by

$$[k] \cdot (z_0, z_1) = \big(\xi^k z_0, \xi^{qk} z_1\big)$$

for $[k] \in \mathbb{Z}_p$ and $(z_0, z_1) \in S^3$. Then we have the covering projection $S^3 \to S^3/\mathbb{Z}_p$. The orbit space $S^3/\mathbb{Z}_p$ is called the

Lens space of type (p, q) and is denoted by $L(p, q)$. Note that $L(1, 1) = S^3$ and $L(2, 1) = \mathbb{RP}^3$.

We shall give an example of G-action on S^{2n+1} which is not proper. Regard S^{2n+1} as the set $\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \dots + |z_n|^2 = 1\}$. We define the action of S^1 on S^{2n+1} by

$$z \cdot (z_0, \dots, z_n) = (zz_0, \dots, zz_n) \quad \text{for } z \in S^1 \quad \text{and} \quad (z_0, \dots, z_n) \in S^{2n+1}.$$

Then we call the orbit space S^{2n+1}/S^1 the **complex projective** n**-space** and denote it by $\mathbb{CP}^n$.

Let $(\widetilde{X}, p)$ be a covering projection of an arcwise connected space X and let $x_0 \in X$. For each $\alpha = [f] \in \pi_1(X, x_0)$ and $\tilde{x} \in p^{-1}(x_0)$, by path lifting property and monodromy theorem, we can define $\alpha \cdot \tilde{x} = \tilde{f}(1)$, where $\tilde{f}$ is the lifting of f with $\tilde{f}(0) = \tilde{x}$. This induces a transitive action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$. The isotropy group of $\tilde{x}_0 \in p^{-1}(x_0)$ is isomorphic to $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ and therefore the cardinality of $p^{-1}(x_0)$ is equal to the index of $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$. Hence for any points $x_0, x_1 \in X$ the fibers $p^{-1}(x_0)$, $p^{-1}(x_1)$ have the same cardinality. In fact, any two fibers are homeomorphic because each fiber is discrete. Therefore we have the definition: the **multiplicity** of $(\widetilde{X}, p)$ is the cardinality of a fiber. If the multiplicity is m, we say that $(\widetilde{X}, p)$ is an m**-sheeted covering** of X or $(\widetilde{X}, p)$ is an m**-fold covering** of X. By this observation, we have that, if $n \geqslant 2$, $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$. Similarly, as the multiplicity of the covering projection $S^3 \to S^3/\mathbb{Z}_p = L(p, q)$ is p for any pair of relatively prime integers p and q, $\pi_1(L(p, q)) \cong \mathbb{Z}_p$.

For existence of lifting we have the following criterion. Let Y be a connected and locally arcwise connected space and let $f : (Y, y_0) \to (X, x_0)$ be a map. If $(\widetilde{X}, p)$ is a covering space of X, then there exists a unique lifting $\tilde{f} : (Y, y_0) \to (\widetilde{X}, \tilde{x}_0)$ of f, where $\tilde{x}_0 \in p^{-1}(x_0)$, if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ (**lifting criterion**).

A **universal covering space** of a space X is a covering space $(\widetilde{X}, p)$ with $\widetilde{X}$ simply connected. $\mathbb{R}$ is a universal covering space of S^1, and the plane $\mathbb{R}^2$ is a universal covering space of the torus. If $n \geqslant 2$, S^n is a universal covering space of $\mathbb{RP}^n$. By lifting criterion, for any two universal covering projections $p : \widetilde{X} \to X$ and $q : \tilde{Y} \to X$, there exists a homeomorphism $h : \widetilde{X} \to \tilde{Y}$ such that $q \circ h = p$.

A space X is **semilocally** 1**-connected** if each $x \in X$ has an open neighbourhood U so that the induced homomorphism $i_* : \pi_1(U, x) \to \pi_1(X, x)$ by the inclusion map $i : U \to X$ is trivial. Let (X, x_0) be a pointed space and let G be a subgroup of $\pi_1(X, x_0)$. If X is connected, locally arcwise connected and semilocally 1-connected, then there exists a covering space $(\widetilde{X}_G, p)$ of X such that $p_*(\pi_1(\widetilde{X}_G, \tilde{x}_0)) = G$. Hence a connected and locally arcwise connected space has a universal covering if and only if it is semilocally 1-connected. It follows that every connected ANR, therefore every connected CW-complex and every connected n-manifold, has a universal covering.

For a covering space $(\widetilde{X}, p)$ of a space X, a **covering transformation** or a **deck transformation** is a homeomorphism $h : \widetilde{X} \to \widetilde{X}$ with $p \circ h = p$. By $G(\widetilde{X}|X)$ we denote the set of all covering transformations of $\widetilde{X}$. $G(\widetilde{X}|X)$ is a group under composition of homeomorphisms and acts on $p^{-1}(x_0)$ for every $x_0 \in X$. Suppose that X is connected and locally arcwise connected. $G(\widetilde{X}|X)$ acts transitively on $p^{-1}(x_0)$ if and only if $p_*(\pi_1(\widetilde{X}, \tilde{x}_0)) = p_*(\pi_1(\widetilde{X}, \tilde{x}_1))$ for all $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$. Such a covering space $(\widetilde{X}, p)$ is said to be a **regular covering space**. A covering space $(\widetilde{X}, p)$ is regular if and only if $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$. If $(\widetilde{X}, p)$ is a regular covering of a connected and locally arcwise connected space, then for $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$, $G(\widetilde{X}|X) \cong \pi_1(X, x_0)/p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$. Hence, if $\pi_1(X, x_0)$ is Abelian, $G(\widetilde{X}|X) \cong \pi_1(X, x_0)/p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$. In particular, if $(\widetilde{X}, p)$ is a universal covering, $G(\widetilde{X}|X) \cong \pi_1(X, x_0)$.

Here covering spaces serve mainly as a tool for calculating fundamental groups. However they are important in their own right, particularly in the study of Riemann surfaces and complex manifolds.

5. Higher homotopy groups

Next, generalizing the idea of fundamental groups, Hurewicz introduced higher homotopy groups. Let X be a pointed space with a base pointx_0 and let n be an arbitrary natural number. Let $\mathbb{I}^n$ be the n-cube,

$$\mathbb{I}^n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid 0 \leqslant t_1, \dots, t_n \leqslant 1\},$$

and let $\dot{\mathbb{I}}^n$ be its boundary. By $\pi_n(X, x_0)$ or $\pi_n(X)$ we denote the set $[\mathbb{I}^n, \dot{\mathbb{I}}^n; X, x_0]$. We may define an addition in $\pi_n(X, x_0)$ as follows. For any two maps $f_1, f_2 : (\mathbb{I}^n, \dot{\mathbb{I}}^n) \to (X, x_0)$, their sum $f_1 + f_2$ is the map defined by

$$(f_1 + f_2)(t) = \begin{cases} f_1(2t_1, t_2, \dots, t_n), & 0 \leqslant t_1 \leqslant 1/2, \\ f_2(2t_1 - 1, t_2, \dots, t_n), & 1/2 \leqslant t_1 \leqslant 1, \end{cases}$$

for every point $(t_1, t_2, \dots, t_n) \in \mathbb{I}^n$. Obviously $f_1 + f_2$ is a map from $(\mathbb{I}^n, \dot{\mathbb{I}}^n)$ to (X, x_0). Then the composition of $[f_1], [f_2] \in \pi_n(X, x_0)$ is defined by

$$[f_1] + [f_2] = [f_1 + f_2].$$

This composition makes $\pi_n(X, x_0)$ a group which is called the n**-th (absolute) homotopy group** of X at x_0. The identity element is the homotopy class of the constant map. The inverse element of $[f]$ is the homotopy class of the map $\bar{f}$ defined by $\bar{f}(t) = f(1 - t_1, t_2, \dots, t_n)$.

Let $S^n = \{(t_1, \dots, t_{n+1}) \in \mathbb{R}^{n+1} : t_1^2 + \dots t_{n+1}^n = 1\}$ be the n-sphere and $z_0 = (1, 0, \dots, 0)$ be the base point. Then there exists a relative homeomorphism $\psi : (\mathbb{I}^n, \dot{\mathbb{I}}^n) \to (S^n, z_0)$ and the induced function $\psi_* : [S^n; X]_0 \to \pi_n(X)$ is a bijection. Therefore we can identify $[S^n; X]_0$ with $\pi_n(X)$. Then the group structure on $[S^n; X]_0$ is the same as the one induced by the H'-group structure of S^n. Hence $\pi_n(X)$, $n \geqslant 2$, is an Abelian group (see 2 of the next section).

Let (X, A, x_0) be a pointed pair. We identify $\mathbb{I}^{n-1}$ with the set $\{(t_1, \dots, t_n) \in \mathbb{I}^n : t_n = 0\}$ and define $\mathbb{J}^{n-1}$ as the closure of $\dot{\mathbb{I}}^n \setminus \mathbb{I}^{n-1}$. By $\pi_n(X, A, x_0)$ or $\pi_n(X, A)$ we denote the set $[\mathbb{I}^n, \dot{\mathbb{I}}^n, \mathbb{J}^{n-1}; X, A, x_0]$. If $n \geqslant 2$, by using the composition defined in the above, we can define a group structure on $\pi_n(X, A, x_0)$, called the n-th **relative homotopy group** of (X, A). In particular, if $n \geqslant 3$, $\pi_n(X, A, x_0)$ is an Abelian group.

We can identify $\pi_n(X, x_0)$ with $\pi_n(X, \{x_0\}, x_0)$. Hence we are discussing absolute homotopy groups as special ones of relative homotopy groups.

Let $B^n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_1^2 + \dots + t_n^n \leqslant 1\}$ be the n-ball. Then the boundary is S^{n-1}. There exists a relative homeomorphism $\psi : (\mathbb{I}^n, \mathbb{J}^{n-1}) \to (B^n, z_0)$. Then $\psi(\mathbb{I}^{n-1}) = S^{n-1}$. Hence ψ induces the bijection $\psi^* : [B^n, S^{n-1}, z_0;\ X, A, x_0] \to \pi_n(X, A, x_0)$.

Let $\varphi : (X, A, x_0) \to (Y, B, y_0)$ be a map. The function $\varphi_* : \pi_n(X, A, x_0) \to \pi_n(Y.B, y_0)$ given by $\varphi([f]) = [\varphi \circ f]$ is a group homomorphism if $n \geqslant 2$ or $n = 1$ and $A = \{x_0\}$. In these cases we call φ_* the **induced homomorphism** by φ. If φ is a homotopy equivalence, φ_* is an isomorphism for every n.

Let G be a given group and let n be a positive integer. Suppose that G is an Abelian group if $n \geqslant 2$. There exists an arcwise connected CW-complex K such that

$$\pi_i(K) = \begin{cases} G & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

We call such a CW-complex an **Eilenberg–MacLane space** of type (G, n) and denote it by $K(G, n)$. The homotopy type of an Eilenberg–MacLane space $K(G, n)$ is unique. The **infinite-dimensional real projective space** $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}_2, 1)$, the infinite-dimensional complex projective space $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$, and the 1-sphere S^1 is a $K(\mathbb{Z}, 1)$.

Let (X, A, x_0) be a pointed pair. For $\alpha = [f] \in \pi_n(X, A, x_0)$, the correspondence $\partial(\alpha) = [f \restriction \mathbb{I}^{n-1}] \in \pi_{n-1}(A, x_0)$ induces a homomorphism $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ which is called the **boundary homomorphism**. Then we have the exact sequence

$$\begin{aligned} \cdots &\xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \\ &\xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \cdots \\ \cdots &\xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{j_*} \pi_1(X, A, x_0) \\ &\xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0), \end{aligned}$$

which we call the **homotopy exact sequence** of the pointed pair (X, A, x_0). Here a sequence of groups

$$\cdots 0 \to G_{n-1} \xrightarrow{f_n} G_n \xrightarrow{f_{n+1}} G_{n+1} \to \cdots$$

is an **exact sequence** if $\operatorname{Im} f_n = \operatorname{Ker} f_{n+1}$ for every n.

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k-8 Homotopy, II

1. Homotopy extension property

Let (X, A) be a pair and let Y be a space. When two homotopic maps $f_0, f_1 : A \to Y$ admit continuous extensions $\tilde{f}_0, \tilde{f}_1$ over X, we may ask whether or under what condition the extensions can be homotopic. This question suggests the **homotopy extension problem**: let $f : X \to Y$ be a map, and let $F : A \times \mathbb{I} \to Y$ be a homotopy of $f \restriction A$ to a map $f' : A \to Y$. Then f and F define a map

$$H : (A \times \mathbb{I}) \cup (X \times \{0\}) \to Y$$

and we may ask whether H admits a continuous extension over $X \times \mathbb{I}$.

When the question is affirmatively solved for every space Y and map H, we say that the pair (X, A) has the **homotopy extension property** or the inclusion map $i : A \to X$ is a **cofibration**. For example, a pair (K, L) of a CW-complex K and its subcomplex L and a pair (X, A) of an ANR X and its closed subset A which is an ANR have the homotopy extension property.

On the other hand, when the question is affirmatively solved for every map H, we say that Y has the **homotopy extension property with respect to** (X, A). Every ANR has the homotopy extension property with respect to all pairs (X, A) of a normal and countably paracompact space X and its closed subset A. This homotopy extension property introduced an important notation in general topology called ***Dowker spaces***. See [MN] for related topics.

2. Group-structures in $[X; Y]_0$

If Y is a ***topological group*** then for any space X, the set $[X; Y]$ has a group-structure induced by the group-structure of Y. For example, for a continuum X, many authors have investigated the group $[X; S^1]$, which is called **Bruschlinsky group** [KII, Chapter 8]. In fact, it is the same as the one-dimensional ***Čech cohomology group*** $\check{H}^1(X)$.

We shall consider pointed spaces Y such that $[X; Y]_0$ admits a group-structure for all pointed spaces X. A pointed space Y (with a base point y_0) is an H**-group** if there are pointed maps $\mu : Y \times Y \to Y$ and $\eta : Y \to Y$ such that

$\mu \circ (\mathrm{id}_Y \times \mu) \simeq \mu \circ (\mu \times \mathrm{id}_Y)$ (homotopy associativity),
$\mu \circ j_1 \simeq \mathrm{id}_Y \simeq \mu \circ j_2$ (existence of homotopy identity),
$\mu \circ (\mathrm{id}_Y, \eta) \simeq c \simeq \mu \circ (\eta, \mathrm{id}_Y)$ (existence of homotopy inverse),

where $j_1, j_2 : Y \to Y \times Y$ are the injections defined by $j_1(y) = (y, y_0)$ and $j_2(y) = (y_0, y)$, and $c : Y \to Y$ is the constant map at y_0. Then for a pointed space X we have a group-structure on the set $[X; Y]_0$ by the formula

$$[f_1] * [f_2] = \bigl[\mu \circ (f_1, f_2)\bigr].$$

Here for maps $f : X \to Y$ and $g : X \to Z$, we define the map $(f, g) : X \to Y \times Z$ by $(f, g)(x) = (f(x), g(x))$ for $x \in X$. If a pair (Y, μ) satisfies only existence of homotopy identity then we call it an H**-space**.

Topological groups can be regarded as a pointed spaces by choosing the identity as a base point and they are H-groups. As another typical H-group, we introduce loop spaces. For a pointed space (Y, y_0) the **loop space** (based at y_0), denoted by $\Omega(Y)$ or $\Omega(Y, y_0)$, is defined to be the space of maps $\omega : (\mathbb{I}, \dot{\mathbb{I}}) \to (Y, y_0)$ topologized by the compact-open topology, where $\dot{\mathbb{I}} = \{0, 1\}$. $\Omega(Y)$ is regarded as a pointed space with the base point ω_0 equal to the constant map of $\mathbb{I}$ to y_0. Then there exists a map $\mu : \Omega(Y) \times \Omega(Y) \to \Omega(Y)$ defined by

$$\mu(\omega, \omega')(t) = \begin{cases} \omega(2t), & 0 \leqslant t \leqslant 1/2, \\ \omega'(2t-1), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

By this multiplication $\Omega(Y)$ is an H-group. For an Abelian group G and a positive integer $n \geqslant 1$ we can take an Eilenberg–MacLane space $K(G, n)$ as an H-group. Note that Adams [1] showed that only in case of $n = 1$, 3 or 7 the n-spheres are H-spaces. In fact, S^1 and S^3 are topological groups but S^7 is not a topological group.

As its dual notation we have the following one. If (X, x_0) and (Y, y_0) are pointed spaces, their **pointed sum** is denoted by $X \vee Y$. Here $X \vee Y$ is regarded as the subspace $X \times \{y_0\} \cup \{x_0\} \times Y$ of the pointed space $(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0))$. For a pointed space Z and pointed maps $f : X \to Z$, $g : Y \to Z$, let $(f, g) : X \vee Y \to Z$ be the map defined by $(f, g) \restriction X = f$ and $(f, g) \restriction Y = g$.

A pointed space (X, x_0) is an H'**-group** if there are pointed maps $\nu : X \to X \vee X$ and $h : X \to X$ such that

$(\mathrm{id}_X \vee \nu) \circ \nu \simeq (\mu \vee \mathrm{id}_X) \circ \nu$ (homotopy associativity),
$q_1 \circ \nu \simeq \mathrm{id}_X \simeq q_2 \circ \nu$ (existence of homotopy identity),
$(\mathrm{id}_X, h) \circ \nu \simeq c \simeq (h, \mathrm{id}_X) \circ \nu$ (existence of homotopy inverse).

Then for a pointed space Y the composition in $[X, Y]_0$ defined by

$$[f_1] \cdot [f_2] = \bigl[(f_1, f_2) \circ \nu\bigr]$$

gives a group-structure.

Let (Z, z_0) be a pointed space. The **reduced suspension** of Z, denoted by $\Sigma(Z)$, is defined to be the quotient space of $Z \times \mathbb{I}$ in which $(Z \times \dot{\mathbb{I}}) \cup (\{z_0\} \times \mathbb{I})$ is identified to a single point. If $(z,t) \in Z \times \mathbb{I}$, we use $[z,t]$ to denote the corresponding point of $\Sigma(Z)$ under the quotient map $Z \times \mathbb{I} \to \Sigma(Z)$. The point $[z_0, 0] \in \Sigma(Z)$ is also denoted by z_0, and $\Sigma(Z)$ is regarded as a pointed space with the base point z_0. We define a map $\nu : \Sigma(Z) \to \Sigma(Z) \vee \Sigma(Z)$ by the formula

$$\nu([z,t]) = \begin{cases} ([z,2t], z_0), & 0 \leqslant t \leqslant 1/2, \\ (z_0, [z, 2t-1]), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then $\Sigma(Z)$ is an H'-group.

In particular, the n-sphere S^n, $n \geqslant 1$, is an H'-group. Hence for every pointed space Y the set $[S^n; Y]_0$ admits a group-structure. If X is an H'-group and Y is an H-space, then the operation on $[X; Y]_0$ determined by the H'-structure on X coincides with the operation induced the H-structure of Y and the group $[X; Y]_0$ is Abelian. Therefore, if $n \geqslant 2$, $[S^n, Y]_0$ admits an Abelian group-structure.

As we noted above, the n-sphere S^n need not be an H-group. However, Borsuk gave an idea to induce a group-structure in $[X; S^n]_0$, and Spanier gave the name, **cohomotopy group**, and investigated duality to homotopy groups and relation with cohomology groups. Suppose that a pointed metric space X has dimension $\leqslant 2n-2$. For two maps $f, g : X \to S^n$ the map $(f,g) : X \to S^n \times S^n$ is homotopic to a map $h : X \to S^n \vee S^n$. Then by the homotopy class of the composition $q \circ h : X \to S^n$ we can define an addition $[f] + [g]$, where the map $q : S^n \vee S^n \to S^n$ is defined by $q(z, z_0) = z = q(z_0, z)$ for $z \in S^n$. We denote the nth cohomotopy group of X by $\pi^n(X)$.

3. Lifting problems and fibrations

The homotopy extension problem suggests a dual problem, called the **lifting problem**: Let (X, A) be a pair and let $p : Y \to B$ be a map. For given maps $f : X \to B$ and $g : A \to Y$ such that $p \circ g = f \restriction A$, we may ask whether there exists a map $\tilde{f} : X \to Y$ such that $p \circ \tilde{f} = f$ and $\tilde{f} \restriction A = g$.

When the problem has an affirmative solution for a pair (X, A), p is called a **soft map with respect to** (X, A). If p is soft with respect to all pairs (X, A) of a metric space X (or a metric space X with dim $X \leqslant n$) and its closed subset A then p is called a **soft map** (or an n**-soft map**). In case when both Y and B are complete metric spaces the map p is 0-soft if and only if it is an open map. Using (n-)soft maps Shchepin [5] and Dranishnikov [3] found interesting results in dimension theory and metrization problems.

The lifting problem in case of a pair $(X \times \mathbb{I}, X \times \{0\})$ and maps $f : X \times I \to B$ and $g : X \times \{0\} \to Y$ is called the **homotopy lifting problem**. When this problem has an affirmative solution for every space X (or for every $\mathbb{I}^n$, $n \geqslant 1$) and pair of maps f and g, we say that p is a **fibration** (or **weak fibration**). These notations are due to Hurewicz and Serre, respectively.

If $p : Y \to B$ is a (weak) fibration, the **fiber** over $b \in B$ is the set $F_b = p^{-1}(b)$. If B is arcwise connected, all the fibers F_b have the same homotopy type. If $p : Y \to B$ is a weak fibration, $b \in B$ and $e \in F = F_b$, p induces an isomorphism p_*: $\pi_n(Y, F, e) \cong \pi_n(B, b)$ for all $n \geqslant 1$. Hence we can define $\bar{\partial} : \pi_n(B, b) \to \pi_{n-1}(F, e)$, $n \geqslant 1$, to be the composition

$$\pi_n(B, b) \xrightarrow{p_*^{-1}} \pi_n(Y, F, e) \xrightarrow{\partial} \pi_{n-1}(F, e).$$

Then we have the exact homotopy sequence of the weak fibration:

$$\begin{aligned} \cdots \longrightarrow \pi_n(F, e) &\xrightarrow{i_*} \pi_n(Y, e) \xrightarrow{p_*} \pi_n(B, b) \\ &\xrightarrow{\bar{\partial}} \pi_{n-1}(F, e) \longrightarrow \cdots \\ \cdots \longrightarrow \pi_1(B, b) &\xrightarrow{\bar{\partial}} \pi_0(F, e) \\ &\xrightarrow{i_*} \pi_0(Y, e) \xrightarrow{p_*} \pi_0(Y, e). \end{aligned}$$

The most useful generalization of covering projections is the following. A **fiber bundle** with **fiber** F is a map $p : E \to B$ for which there exists an open covering $\mathcal{V}$ of B and homeomorphisms $\varphi_V : V \times F \to p^{-1}(V)$ for all $V \in \mathcal{V}$ such that $p \circ \varphi_V(v, x) = v$ for all $(v, x) \in V \times F$. The open sets $V \in \mathcal{V}$ are called **coordinate neighbourhood**s. Note that a fiber bundle $p : E \to B$ with fiber F is a weak fibration, and if B is paracompact and Hausdorff, p is a fibration.

Every covering projection $p : \widetilde{X} \to X$ is a fiber bundle because all fibers $p^{-1}(x)$, $x \in X$, are homeomorphic discrete spaces, any of one of which we may denote by F. Let B and F be spaces. Let $E = B \times F$, and let $p : E \to B$ be the projection. Then $p : E \to B$ is a fiber bundle. In this case $p : E \to B$ is called the **trivial bundle**.

The natural projection $p : S^{2n+1} \to \mathbb{CP}^n$ is a fiber bundle with fiber S^1. We call the map p the **Hopf bundle**. Applying the exact homotopy sequence of such p, we have that $\pi_n(S^3) \cong \pi_n(S^2)$ for all $n \geqslant 3$. Especially, $\pi_3(S^2) \cong \mathbb{Z}$.

Let $p : E \to B$ and $f : D \to B$ be maps. A **pullback** of p and f is a triple (X, q, g), where $q : X \to D$ and $g : X \to E$ are maps, such that $f \circ q = p \circ g$, and for any tripe (Z, r, h) of maps $r : Z \to D$ and $h : Z \to E$ such that $f \circ r = p \circ h$ there exists a unique map $\theta : Z \to X$ such that $q \circ \theta = r$ and $g \circ \theta = h$. In fact, the pullback is topologically unique and given by the space

$$\widetilde{E} = \{(x, u) \in D \times E \colon f(x) = p(u)\}$$

and the coordinatewise projections $q : \widetilde{E} \to D$, $g : \widetilde{E} \to E$. If p is a fibration or a fiber bundle with fiber F, then q is also a fibration or a fiber bundle with fiber F, respectively. $q : \widetilde{E} \to D$ is called the **induced fibration** or **induced fiber bundle** of p by f, respectively. We sometimes call the space $\widetilde{E}$ the **fiber product** of p and f.

For a map $f : X \to Y$ let us take the path-fibration $p : Y^{\mathbb{I}} \to Y$ defined by $p(\omega) = \omega(1)$, where $Y^{\mathbb{I}}$ is the space of all paths in Y with the compact-open topology. Then we consider the pullback (F_p, q, g), where $F_p = \{(x, \omega) \in$

$X \times Y^{\mathbb{I}}$: $f(x) = \omega(1)\}$, and $q(x, \omega) = x$, $g(x, \omega) = \omega$ for $(x, \omega) \in F_p$. The composition $\pi = p \circ g : F_p \to B$ is a fibration and q has the **section** $s : X \to F_p$, i.e., $q \circ s = \mathrm{id}_X$, given by $s(x) = (x, c_{f(x)})$ such that $s \circ q \simeq \mathrm{id}_{F_p}$. In particular, q is a homotopy equivalence. Therefore every map $f : X \to Y$ has a factorization $X \xrightarrow{s} F_p \xrightarrow{\pi} Y$ such that s is a homotopy equivalence and π is a fibration.

4. Several remarkable theorems

Let (X, A) be a pair and (Z, C) be a pointed pair with a base point z_0. Let $\omega : \mathbb{I} \to A$ be a path in A. Two maps $f_0, f_1 : (Z, C) \to (X, A)$ with $f_0(z_0) = \omega(0)$ and $f_1(z_0) = \omega(1)$ is **ω-homotopic** if there is a homotopy

$$H : (Z, C) \times \mathbb{I} \to (X, A)$$

such that $H(z, 0) = f_0(z)$, $H(z, 1) = f_1(z)$ and $H(z_0, t) = \omega(t)$.

For any map $f_0 : (\mathbb{I}^n, \dot{\mathbb{I}}^n, \mathbb{J}^{n-1}) \to (X, A, \omega(0))$, by the homotopy extension property of $(\dot{\mathbb{I}}^n, \mathbb{J}^{n-1})$ and $(\mathbb{I}^n, \dot{\mathbb{I}}^n)$, there is a map $f_1 : (\mathbb{I}^n, \dot{\mathbb{I}}^n, \mathbb{J}^{n-1}) \to (X, A, \omega(1))$ such that f_0 is ω-homotopic to f_1. The homotopy class of f_1 depends on only homotopy classes of f_0 and ω. Hence we can define the function $h_{[\omega]} : \pi_n(X, A, \omega(0)) \to \pi_n(X, A, \omega(1))$ by $h_{[\omega]}([f_0]) = [f_1]$. In fact, $h_{[\omega]}$ is a homomorphism. Then $h_{[\omega * \omega']} = h_{[\omega']} \circ h_{[\omega]}$ for a pair of paths ω, ω' in A with $\omega(1) = \omega'(1)$, and for the constant path c_{x_0} at $x_0 \in A$, $h_{[c_{x_o}]} = \mathrm{id}_{\pi_n(X,A)}$. Hence $h_{[\omega]}$ is an isomorphism. Therefore, if A is arcwise connected, $\pi_n(X, A, x_0)$ does not depend on choosing base points and we can write $\pi_n(X, A)$ instead of $\pi_n(X, A, x_0)$. If $[\omega] \in \pi_1(A, x_0)$, then $h_{[\omega]}$ induces an action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$ and we specify it as the **action of** $\pi_1(A, x_0)$.

For a pair (X, A), if X, A are arcwise connected and $\pi_1(A)$ acts trivially on $\pi_n(X, A)$, then we say that (X, A) is **n-simple**. Similarly an arcwise connected space X is said to be **n-simple** if $\pi_1(X)$ acts trivially on $\pi_n(X)$. We usually say **simple** instead of 1-simple. An arcwise connected space X is simple if and only if $\pi_1(X)$ is Abelian. An arcwise connected H-space is n-simple for every $n \geqslant 1$. A space X is **n-connected** if $\pi_i(X) = 0$ for all $i = 0, 1, \ldots, n$. 0-connectedness is equivalent to arcwise connectedness and 1-connectedness is the same as ***simply connectedness***. Simply connected spaces are n-simple for every $n \geqslant 1$. A pair (X, A) is **n-connected** if both X and Y are arcwise connected and $\pi_i(X, A) = 0$ for all $i = 1, \ldots, n$. Contractible spaces are n-connected for all $n \geqslant 0$. Each S^n is $(n-1)$-connected and each (B^n, S^{n-1}), $n \geqslant 2$, is $(n-1)$-connected.

For a pointed pair (X, A), the **Hurewicz homomorphism** $\varphi : \pi_n(X, A) \to H_n(X, A)$ is defined by

$$\varphi([f]) = f_*(\iota_n) \quad \text{for } [f] \in \pi_n(X, A),$$

where ι_n is a generator of $H_n(\mathbb{I}^n, \dot{\mathbb{I}}^n)$.

Hurewicz isomorphism theorem
If a pair (X, A) is $(n-1)$-connected and n-simple, where $n \geqslant 2$, then $H_i(X, A) = 0$ for all $i = 0, 1, \ldots, n-1$, and $\varphi_n : \pi_n(X, A) \cong H_n(X, A)$.

In particular, if X is $(n-1)$-connected, where $n \geqslant 2$, then $H_i(X) = 0$ for all $i = 0, 1, \ldots, n-1$, $\varphi_n : \pi_n(X) \cong H_n(X)$, and $\varphi_{n+1} : \pi_{n+1}(X) \to H_{n+1}(X)$ is an epimorphism.

It follows that $\pi_n(S^n) \cong H_n(S^n) \cong \mathbb{Z}$. Note that for $n \geqslant 2$, $\pi_{n+1}(S^n) \neq 0$ although $H_i(S^n) = 0$ for all $i \geqslant n+1$. In fact, for $n \geqslant 2$, $\pi_i(S^n) \neq 0$ for infinitely many i [4].

A map $f : X \to Y$ is called an **n-equivalence** for $n \geqslant 1$ if f induces a bijective correspondence between the arcwise components of X and of Y and if for every $x \in X$, $f_* : \pi_i(X, x) \to \pi_i(Y, f(x))$ is an isomorphism for $i = 1, \ldots, n-1$ and $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an epimorphism. A map $f : X \to Y$ is called a **weak homotopy equivalence** if f is an n-equivalence for all $n \geqslant 1$.

Whitehead theorem I
Let $f : X \to Y$ be a map between arcwise connected spaces. If f is an n-equivalence, then $f_* : H_i(X) \to H_i(Y)$ is an isomorphism for $i = 1, \ldots, n-1$ and $f_* : H_n(X) \to H_n(Y)$ is an epimorphism.

Conversely, if X and Y are simply connected, and f_*: $H_i(X) \to H_i(Y)$ is an isomorphism for $i = 1, \ldots, n-1$ and f_*: $H_n(X) \to H_n(Y)$ is an epimorphism, then f is an n-equivalence.

Whitehead theorem II
A map $f : X \to Y$ is an n-equivalence if and only if

(1) for every CW-complex K with $\dim K < n$,

$$f_* : [K; X] \to [K; Y]$$

is an injection,

(2) for every CW-complex K with $\dim K \leqslant n$,

$$f_* : [K; X] \to [K; Y]$$

is a surjection.

Therefore, a map $f : K \to L$ between CW-complexes is a homotopy equivalence if and only if it is an n-equivalence and $\max\{\dim K, \dim L + 1\} \leqslant n$ or it is a weak homotopy equivalence.

For a pointed map $f : X \to Y$ we define the reduced suspension $\Sigma(f) : \Sigma(X) \to \Sigma(Y)$ by $\Sigma(f)([x, t]) = [f(x), t]$. This correspondence induces the homomorphisms

$$E : \pi_n(X) \to \pi_{n+1}(\Sigma(X)), \quad n = 1, 2, \ldots,$$

called the **suspension homomorphism**.

Freudenthal suspension theorem
Suppose that a pointed space (X, x_0) is n-connected with $n \geqslant 0$ and the pair $(X, \{x_0\})$ has the homotopy extension property. Then the suspension homomorphism $E : \pi_i(X) \to$

$\pi_{i+1}(\Sigma(X))$ is an isomorphism for every $i = 1, \ldots, 2n$ and is an epimorphism for $i = 2n+1$.

It follows that $\pi_4(S^3) \cong \pi_5(S^4) \cong \cdots \cong \mathbb{Z}_2$.

5. Obstruction theory

In order to clarify extendability of maps by algebraic words, Eilenberg introduced **obstruction theory**. Let (K, L) be a ***CW-pair*** with $K = K^{(n+1)} \cup L$ and $K^{(n)} \subseteq L$, and let Y be an arcwise connected and n-simple space. Let $f: L \to Y$ be a map. For an $(n+1)$-cell σ in $K \setminus L$ with the ***characteristic map*** φ_σ, the map $f \circ \varphi_\sigma \restriction S^n$ defines a certain element $c^{n+1}(f, \sigma)$ of $\pi_n(Y)$. This correspondence induces the $(n+1)$-dimensional ***cocycle*** $c^{n+1}(f) \in Z^{n+1}(K, L; \pi_n(Y))$, called the **obstruction cochain**. Then f can be extended to K if and only if $[c^{n+1}(f)] = 0$ in $H^{n+1}(K, L; \pi_n(Y))$. Suppose that we have two continuous extensions $f_0, f_1: K \to Y$ of f. We define the map $F: (K \times \dot{\mathbb{I}}) \cup (L \times \mathbb{I}) \to Y$ by $F(x, 0) = f_0(x)$, $F(x, 1) = f_1(x)$ for $x \in K$, and $F(a, t) = f(a)$ for $a \in L$, $t \in \mathbb{I}$. Identifying $Z^{n+2}(K \times I, K \times \dot{\mathbb{I}} \cup L \times \mathbb{I}; \pi_{n+1}(Y))$ with $Z^{n+1}(K, L; \pi_{n+1}(Y))$, we have the **difference cochain** $d^{n+1}(f_0, f_1) = c^{n+2}(F)$. Then $f_0 \simeq f_1$ rel L if and only if $[d^{n+1}(f_0, f_1)] = 0$ in $H^{n+1}(K, L; \pi_{n+1}(Y))$.

Hopf–Eilenberg classification theorem
Let (K, L) be a CW-pair with $\dim(K \setminus L) \leqslant k$. Suppose that an arcwise connected space Y is $(n-1)$-connected, r-simple for $n+1 \leqslant r \leqslant k$,

- $H^{r+1}(K, L; \pi_r(Y)) = 0$ for $n+1 \leqslant r < k$, and
- $H^r(K, L; \pi_r(Y)) = 0$ for $n+1 \leqslant r \leqslant k$.

Let $f_0: K \to Y$ be a map. Then the correspondence $f \mapsto d^n(f_0, f)$ induces a one-to-one correspondence between the set $\{[f]_L \colon f \in C(K, L)$ and $f \restriction L = f_0 \restriction L\}$ and the group $H^n(K, L; \pi_n(Y))$.

Hopf classification theorem
Let K be an n-dimensional CW-complex and let Y is $(n-1)$-connected n-simple space. Then $[K; Y] \cong H^n(K; \pi_n(Y))$ as sets.

Hence Eilenberg–MacLane spaces may be considered as **representing space**s for cohomology groups in the following sense: for an Abelian group G, a positive integer n and a CW-complex K, $[K; K(G, n)] \cong H^n(K; G)$. This result can be generalized to Čech cohomology groups of general spaces. Representation theorems for generalized cohomology theories were developed by Brown [2].

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k-9 Shape Theory

Standard notions of homotopy theory are not adequate to describe and study global properties of spaces with "bad" local behavior. Shape theory is designed to correct this shortcoming of homotopy theory, by replacing homotopy classes of maps by shape morphisms. This is done in such a way that, in the case of spaces which belong to the class HPol of *spaces having the homotopy type of polyhedra*, shape morphisms coincide with homotopy classes. Since the class HPol includes all ***CW-complex***es, as well as all ANRs (***absolute neighborhood retracts*** for metric spaces), shape theory can be viewed as an extension of homotopy theory from the realm of CW-complexes to arbitrary ***topological space***s. In particular, shape theory is applicable to arbitrary ***metric compacta***.

Many constructions in topology lead naturally to locally "bad" spaces, even when one initially considers locally "good" spaces, e.g., manifolds. Standard examples include fibers of maps, sets of fixed points, attractors of dynamical systems, spectra of operators, boundaries of certain groups. In all these areas shape theory has proved useful. Moreover, methods developed in shape theory have suggested new techniques applicable to some areas which do not belong to topology, e.g., C*-algebras.

The two basic concepts of shape theory are the **shape category** Sh(Top) and the **shape functor** $S : \mathrm{H(Top)} \to \mathrm{Sh(Top)}$, defined on the **homotopy category** H(Top) of topological spaces and homotopy classes $[f]$ of maps $f : X \to Y$. The objects of Sh(Top) are all topological spaces X. Morphisms $F : X \to Y$ of Sh(Top), called **shape morphisms**, are functions which to every homotopy class $[\varphi] : Y \to P$, where $P \in \mathrm{HPol}$, assign a homotopy class $F[\varphi] : X \to P$ in such a way that, whenever $Q \in \mathrm{HPol}$ and $[p] : P \to Q$ is a homotopy class of maps, then $F([p][\varphi]) = [p]F[\varphi]$. If $G : Y \to Z$ is another shape morphism, then $GF : X \to Z$ is defined by $(GF)[\psi] = F(G[\psi])$, where $[\psi] : Z \to P$, $P \in \mathrm{HPol}$. The shape functor S keeps objects X fixed and to a homotopy class of maps $[f] : X \to Y$ assigns the shape morphism $F = S[f] : X \to Y$, defined by $F[\varphi] = [\varphi][f] : X \to P$, for $[\varphi] : Y \to P$. If $Y = P \in \mathrm{HPol}$ and $F : X \to Y$ is a shape morphism, then there is a unique homotopy class $[f] : X \to Y$ such that $S[f] = F$. This is the class $[f] = F[\mathrm{id}_P]$. Consequently, shape morphisms F into spaces Y from HPol can be viewed as homotopy classes of maps. In the definitions of Sh(Top) and S it suffices to consider as test spaces P all polyhedra (equivalently, all CW-complexes or all ANRs).

Spaces X and Y, which are isomorphic objects in Sh(Top), are said to have the same **shape**, which is denoted by $\mathrm{sh}(X) = \mathrm{sh}(Y)$. Maps $f : X \to Y$, for which $S[f]$ is an isomorphism in Sh(Top), are called **shape equivalence**s. Clearly, homotopy equivalences are always shape equivalences and spaces having the same homotopy type also have the same shape. A map $f : X \to Y$ is a shape equivalence if and only if it has the following two properties. For every $P \in \mathrm{HPol}$ and every map $\varphi : X \to P$, there is a map $\psi : Y \to P$ such that $\varphi \simeq \psi f$. If $\psi_0, \psi_1 : Y \to P$ are maps such that $\psi_0 f \simeq \psi_1 f$, then $\psi_0 \simeq \psi_1$. Basic notions of the shape theory of spaces are easily generalized to the case of pairs of spaces as well as to pointed spaces.

In concrete situations it is difficult to apply the above stated categorical definition of shape, because there are very many homotopy classes $[\varphi] : X \to P$ and $[p] : P \to Q$, for $P, Q \in \mathrm{HPol}$. This difficulty is circumvented in the inverse system approach to shape, initiated in 1970 by S. Mardešić and J. Segal (for ***compact Hausdorff*** spaces) and further developed in 1975 by K. Morita (for topological spaces) (see [6]). An **inverse system** of spaces $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ consists of an index set Λ, endowed with a **directed ordering** $\leqslant$ (any two elements have an upper bound), of spaces X_λ, for $\lambda \in \Lambda$, and of maps $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$, for $\lambda \leqslant \lambda'$. The functorial requirements $p_{\lambda\lambda'} p_{\lambda'\lambda''} = p_{\lambda\lambda''}$, for $\lambda \leqslant \lambda' \leqslant \lambda''$, and $p_{\lambda\lambda} = \mathrm{id}$, for $\lambda \in \Lambda$, are imposed. $\mathbf{X}$ is a **cofinite system** if every element of Λ has only a finite number of predecessors. A (compact) HPol**-system** is a system $\mathbf{X}$, for which every $X_\lambda \in \mathrm{HPol}$ (and is compact). Inverse systems $\mathbf{X}$ are called **inverse sequence**s if $\Lambda = \mathbb{N}$.

A **map** $\mathbf{f} = (f, f_\mu) : \mathbf{X} \to \mathbf{Y}$ between two cofinite inverse systems $\mathbf{X}$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ consists of an increasing function $f : M \to \Lambda$ and of maps $f_\mu : X_{f(\mu)} \to Y_\mu$, $\mu \in M$, such that, for $\mu \leqslant \mu'$, the following diagram commutes.

$$\begin{array}{ccc} X_{f(\mu)} & \xleftarrow{\;p_{f(\mu)f(\mu')}\;} & X_{f(\mu')} \\ {\scriptstyle f_\mu}\downarrow & & \downarrow{\scriptstyle f_{\mu'}} \\ Y_\mu & \xleftarrow[\;q_{\mu\mu'}\;]{} & Y_{\mu'} \end{array} \tag{1}$$

An example is provided by the **identity map** $\mathrm{id} : \mathbf{X} \to \mathbf{X}$, which consists of the identity function on Λ and of the identity maps on X_λ. A special case is that of a map $\mathbf{p} = (p_\lambda) : X \to \mathbf{X}$ of a space X to a system $\mathbf{X}$. It consists of maps $p_\lambda : X \to X_\lambda$, $\lambda \in \Lambda$, such that $p_{\lambda\lambda'} p_{\lambda'} = p_\lambda$, for $\lambda \leqslant \lambda'$.

More general are **homotopy maps** $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ between systems. They consist again of f and f_μ, but diagram (1) is required to commute only up to homotopy. Two homotopy maps $\mathbf{f}$ and $\mathbf{f}'$ are **equivalent**, $\mathbf{f} \simeq \mathbf{f}'$, if there exists an increasing function $f'' : M \to \Lambda$ such that $f'' \geqslant f, f'$ and, for every $\mu \in M$,

$$f_\mu p_{f(\mu)f''(\mu)} \simeq f_\mu p_{f'(\mu)f''(\mu)}. \tag{2}$$

Composition **gf** of homotopy maps is defined by composing homotopy commutative diagrams of type (1). Composition of equivalence classes $[\mathbf{f}]$ of homotopy maps $\mathbf{f}$ is defined by putting $[\mathbf{g}][\mathbf{f}] = [\mathbf{gf}]$. Cofinite inverse systems of spaces and equivalence classes of homotopy maps between systems form the **prohomotopy category** pro-H(Top).

A map $\mathbf{p}: X \to \mathbf{X}$ is a **homotopy expansion** of X provided it satisfies the following conditions of Morita.

(M1) For every map $f: X \to P$ into a polyhedron (or an ANR), there exist a $\lambda \in \Lambda$ and a map $h: X_\lambda \to P$ such that $hp_\lambda \simeq f$.

(M2) If for some $\lambda \in \Lambda$ and some maps $h_0, h_1: X_\lambda \to P$ one has $h_0 p_\lambda \simeq h_1 p_\lambda$, then there exists a $\lambda' \geqslant \lambda$ such that $h_0 p_{\lambda\lambda'} \simeq h_1 p_{\lambda\lambda'}$.

If all X_λ belong to HPol, one speaks of an HPol-**homotopy expansion**.

A shape morphism $F: X \to Y$ of topological spaces is given by cofinite HPol-homotopy expansions $\mathbf{p}: X \to \mathbf{X}$, $\mathbf{q}: Y \to \mathbf{Y}$ and by a morphism $[\mathbf{f}]: \mathbf{X} \to \mathbf{Y}$ of pro-H(Top).

$$\begin{array}{ccc} \mathbf{X} & \xleftarrow{\ \mathbf{p}\ } & X \\ {\scriptstyle[\mathbf{f}]}\big\downarrow & & \vdots\, {\scriptstyle F} \\ \mathbf{Y} & \xleftarrow[\ \mathbf{q}\ ]{} & Y \end{array} \tag{3}$$

Composition GF is defined by composing diagrams of type (3). If $f: X \to Y$ is a map, there exists a unique morphism $[\mathbf{f}]: \mathbf{X} \to \mathbf{Y}$ of pro-H(Top) such that

$$\mathbf{fp} \simeq \mathbf{q} f. \tag{4}$$

The shape functor S maps $[f]$ to the shape morphism $F: X \to Y$, given by $\mathbf{p}$, $\mathbf{q}$ and $[\mathbf{f}]$.

The key to this approach to shape is the fact that every topological space X admits a cofinite HPol-homotopy expansion $\mathbf{p}: X \to \mathbf{X}$. If X is a compact Hausdorff space, it suffices to consider a cofinite inverse system of compact polyhedra $\mathbf{X}$ having X for its **inverse limit**, i.e., X should be the set of all points $x = (x_\lambda)$ in the direct product $\prod X_\lambda$ such that $p_{\lambda\lambda'}(x_{\lambda'}) = x_\lambda$, for $\lambda \leqslant \lambda'$ [6]. In the general case of a topological space X, it suffices to consider a cofinite inverse system of polyhedra $\mathbf{X}$ and a map $\mathbf{p}: X \to \mathbf{X}$, which is a **resolution** of X, i.e., it has the following two properties (see [6]).

(R1) For every polyhedron P, ***open cover*** $\mathcal{V}$ of P and map $f: X \to P$, there exist a $\lambda \in \Lambda$ and a map $h: X_\lambda \to P$ such that the maps hp_λ and f are $\mathcal{V}$-near, i.e., every $x \in X$ admits a $V \in \mathcal{V}$ such that $hp_\lambda(x), f(x) \in V$.

(R2) There exists an open cover $\mathcal{V}'$ of P, such that whenever, for a $\lambda \in \Lambda$ and for two maps $h_0, h_1: X_\lambda \to P$, the maps $h_0 p_\lambda, h_1 p_\lambda$ are $\mathcal{V}'$-near, then there exists a $\lambda' \geqslant \lambda$, such that the maps $h_0 p_{\lambda\lambda'}, h_1 p_{\lambda\lambda'}$ are $\mathcal{V}$-near.

Foundations of the theory of shape were laid in 1968, when K. Borsuk defined the shape category and the shape functor for compact metric spaces, embedded in the Hilbert cube $Q = I^\infty$, $I = [0,1]$. His definitions are more geometric and are based on **fundamental sequence**s $\mathbf{f}: X \to Y$. These are sequences $\mathbf{f} = (f_n)$ of maps $f_n: Q \to Q$, $n = 1, 2, \ldots$, which have the property that every neighbourhood V of Y in Q admits a neighbourhood U of X in Q and admits an integer n_V such that the restrictions $f_n|U$ and $f_m|U$ are homotopic in V, for all $n, m \geqslant n_V$. Two fundamental sequences $\mathbf{f}$ and $\mathbf{f}' = (f'_n)$ are considered **equivalent** provided every neighbourhood V of Y in Q admits a neighbourhood U of X in Q and admits an integer m_V such that $f_n|U \simeq f'_n|U$ in V, for all $n \geqslant m_V$. The **shape morphisms** $F: X \to Y$ are defined as equivalence classes $[\mathbf{f}]$ of fundamental sequences $\mathbf{f}: X \to Y$. The shape functor S maps the homotopy class $[f]: X \to Y$ to the equivalence class of the fundamental sequence $\mathbf{f} = (f_n): X \to Y$, where all f_n coincide with an arbitrary extension $\tilde{f}: Q \to Q$ of the map $f: X \to Y$. Borsuk's approach to shape was generalized in 1972 by R.H. Fox to ***metric spaces*** and in 1974 by A. Šostak to ***paracompact p-spaces***. In 1992 J.M.R. Sanjurjo introduced yet another approach to shape of metric compacta. The basic idea was to replace maps by ***multivalued functions***, where points map into sets of controlled size. In 1995 Sanjurjo's construction was generalized to topological spaces by Z. Čerin.

In developing shape theory for metric compacta, K. Borsuk introduced two important shape invariant classes of metric compacta, called FANRs (abbreviation for **fundamental absolute neighborhood retract**) and **movable compacta** (see [1, 6]). A metric compactum X is an FANR provided, for every metric compactum Y containing X, there exist a neighbourhood U of X in Y and a **shape retraction** $R: U \to X$, i.e., a shape morphism such that $RS[i] = \mathrm{id}_X$, where $i: X \to U$ denotes the inclusion map. In 1976 D.A. Edwards and R. Geoghegan proved that every (pointed) FANR has the shape of a polyhedron (see [6], II.9.5, Theorems 15 and 19). There exist FANRs which do not have the shape of a compact polyhedron [6, II.9.5, Example 2].

A metric compactum X, contained in the Hilbert cube Q, is **movable** provided every neighbourhood U of X in Q admits a neighbourhood $V \subseteq U$ such that V can be deformed within U arbitrarily close to X, i.e., for every neighbourhood $W \subseteq U$ of X in Q there exists a homotopy $H: V \times I \to U$ such that $H|(V \times 0)$ is the inclusion $V \hookrightarrow U$ and $H(V \times 1) \subseteq W$. More general are n-**movable compacta**. If embedded in Q they are characterized by the property that every neighbourhood U admits a neighbourhood $V \subseteq U$ such that, for every neighbourhood $W \subseteq U$, every polyhedron P of dimension $\dim P \leqslant n$ and every map $h: P \to V$, there exists a map $r: P \to W$ such that $r \simeq h$ in U. A highly nontrivial result of H.M. Hastings and A. Heller (1982) asserts that connected FANRs and pointed FANRs coincide. Whether movable continua and pointed movable continua coincide is still an open question. The FANR property is easily extended to the class of metric spaces, while movability extends to arbitrary topological spaces.

Classical algebraic invariants of homotopy theory have their analogues in shape theory. Application of the homotopy group functor π_n to inverse systems of pointed spaces

$(\mathbf{X}, *)$ yields their homotopy progroups $\pi_n(\mathbf{X}, *)$. Moreover, maps and homotopy maps $\mathbf{f}:(\mathbf{X}, *) \to (\mathbf{Y}, *)$ yield **morphisms** of progroups $\mathbf{f}_\#:\pi_n(\mathbf{X}, *) \to \pi_n(\mathbf{Y}, *)$, which consist of f and of homomorphisms $f_{\mu\#}:\pi_n(X_{f(\mu)}, *) \to \pi_n(Y_\mu, *)$, induced by $f_\mu:(X_{f(\mu)}, *) \to (Y_\mu, *)$. The **homotopy progroup** $\underline{\pi}_n(X, *)$ of a space X is defined as the progroup $\pi_n(\mathbf{X}, *)$, where $(\mathbf{X}, *)$ is any pointed HPol-homotopy expansion of $(X, *)$. If a shape morphism $F:(X, *) \to (Y, *)$ is given by pointed HPol-homotopy expansions $\mathbf{p}:(X, *) \to (\mathbf{X}, *)$, $\mathbf{q}:(Y, *) \to (\mathbf{Y}, *)$ and by a morphism $[\mathbf{f}]:(\mathbf{X}, *) \to (\mathbf{Y}, *)$ in pointed pro-H(Top), one defines the induced homomorphism $F_*:\underline{\pi}_n(X, *) \to \underline{\pi}_n(Y, *)$ as $\mathbf{f}_*:\pi_n(\mathbf{X}, *) \to \pi_n(\mathbf{Y}, *)$. Similarly, one defines **homology progroups** of an inverse system $H_n(\mathbf{X}; G)$ and homology progroups of a space $\underline{H}_n(X; G)$.

Using progroups instead of groups, classical results of homotopy theory (the Whitehead theorem, the Hurewicz theorem, classification of covering maps) generalize from CW-complexes to arbitrary spaces. E.g., the Whitehead theorem assumes the following form. If $F:(X, *) \to (Y, *)$ is a shape morphism of pointed connected finite-dimensional spaces, which induces isomorphisms of all homotopy progroups, then F is a shape equivalence of pointed spaces. The assumption $\dim X, \dim Y < \infty$ can be weakened by assuming that the shape dimensions of X and Y be finite, but it cannot be omitted. By definition, the **shape dimension** (also called **fundamental dimension**) of a space $\operatorname{sd} X \leqslant n$ provided every map $f:X \to P$ to a polyhedron P admits a factorization up to homotopy through a polyhedron Q of dimension $\leqslant n$. In the presence of movability these theorems simplify considerably. In particular, homotopy progroups (homology progroups) can often be replaced by their limits, the **shape groups** (the **Čech homology groups**).

An important part of shape theory are the **complement theorems**. The first one was discovered in 1972 by T.A. Chapman. It asserts that two compacta X and Y, embedded in the Hilbert cube Q as ***Z-set***s (see [8]), have the same shape if and only if their complements $Q\backslash X$ and $Q\backslash Y$ are homeomorphic (see [2]). Corresponding results for compacta embedded in $\mathbb{R}^n$ are more complicated and involve the inessential loop condition of embedded compacta (see [6]) as well as ***r*-shape connectedness** of X and Y, i.e., the vanishing of the homotopy progroups in dimensions $q \leqslant r$ (R.B. Sher, 1981 and 1987).

Beside ordinary shape there exists a finer theory, presently called strong shape theory. Its basic notions are the **strong shape category** SSh(Top) and the strong shape functor $\overline{S}:\mathrm{H(Top)} \to \mathrm{SSh(Top)}$. The objects of this new category are again topological spaces, but the morphisms have more structure than in the case of ordinary shape. Therefore, there is a “forgetful functor” $E:\mathrm{SSh(Top)} \to \mathrm{Sh(Top)}$. Strong shape has an intermediate position between homotopy and ordinary shape, because $S = E\overline{S}$.

Using simplicial classes (a generalization of simplicial sets), F.W. Bauer and B. Günther have constructed the category SSh(Top) by rigidifying the categorical definition of Sh(Top). The inverse system approach to strong shape requires strong expansions instead of homotopy expansions and a category Ho(pro-Top) instead of pro-H(Top). The category Ho(pro-Top) is obtained from the category pro-Top by localization at morphisms which are **homotopy level equivalences** ($f = \mathrm{id}$ and each f_μ is a homotopy equivalence) or equivalently, by using coherent maps instead of homotopy maps (see [4]). **Strong expansions** are defined by properties (M1) and (S2) (a stronger form of (M2)): If $h_0, h_1:X_\lambda \to P$ are maps and $F:X \times I \to P$ is a homotopy, which connects $h_0 p_\lambda$ to $h_1 p_\lambda$, then there exist a $\lambda' \geqslant \lambda$ and a homotopy $H:X_{\lambda'} \times I \to P$, which connects $h_0 p_{\lambda\lambda'}$ to $h_1 p_{\lambda\lambda'}$, and the homotopies $H(p_{\lambda'} \times 1):X \times I \to P$ and $F:X \times I \to P$ are connected by a homotopy $(X \times I) \times I \to P$, which is fixed on $X \times \partial I$.

It is a consequence of Chapman’s complement theorem that two metric compacta have the same shape if and only if they have the same strong shape. However, in the two categories the sets of morphisms $F:X \to Y$ differ. E.g., for every metric continuum Y, there is only one shape morphism of the one-point space $\{*\}$ to Y. However, if Y is not pointed 1-movable, there are $2^{\aleph_0}$ different strong shape morphisms $\{*\} \to X$ (R. Geoghegan and J. Krasinkiewicz, 1991). In 1976 D.A. Edwards and H.M. Hastings showed that the strong shape category of compact Z-sets X of the Hilbert cube Q is isomorphic to the proper homotopy category of their complements $Q\backslash X$ [3].

Appropriate variations of the basic ideas of shape led to new types of shape theories. In particular, there is *proper shape* (B.J. Ball and R.B. Sher, 1974, B.J. Ball, 1975, Z. Čerin, 1994), *fibered shape* (H. Kato, 1981, T. Yagasaki, 1985), *equivariant shape* (S.A. Antonian and S. Mardešić, 1987, Z. Čerin, 1995), *stable shape* (S. Nowak, 1987, F.W. Bauer, 1995), *n-shape* (A.Ch. Chigogidze, 1989), *uniform shape* (J. Segal, S. Śpiez and B. Günther, 1993, T. Miyata, 1994), *extension shape* (I. Ivanšić and L.R. Rubin, 1999).

Shape theory has interesting applications in General topology (see [7]). Most applications to metric continua use movability and its variants. A ***circle-like*** metric continuum embeds in the plane if and only if it is movable. Every nonmovable circle-like continuum is ***indecomposable*** (M.C. McCord, 1967 and J. Krasinkiewicz, 1976). All ***hereditarily indecomposable*** continua have **trivial shape**, i.e., the shape of a point (J. Krasinkiewicz and M. Smith, 1983). Pointed 1-movability is preserved under continuous maps (D.R. McMillan Jr, 1975 and J. Krasinkiewicz and P. Minc, 1979). Every ***hereditarily decomposable*** continuum is pointed movable (J. Krasinkiewicz, 1978). A continuum X admits a ***locally connected*** **shape representative**, i.e., a continuum Y of the same shape, if and only if X is pointed 1-movable (J. Krasinkiewicz, 1977). Every 1-dimensional continuum has a hereditarily indecomposable shape representative. For $n \geqslant 2$, the sphere S^n does not admit a hereditarily indecomposable shape representative (J. Krasinkiewicz, 1980).

With every compact Hausdorff space X is associated the compact Hausdorff ***hyperspace*** 2^X. It consists of all nonempty closed subsets $A \subseteq X$ and is endowed with the ***Vietoris topology***. If X is connected one also considers the hyperspace $C(X) \subseteq 2^X$, which consists of all subcontinua of X. If X is metric, so are 2^X and $C(X)$. For all X, the spaces 2^X and $C(X)$ have trivial shape. Shape theory proved particularly useful in studying **Whitney continua** in $C(X)$. These are fibers of **Whitney maps**, i.e., maps $\omega : C(X) \to \mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geqslant 0\}$, which have the property that $\omega(A) < \omega(B)$, whenever A is a proper subset of B, and $\omega(\{x\}) = 0$, for every point $x \in X$. A property P is a **Whitney property** provided, for every continuum X which has property P, all Whitney continua $\omega^{-1}(t)$, where $t < \omega(X)$, also have property P. H. Kato proved in 1986 that pointed 1-movability is a Whitney property. A property P is **strongly Whitney-reversible** provided X has property P, whenever there exists a Whitney map $\omega : C(X) \to \mathbb{R}^+$ such that all $\omega^{-1}(t)$, $t < \omega(X)$, have property P. In 1989 Kato proved that being like a family of compact connected ANRs is such a property. Shape theory has also suggested new metrics for hyperspaces, which have the property that convergence of sequences is equivalent to various kinds of ***regular convergence*** (Z. Čerin, 1980 and 1983).

For a Tychonoff space X, let $i : X \to \beta(X)$ denote the natural inclusion in its ***Čech–Stone compactification*** βX. For connected X, i is a shape equivalence if and only if X is ***pseudocompact***. Several results concerning subcontinua K of the remainder $\beta X \backslash X$ were obtained by J.E. Keesling using shape-theoretic techniques. If X is ***realcompact*** and $f : K \to Y$ is a surjection which induces an isomorphism of the first Čech cohomology groups $f^* : \check{H}^1(Y; \mathbb{Z}) \to \check{H}^1(K; \mathbb{Z})$, then f is a ***homeomorphism***. If X is ***Lindelöf*** and K is infinite-dimensional, then there exists a map $f : K \to \prod_\alpha S^1$ to a product of $2^{\aleph_0}$ copies of S^1, which is homotopically surjective, i.e., it is not homotopic to a map g, whose image is a proper subset of $\prod_\alpha S^1$. Consequently, X has dimension $2^{\aleph_0}$ (in the sense of essential families of maps). More recently, Keesling (1994) has successfully applied shape theory to the study of the Higson compactification and its corona.

There exist ***contractible*** compacta X which fail to have the **fixed-point property**, i.e., they admit a self-map $f : X \to X$ without ***fixed points***. Shape theory inspired K. Borsuk to introduce in 1975 the class of nearly extendable maps $f : X \to Y$ and prove that, for compacta X of trivial shape such a map $f : X \to X$ always has a fixed point. The conclusion also holds for FANRs X, provided the Lefschetz number $\Lambda(f) \neq 0$. For compacta X and Y, contained in the Hilbert cube Q, **nearly extendable** maps $f : X \to Y$ are defined as maps which admit an extension $\tilde{f} : Q \to Q$ having the following property. For every $\varepsilon > 0$, there is a neighbourhood U of X in Q such that, for every neighbourhood V of Y, there is a map $g : U \to V$ with distance $d(\tilde{f}|U, g) \leqslant \varepsilon$. Combining the theory of ***approximate resolution***s (developed by S. Mardešić, L. Rubin and T. Watanabe in 1989) with the theory of ***universal maps*** (developed by J. Mioduszewski and M. Rochowski in 1962 and W. Holszyński in 1964), J. Segal and T. Watanabe introduced in 1992 cosmic approximate polyhedral resolutions and proved that every compact Hausdorff space which admits such a resolution has the fixed-point property. A consequence of this result is that, for a locally connected Hausdorff continuum X, the hyperspaces 2^X and $C(X)$ have the fixed-point property.

Some of the most interesting applications of shape theory refer to ***dynamical system***s, more precisely, to ***flow***s (actions of $\mathbb{R}$) and ***discrete dynamical system***s (actions of $\mathbb{Z}$). The **Conley index** of an isolated invariant set S of a flow $g : \mathbb{R} \times X \to X$ on X was introduced by C.C. Conley under the name **Morse index** and gives useful information concerning the dynamical structure of S. One can associate with S index pairs (N_1, N_2) which have the property that the homotopy type of the quotient $N_1/N_1 \cap N_2$ is completely determined by the flow. By definition this is the Conley index of S. This definition is not applicable to discrete systems. In 1988 J.W. Robbin and D. Salamon modified the definition and obtained an index of isolated invariant sets for diffeomorphisms. By definition, this index is a shape type of metric compacta. Recently, the theory of the shape index was further generalized by F.R. Ruiz del Portal and J.M. Salazar to discrete systems on noncompact metric spaces. Moreover, J.M.R. Sanjurjo successfully applied the shape-theoretic Lusternik–Schnirelmann category (introduced by K. Borsuk in 1978) to the study of the Conley index. He obtained simple inequalities which relate the categories of isolated invariant sets, of isolating neighbourhoods, of the exit sets and of the ***unstable manifolds*** of invariant sets and Morse sets.

Shape theory made it possible to characterize finite-dimensional metric compacta X which embed in a topological manifold M as attractors of a flow on M. According to B. Günther and J. Segal (1993) a necessary and sufficient condition is that X has the shape of a compact polyhedron. The same result holds also for differentiable flows of class C^r, $1 \leqslant r < \infty$ (Günther, 1995). For $r = 1$ the necessity of the condition was already established in 1989 by S.A. Bogatyĭ and V.I. Gutsu. The dyadic solenoid is the attractor of a discrete system on a certain 3-manifold in spite of the fact that it fails to be movable and thus, does not have the shape of a compact polyhedron. On the other hand, there exist solenoids which are not attractors of any self-map $f : M \to M$ of a manifold M (Günther, 1994).

Sanjurjo (1995) studied attractors A of flows g on locally compact ANRs. He proved that every $T > 0$ and $\varepsilon > 0$ admit a $\delta > 0$ such that every other flow g' with $d(g(t, x), g'(t, x)) < \delta$, for $t \in [0, T]$ and $x \in X$, admits an attractor K' such that K' is in the ε-neighbourhood of K and its shape $\mathrm{sh}(K') = \mathrm{sh}(K)$. L. Kapitanski and I. Rodnianski (1999) have studied shape properties of attractors in the more general case of semi-dynamical systems on complete metric spaces.

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k-10 Manifold

An n**-manifold** is a ***Hausdorff space*** in which each point has a neighbourhood homeomorphic to $\mathbb{R}^n$, Euclidean n-space, and similarly an n**-manifold with boundary** is a Hausdorff space in which each point has a closed neighbourhood homeomorphic to the **unit** n**-ball** $B^n = \{x \in \mathbb{R}^n: \|x\| \leqslant 1\}$. (The reader should be aware that this terminology is applied inconsistently throughout the current literature; some writers use "**manifold**" as defined here, while others employ it in a more inclusive sense, allowing boundary.) Open subsets of n-manifolds (with boundary) are n-manifolds (with boundary); the Cartesian product of any n-manifold (with boundary) and any m-manifold (with boundary) is an $(n+m)$-manifold (with boundary).

Every n-manifold with boundary W has both an intrinsic **manifold interior**, written $\operatorname{Int} W$ and defined as the maximal open subset of W which is an n-manifold, and an intrinsic **manifold boundary**, written ∂W and defined as $\partial W = W - \operatorname{Int} W$. In algebraic terms, ∂W consists of all points $w \in W$ for which the pair $(W, W - \{w\})$ has trivial nth homology (with any nontrivial Abelian coefficient group G), whereas $H_n(W, W - \{w\}; G) \cong G$ for $w \in \operatorname{Int} W$. When non-empty, the boundary of an n-manifold with boundary is necessarily an $(n-1)$-manifold (without boundary). For example,

$$\partial B^n = \left\{x \in \mathbb{R}^n: \|x\| = 1\right\} = S^{n-1}$$

(called the $(n-1)$-sphere), and $\operatorname{Int} B^n$ is homeomorphic to $\mathbb{R}^n$.

Being locally compact, all n-manifolds M^n are regular spaces. The long ray – the Cartesian product of the ordinal space $[0, \omega_1)$ with $[0, 1)$ under the lexicographical ordering – is a normal but non-metrizable 1-manifold with boundary. There exists a separable 2-manifold W^2 which contains an uncountable, discrete closed subset; hence, W^2 cannot even be normal. Whether ***perfectly normal*** manifolds are metrizable is undecidable: M.E. Rudin and P. Zenor [13] showed that the ***Continuum Hypothesis*** (CH) implies the existence of a nonmetrizable, perfectly normal manifold; the first author [12] then showed that the combination of ***Martin's Axiom*** and the negation of CH implies the metrizability of any perfectly normal manifold. V.V. Fedorchuk [5] proved that CH implies the existence of a perfectly normal 4-manifold W^4 whose ***covering dimension*** is larger than 4.

Each n-manifold with boundary W embeds in an n-manifold U as a retract. One can form U from the disjoint union of two copies W' and W'' of W by identifying each $x' \in \partial W'$ with the corresponding point $x'' \in \partial W''$, and can define a retraction r of U to the image of W' by folding W'' onto W', i.e., by setting $r(w'') = w'$ for points w'' in the image of W'' and their corresponding points w' in the other image.

From now on we consider only metrizable manifolds, with or without boundary. In the case of a metric manifold with boundary, W, its boundary, ∂W, is **collared**: ∂W has a neighbourhood in W homeomorphic to $\partial W \times [0, 1)$ [3].

Manifolds of dimensions 1 and 2 are well-understood and completely classified. For instance, two compact 2-manifolds are topologically equivalent if and only if they have isomorphic homology groups; alternatively, two compact, connected 2-manifolds are topologically equivalent provided they have equal Euler characteristics and the same orientability classes (defined in the subsequent paragraph). Similarly, two compact 2-manifolds with boundary are equivalent if and only if, in addition to the above, their boundaries have an equal number of components. Noncompact 2-manifolds lend themselves to related, but more complicated, characterizations, necessarily also taking into account the behavior near infinity.

Orientability is a recurring topic in the theory. An n-manifold with boundary W is **nonorientable** if and only if any two embeddings $e_0, e_1 : B^n \to W$ are **isotopic**, in the sense that there is a map $\Phi : B^n \times [0, 1] \to W$ such that $\Phi_0 = e_0$, $\Phi_1 = e_1$ and Φ_t is $1-1$ for all $t \in [0, 1]$; otherwise, W is **orientable**. A manifold with boundary is orientable if and only if its manifold interior is orientable. Open subsets of orientable manifolds with boundary are orientable. A product of manifolds with boundary is orientable if and only if each of the factors is. If a compact, connected n-manifold M is orientable, $H_n(M; G) \cong G$, for any Abelian coefficient group G; if it is nonorientable, $H_n(M; \mathbb{Z}) \cong 0$. The Möbius band, $M\ddot{o}b$, embodies nonorientability: sliding an embedded 2-cell once around the Möbius band back to its initial position (setwise) amounts to reflection through one coordinate of the original embedding. Regardless of any compactness or boundary considerations, the presence of an embedded $M\ddot{o}b \times B^{n-2}$ in a given n-manifold is a precise detector of its nonorientability. Partially based on the analysis of the central loops of such $M\ddot{o}b \times B^{n-2}$, it turns out that every nonorientable manifold admits a 2-fold covering by an orientable one; hence, a necessary condition for nonorientability of W is that $\pi_1(W)$ contains an index 2 (normal) subgroup. Consequently, every ***simply connected*** manifold is orientable.

Straightforward classifications are unavailable for manifolds of dimension greater than 2. Back in 1904 H. Poincaré [9] presented a compact 3-manifold different from S^3 but with identical homology groups, and simultaneously he put forward the celebrated, still unsettled **Poincaré Conjecture** – that every compact, simply connected 3-manifold is homeomorphic to S^3. It is known that equivalence of homology and fundamental groups does not imply topological equivalence for 3-manifolds: among a fairly simple family of compact 3-manifolds, the **Lens spaces** – obtained by identifying together two copies of $S^1 \times B^2$ via an arbitrary homeomorphism between boundaries – there are topologically distinct objects having the same homology and fundamental groups. Even worse, non-homeomorphic Lens spaces can be homotopy equivalent.

There do exist algorithms for deciding whether a given compact 3-manifold is S^3 [11, 16]. These have not proved effective for addressing the Poincaré Conjecture, however, because as yet there is no algorithm for deciding simple-connectedness of a 3-manifold.

Typically, unstructured manifolds have been resistant to thorough analysis. Instead, progress has come about by exploiting the presence of some kind of additional structure. Given a class C of homeomorphisms defined for the collection of all open subsets of $\mathbb{R}^n$, one defines a C-structure on n-manifolds as a maximal collection of homeomorphisms $h_\alpha : U_\alpha \to V_\alpha$, where U_α denotes an open subset of $\mathbb{R}^n$ and V_α an open subset of M^n, such that

$$h_\beta^{-1} h_\alpha \colon h_\alpha^{-1}\big(h_\alpha(U_\alpha) \cap h_\beta(U_\beta)\big) \to h_\beta^{-1}\big(h_\alpha(U_\alpha) \cap h_\beta(U_\beta)\big)$$

belongs to C, for all α and β in the index set. Two such structures

$$\Gamma = \{g_\alpha \colon \alpha \in A\}, \qquad \Gamma' = \{h_\gamma \colon \gamma \in A'\}$$

are **equivalent** if there exists a homeomorphism $\psi : M \to M$ such that, for all $g_\alpha \in \Gamma$ one has $\psi g_\alpha \in \Gamma'$.

For many years the most influential and common structures were the **smooth**, determined by the class C of infinitely differentiable homeomorphisms (called **diffeomorphism**s), and the **PL**, determined by the class of PL homeomorphisms (each point $x \in U_\alpha$ has a neighbourhood $W_\alpha \subset U_\alpha$ such that for $y \in W_\alpha$ and $t \in [0, 1]$, $f(t \cdot x + (1-t) \cdot y) = t \cdot f(x) + (1-t) \cdot f(y)$). The presence of a PL structure on a manifold with boundary W allows one to regard W as a ***simplicial complex*** K in some Euclidean space, where for each vertex $v \in K$, its **star**, defined as $\bigcup\{\sigma \in K \colon v \in \sigma\}$, is PL homeomorphic to $I^n = [0, 1]^n$. In recent years many diverse geometric structures have provided important applications. These arise in another way, where one knows some geometric structure $\mathcal{G}$ on a model simply connected n-manifold $\mathcal{M}$, and where then an n-manifold N^n is said to have **geometric structure** $\mathcal{G}$ if $\mathcal{M}$ is the universal cover of N^n and if the group of covering transformations consists of ***isometries*** with respect to the metric for $\mathcal{M}$. Of course, the metric associated with the geometric structure $\mathcal{S}$ has a foundational role. For example, Hyperbolic n-space and Euclidean n-space – two appropriate model manifolds $\mathcal{M}$ – though drastically different geometrically, are topologically equivalent, and the relevant metrics generate equivalent topologies. With compact 3-manifolds, for instance, the hyperbolic ones are much richer than and quite unlike the Euclidean ones; among other features, the latter all have a finite-sheeted covering by $S^1 \times S^1 \times S^1$, while the fundamental groups of the former never contain non-cyclic free Abelian subgroups. Thurston [17] exposed the powerful impact of geometric structures for our understanding of 3-manifolds.

Returning to the topic of classification, according to [2] deciding whether two n-manifolds with appropriate structures are (1) homeomorphic, (2) diffeomorphic, (3) PL homeomorphic, or (4) homotopy equivalent is recursively unsolvable for $n \geqslant 4$. Indeed, for each recursively enumerable degree of unsolvability D, there is a class of n-manifolds with appropriate structures such that the problem of deciding whether two members of the class are equivalent in any of the four above senses is of degree D.

Nevertheless, certain specific manifolds, or groups of manifolds, allow more focused, effective characterization. Given a particular manifold M one can seek conditions under which another manifold M' is equivalent to M. An historically important approach involves a concept of cobordism. Two compact n-manifolds M_0 and M_1 are ***h*-cobordant manifolds** if there exists an $(n + 1)$-manifold with boundary W, where ∂W is the (disjoint) union of M_0 and M_1, and the inclusions $M_i \to W$, $i = 1, 2$, are homotopy equivalences. Such a triple (W, M_0, M_1) is called an ***h*-cobordism**. It turns out that M_0 and M_1 are h-cobordant if and only if $M_0 \times \mathbb{R}$ and $M_1 \times \mathbb{R}$ are homeomorphic. The classical h-cobordism Theorem [14, 15] promises that if (W^{n+1}, M_0, M_1) is an h-cobordism, $n \geqslant 5$, where W^{n+1} is a simply connected PL (respectively, smooth) n-manifold with boundary, then W^{n+1} is PL homeomorphic (respectively, diffeomorphic) to $M_0 \times [0, 1]$. As a corollary, every compact PL n-manifold Σ^n homotopy equivalent to S^n, $n \geqslant 5$, is PL homeomorphic to S^n. There is a related s-cobordism theorem, which promises that a given h-cobordism (W, M_0, M_1) is a product $M_0 \times [0, 1]$ if and only if a certain obstruction $\tau(W^{n+1}) \in Wh(\pi_1(W^{n+1}))$ vanishes. Here $Wh(\pi_1(W^{n+1}))$ denotes a certain group called the **Whitehead group of** $\pi_1(W^{n+1})$. Finitely generated free Abelian groups are among the many groups G for which $Wh(G)$ is known to be trivial. The obstruction theory is known to measure essential features, in that all possible obstructions can be realized: for each compact n-manifold M_0, $n \geqslant 5$, with fundamental group Γ and each $\tau_0 \in Wh(\Gamma)$ there corresponds an h-cobordism (W^{n+1}, M_0, M_1) such that $\tau(W^{n+1}) = \tau_0 \in Wh(\Gamma)$.

Classical information assures that topological m-manifolds, $m \leqslant 3$, admit unique smooth and PL structures. For $m \geqslant 5$ a topological m-manifold M^m admits a PL structure if and only if an obstruction $\Delta(M^m) \in H^4(M^m; \mathbb{Z}_2)$ vanishes; when one such structure exists, the others are classified by elements of $H^3(M^m; \mathbb{Z}_2)$ [8].

In the smooth category, Smale's work [14] established that a compact contractible, smooth n-manifold with simply connected boundary is diffeomorphic to B^n, $n \geqslant 5$. Diffeomorphic equivalence does not always hold for smooth structures on the n-sphere, however. Kervaire and Milnor [7] studied the collection Θ_n of differentiable structures on S^n, which is an Abelian group under the operation of connected sum, showing Θ_n to be finite and often nontrivial for $n > 4$.

A major issue is to decide which homotopy equivalences between manifolds are homotopic to homeomorphisms. It is useful to identify the **manifold structure set** $S(M)$ associated with a given n-manifold M, namely, the set of homotopy equivalences $f : N \to M$, where N is another manifold and where two such objects $f_i : N_i \to M$, $i \in \{1, 2\}$, are considered equivalent provided there exists a homeomorphism $h : N_1 \to N_2$ with $f_2 h$ homotopic to f_1. Bieberbach [1] showed that for any compact n-manifold M with **flat** geometric structure, that is, with the geometric structure of $\mathbb{R}^n$, every homotopy equivalence $N \to M$ is homotopic to a (special type of) homeomorphism; in other words, $S(M)$ is trivial in this case. Based on this and other evidence, Borel conjectured that compact aspherical n-manifolds M, $n > 4$ all have trivial structure set. The Borel Conjecture has been established for many classes of geometric structures, in the sense that $S(M)$ has been shown to be trivial for all manifolds M possessing the given structure (cf. [4]).

Startling anomalies arise in the 4-dimensional world. For instance, for $n \neq 4$ any smooth (or PL) n-manifold W^n which is topologically homeomorphic to $\mathbb{R}^n$ is diffeomorphic (or PL homeomorphic) to it, but there are uncountably many different (non-diffeomorphic) smooth 4-manifolds W^4 which are topologically equivalent to $\mathbb{R}^4$. Freedman [6] established a beautiful topological classification of compact simply connected 4-manifolds M^4, in terms of the bilinear forms determined by the cup product $H^2(M^4) \times H^2(M^4) \to H^4(M^4)$ and of one extra ingredient – whether the two objects both do or both do not admit smooth structures. The 4-dimensional Poincaré Conjecture is a corollary: every compact 4-manifold Σ^4 homotopy equivalent to S^4 is topologically equivalent to S^4. Whether when endowed with a smooth structure then Σ^4 must be diffeomorphic to S^4 remains unsettled, but other compact 4-manifolds are known to admit non-diffeomorphic smooth structures.

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k-11 Infinite-Dimensional Topology

It is not an easy task to define the scope of infinite-dimensional (**i-d**) topology. Initially, in the thirties, the objects of the theory were i-d **metric linear spaces** (i.e., topological vector spaces whose topology is metrizable) and their convex subsets. The standard examples of such objects were the separable i-d Hilbert space $\ell_2 = \{x = (x_n) \in \mathbb{R}^\omega \colon \|x\| = (\sum_{n\in\omega} x_n^2)^{1/2} < \infty\}$ and the **Hilbert cube** $Q = [-1,1]^\omega$, the countable product of the unit intervals. Beginning in the late sixties, the collection of rather isolated results concerning the topology of those spaces evolved into the theory of **manifold**s modeled on i-d metric linear spaces and their i-d convex subsets (i.e., spaces that are locally homeomorphic with the model). This approach led to the topological identification of certain i-d spaces such as ***topological group***s, function spaces, or even spaces without any natural algebraic or convex structure like ***hyperspaces*** of compact sets. We focus on this part of the theory, and we omit the area of i-d topology that concerns non-metrizable spaces (see [8] and [6]) and aspects of the theory more closely related to geometric, algebraic, and differential topology.

Early results on i-d topology show that some topological intuitions concerning Euclidean spaces $\mathbb{R}^n$ fail in the case of ℓ_2, the space that is viewed as a natural i-d counterpart of $\mathbb{R}^n$. In particular, we have: The closed unit ball B_{ℓ_2} of ℓ_2 does not have the ***fixed-point property*** (Kakutani). The unit sphere S_{ℓ_2} of ℓ_2 is a retract of B_{ℓ_2}, therefore S_{ℓ_2} is contractible (Tychonoff). Furthermore, the sphere S_{ℓ_2} is homeomorphic with the whole space ℓ_2 (Klee). The compacta are topologically **negligible** in ℓ_2, i.e., the space ℓ_2 is homeomorphic with the complement $\ell_2 \setminus K$ for any compact set $K \subset \ell_2$ (Klee). The Hilbert space ℓ_2 has the **homeomorphism extension property** for compacta, meaning that every homeomorphism $\varphi : K \to L$ between compact subsets of ℓ_2 can be extended to a homeomorphism $\Phi : \ell_2 \to \ell_2$ (Klee). The Hilbert cube Q is ***homogeneous*** (Keller). For more details, see [2], the first monograph on i-d topology, and [10].

One of the most important and natural questions of i-d topology is the problem of the topological classification of i-d metric linear spaces and i-d convex sets. Recall that a topological vector space X is **locally convex** if X has a base of neighbourhoods of 0 consisting of convex sets. The local convexity is an essential property, because every convex subset of a locally convex metric linear space is an ***absolute retract*** (AR) (Borsuk, Dugundji). A ***completely metrizable*** locally convex topological vector space is called a **Fréchet space**. The countable product of real lines $\mathbb{R}^\omega$, a Fréchet space which is not a Banach space, plays an important role in i-d topology. The question whether all separable i-d Fréchet spaces are homeomorphic, which is attributed to Fréchet and Banach, is considered as the starting point of i-d topology and has had an enormous impact on the area. The affirmative answer to this question was obtained from the combined work of Kadec, Anderson, and Bessaga–Pełczyński. In particular, the final result (often called the **Kadec–Anderson theorem**) demonstrated that the topological structure of a Fréchet space carries no information on its linear structure. Kadec, employing geometry of Banach spaces, proved that all i-d separable Banach spaces are homeomorphic to ℓ_2. Anderson, using purely topological methods and developing new important tools for the i-d topology, showed that ℓ_2 is homeomorphic to $\mathbb{R}^\omega$. Among other things, he introduced the concept of a Z**-set**, which plays a central role in the theory. The following simple way to define this notion is due to Toruńczyk: A closed subset A of an ***absolute neighborhood retract*** (ANR) X is a Z-set in X if, for every $n \in \omega$, every continuous map $f : [0,1]^n \to X$ can be arbitrarily closely approximated by maps $g : [0,1]^n \to X \setminus A$. For example, the compacta in ℓ_2 are Z-sets. Since in $\mathbb{R}^n$ there are no nonempty Z-sets, this again demonstrates the difference between the i-d and finite-dimensional cases. The homeomorphism extension property mentioned above holds also for the class of Z-sets both in ℓ_2 and Q, respectively. This fact is sometimes referred to as the Z**-set unknotting theorem**. The Kadec–Anderson theorem gives rise to the theory of the so-called ℓ_2**-manifolds**. These are separable metrizable manifolds modeled on ℓ_2, equivalently, modeled on any i-d separable Fréchet space (hence, sometimes called **Fréchet manifold**s). It follows from Anderson's result that such manifolds locally possess the structure of an infinite product of the real lines. This important property is reflected in yet another term used for these objects, namely, the $\mathbb{R}^\omega$**-manifold**s. Those manifolds together with Q**-manifold**s (that is, separable manifolds modeled on Q) have been the objects intensely investigated in i-d topology. Clearly, both classes of those manifolds consist of separable, completely metrizable spaces. Also they are ANRs because a space that is locally an ANR is an ANR, and an open subset of an ANR is an ANR. Q-manifolds are ***locally compact***, while ℓ_2-manifolds are not. The fundamental results concerning those manifolds are their celebrated characterizations due to Toruńczyk [12] and [13] (see also [8] and [6]), whose statements employ the following notions. Given an ***open cover*** $\mathcal{U}$ of a space X, we say that maps $f, g : Y \to X$ are $\mathcal{U}$**-close** if, for every $y \in Y$, the set $\{f(y), g(y)\}$ is contained in some $U \in \mathcal{U}$. This kind of approximation plays an important role in i-d topology, and is used in the case where maps have noncompact domains. This can be illustrated by the following fact. A is a Z-set in an ANR space X if and only if every continuous map $f : X \to X$ can be arbitrarily closely (in the above sense) approximated by maps

$g: X \to X \setminus A$. In case X is either a Q-manifold or an ℓ_2-manifold, one can require that the closure of $g(X)$ is disjoint with A. Closed subsets of ANRs that enjoy the last property are called **strong Z-sets**; not all Z-sets are strong Z-sets. Though strong Z-sets explicitly do not appear in the forthcoming characterizations, their role in obtaining these results, as well as, in topology of incomplete i-d spaces is significant. The space X has the **strong discrete approximation property** (**SDAP**) if, for every continuous map f of the ***topological sum*** $\bigoplus_{n\in\omega}[0,1]^n$ into X and every open cover $\mathcal{U}$ of X, there exists a continuous $g: \bigoplus_{n\in\omega}[0,1]^n \to X$ such that the family $\{g([0,1]^n): n \in \omega\}$ is ***discrete*** in X, and f and g are $\mathcal{U}$-close. The space X has the **disjoint cells approximation property** (**DCAP**) if, for every $n \in \omega$, every continuous map $f: [0,1]^n \times \{0,1\} \to X$ can be arbitrarily closely approximated by continuous maps $g: [0,1]^n \times \{0,1\} \to X$ such that $g([0,1]^n \times \{0\}) \cap g([0,1]^n \times \{1\}) = \emptyset$. Here, by arbitrary close approximation we mean the usual uniform (ε-close) approximation which, in the case of compact domain, is equivalent to the above strong ($\mathcal{U}$-close) approximation. A separable metrizable space X is an ℓ_2-manifold if and only if X is a completely metrizable ANR with SDAP. Here, if we replace ANR by AR, we obtain a characterization of ℓ_2. A separable metrizable space X is a Q-manifold if and only if X is a locally compact ANR with DCAP. Moreover, a metrizable space X is homeomorphic to Q if and only if X is a compact AR with DCAP. Toruńczyk's characterizations are the standard tools for recognizing ℓ_2- and Q-manifolds among i-d completely metrizable spaces. For instance, the ***cone*** over ℓ_2 (respectively, over Q) is homeomorphic to ℓ_2 (respectively, to Q); here, the cone over a metrizable space X is meant to be the space $X \times [0,1) \cup \{\infty\}$ with the usual product topology at the points (x,t), and a basic neighbourhood of ∞ is of the form $X \times (t,1) \cup \{\infty\}$, $0 < t < 1$.

An application of the characterization of ℓ_2 yields that every i-d separable complete metric linear space X is homeomorphic to ℓ_2 if and only if X is AR. This generalizes the Kadec–Anderson theorem and indicates the importance of the question whether all metric linear spaces are ARs. A negative solution to this problem is due to Cauty, who constructed a separable complete metric linear space which is not an AR [4]. This example provides a negative answer to the question of Fréchet who apparently asked whether all i-d separable complete metric linear spaces are homeomorphic. Cauty's example also refuted the conjecture stating that a separable completely metrizable topological group G which is ***locally contractible*** and is not ***locally compact*** must be an ℓ_2-manifold. However, the conjecture holds when the group G has the Absolute Neighborhood Retract property, which is stronger than the local contractibility. More precisely, a separable completely metrizable topological ANR group G either has a structure of a ***Lie group*** (in case G is locally compact, equivalently, G is finite-dimensional), or is an ℓ_2-manifold. Sometimes it is difficult to determine whether a given locally contractible group is an ANR. For $n \in \mathbb{N}$, by $H([0,1]^n)$ we denote the group of all autohomeomorphisms of $[0,1]^n$ that are the identity on the boundary of $[0,1]^n$, equipped with the uniform convergence topology. The local contractibility of $H([0,1]^n)$ was established by Černavskiĭ and Edwards–Kirby. The problem whether the space $H([0,1]^n)$, $n > 2$, is an AR (which would imply that $H([0,1]^n)$ is homeomorphic to ℓ_2) remains one of the fundamental open questions in i-d topology. The cases of $n = 1, 2$ have been affirmatively settled. It is known that $H(Q)$, the group of all autohomeomorphisms of Q, is homeomorphic to ℓ_2. In general, an intrinsic characterization of ℓ_2-manifolds that admit a group structure is unknown.

Among numerous other applications of Toruńczyk's characterization of ℓ_2 we have selected the following list of spaces homeomorphic to ℓ_2: all convex closed and non-locally compact subsets of a separable Fréchet space; the space of all ***pseudo-arc***s $P(\mathbb{R}^2)$ considered as a subspace of the hyperspace of compact subsets of $\mathbb{R}^2$; every product $\prod_{n\in\omega} X_n$ of separable completely metrizable non-compact ARs X_n; the product $X \times \ell_2$, where X is any separable completely metrizable Absolute Retract (more generally, if X is an ANR then $X \times \ell_2$ is an ℓ_2-manifold); the function space $C(X,Y)$ of all continuous maps from an infinite compact metrizable space X into a non-trivial separable completely metrizable Absolute Retract Y, endowed with the uniform convergence topology (in general, if Y is an ANR without isolated points, $C(X,Y)$ is an ℓ_2-manifold). Another interesting example of a function space that is an ℓ_2-manifold is the space $R(M)$ of all ***retractions*** of the compact Q-manifold M regarded as a subspace of $C(M,M)$. Similar results hold for one- or two-dimensional M, (see [vMR, p. 562]). The case of other dimensions is not settled.

For a metrizable space X, M_X denotes the space of equivalence classes of **Borel-measurable maps** from $[0,1]$ into X (here, maps which are a.e. equal with respect to the Lebesgue measure are identified and a map is Borel-measurable if preimages of ***Borel sets*** are Borel). M_X is equipped with the **topology of convergence in measure** (this topology is metrizable by the metric $\rho(f,g) = \int_0^1 d(f(t), g(t))\,dt$, where d is any bounded metric on X). The space M_X, for a non-trivial separable complete metrizable X is yet another example of a topological copy of ℓ_2 (Bessaga–Pełczyński [2]). For $X = \{0,1\}$ the space M_X can be identified with the topological space of the measure algebra for the Lebesgue measure. From Bessaga–Pełczyński's result it can be derived that every separable (completely) metrizable topological group G is group-topological isomorphic to a (closed) subgroup of a topological group H homeomorphic to ℓ_2; consequently, every such group G admits a ***free action*** on ℓ_2 (West).

We start the list of results on identifying Q-manifolds among compact and locally compact i-d spaces with the classical theorem of Keller stating that every i-d compact convex subset of a Banach space is homeomorphic to Q (Klee extended this result for subsets of Fréchet spaces). Toruńczyk's characterization allows to strengthen this as follows. An i-d compact convex subset K of a metric linear space is homeomorphic to the Hilbert cube if and only if K is an AR. One of the central problems of i-d topology is the question whether every compact convex subset of a metric linear space is an AR. A convex locally compact closed subset C of a Fréchet space is homeomorphic with either $[0,1]^n \times \mathbb{R}^k$ or $[0,1]^n \times [0,\infty)$, where $0 \leqslant n \leqslant \omega$ and $k \in \omega$. The same conclusion holds if one replaces the Fréchet space by any metric linear space and additionally assumes that C is an AR. Every (not necessarily closed) i-d locally compact convex subset C of a Fréchet space is homeomorphic to a **solid** Q-manifold, that is, a manifold of the form $Q \setminus A$, where A is a Z-set in Q (the converse is also true, each such Q-manifold is homeomorphic to some convex C).

The product $\prod_{n\in\omega} X_n$ of non-trivial compact ARs X_n is homeomorphic to Q; if finitely many of X_n are locally compact ANRs then $\prod_{n\in\omega} X_n$ is a Q-manifold. In particular, this yields Edward's factor theorem saying that the product of every locally compact absolute neighbourhood retract X with the Hilbert cube is a Q-manifold. The first contribution in this direction was Anderson's solution to the Scottish Book problem of Borsuk whether the product of a ***triod*** with Q is a Hilbert cube (this problem is believed to be one of the origins of i-d topology). Two famous results on identifying topological copies of the Hilbert cube among hyperspaces of compact sets are due to West, Curtis, and Schori, see [10, 11]. For a compact space X, the hyperspace 2^X is homeomorphic to Q if and only if X is a non-degenerate ***Peano continuum***. The hyperspace of subcontinua $C(X)$ of a continuum X is homeomorphic to Q if and only if X is a non-degenerate Peano continuum without free arcs (a **free arc** is an open subset homeomorphic to the real line). Long ago, Wojdysławski established the AR-property of 2^X (and $C(X)$) for a Peano continuum X, and asked whether 2^X was homeomorphic to Q.

The following list of fundamental results of the theory of ℓ_2-manifolds can be found in [2]. These results follow from the work of Anderson, Henderson, Schori, West, and others, which was done at the end of the sixties and at the beginning of the seventies. For ℓ_2-manifolds M and N, we have: M and N are homeomorphic if and only if they are homotopy equivalent (**classification by homotopy type**); M embeds onto a closed subset of ℓ_2 (**closed embedding theorem**); M embeds onto an open subset of ℓ_2 (**open embedding theorem**); every homeomorphism $h: A \to B$ between Z-sets of M that is homotopic to the identity on A extends to a homeomorphism of M (**Z-set unknotting theorem**); M is homeomorphic to the product $K \times \ell_2$ for some locally compact metrizable ***polyhedron*** K (**triangulation theorem**); M is homeomorphic to $M \times \ell_2$ (**stability theorem**); for every set A that is a countable union of Z-sets in M, $M \setminus A$ and M are homeomorphic (**negligibility theorem**). Estimated (controlled) versions of those theorems are also known. Here are two transparent applications of those results (stated in the simplest way). If a topological group G **acts effectively** on ℓ_2 (i.e., if $g_1 x = g_2 x$ for every x, then $g_1 = g_2$), then G admits also a free action on ℓ_2 (West). Every autohomeomorphism h of ℓ_2 is **stable**; more precisely, there exist autohomeomorphisms h_1 and h_2 such that $h = h_1 \circ h_2$, and h_i is the identity on a nonempty open set U_i, $i = 1, 2$ (Wong). All of the above fundamental results on ℓ_2-manifolds hold true for connected metrizable nonseparable **Hilbert manifolds** (i.e., spaces locally homeomorphic to a given Hilbert space). Together with the characterization of ℓ_2-manifolds, Toruńczyk [13] also gave a characterization of nonseparable Hilbert manifolds. As an application, he showed that two i-d Fréchet spaces are homeomorphic if they have the same ***weight*** (equivalently, the same ***density***). This shows that the terms Fréchet manifold or Hilbert manifold describe the same objects. Other applications are also known.

The theory of Q-manifolds is more complex, see [5]; all essential results in this area are due to Chapman. In general, only the counterparts of the Z-unknotting, triangulation, and stability theorems hold. Specifically, the triangulation theorem states that for every Q-manifold M there exists a locally compact metrizable polyhedron K such that M and $K \times Q$ are homeomorphic. This is an essential ingredient in West's [14] affirmative solution of Borsuk's conjecture stating that every compact metrizable ANR has the homotopy type of a finite ***polyhedron***, i.e., one with finitely many ***simplices***. For obvious reasons the closed embedding, open embedding, and negligibility theorems fail, in general, for Q-manifolds. All those theorems, including classification by the homotopy type, hold for the class of $[0,1)$**-stable Q-manifold**s M (that is, in case M is homeomorphic to $M \times [0,1)$). Representing two Q-manifolds M and N as $K \times Q$ and $L \times Q$, respectively, where K and L are locally compact metrizable polyhedra (obtained from the triangulation theorem), the manifolds M and N are homeomorphic if and only if K and L have the same infinite ***simple-homotopy type***. As a corollary, every homeomorphism between compact polyhedra is a simple-homotopy equivalence (the affirmative answer to the topological invariance of Whitehead torsion problem). A compact contractible Q-manifold is necessarily homeomorphic to Q. The solid Q-manifolds are classified in terms of ***shapes***. Namely, two such manifolds $Q \setminus A$ and $Q \setminus B$ (A and B are Z-sets in Q) are homeomorphic if and only if A and B have the same shape.

A map $p: Y \to Z$ is called a **near-homeomorphism** if for every open cover $\mathcal{V}$ of Z there exists a homeomorphism $h: Y \to Z$ that is $\mathcal{V}$-close to p. A surjective ***perfect map*** $f: Y \to Z$ is called a **cell-like map** (**CE-map**) if every point inverse $f^{-1}(z)$ has ***trivial shape*** (i.e., $f^{-1}(z)$ deforms to a point in each of its neighbourhoods in some, hence every, ANR in which it is embedded). CE maps and near-homeomorphisms lie at the heart of i-d topology, e.g., they are basic tools in obtaining the topological characterizations of Hilbert manifolds and Q-manifolds. CE maps are near-homeomorphisms if their domains and images are "regular" spaces. For example, a surjective map between Q-manifolds (respectively, n-dimensional manifolds, $n \neq 3$) is CE if and only if it is a near-homeomorphism. According to **Bing's shrinking**

criterion (and its generalizations due to Toruńczyk), a continuous surjection $p: Y \to Z$ between completely metrizable spaces is a near-homeomorphism if and only if p is **shrinkable**. The latter, in case of compact metrizable Y and Z, means that for every $\varepsilon > 0$ and for every autohomeomorphism h of Y there exists an autohomeomorphism g of Y such that $p \circ h$ and $p \circ g$ are ε-close, and $p \circ g$ is an ***ε-map*** (i.e., all its point-inverses have diameter $< \varepsilon$). For a manifold M modeled on an i-d Hilbert space H, the projection $M \times H \to M$ is a near-homeomorphism, a strengthening of the stability theorem. By proving that, for a separable completely metrizable ANR space X, the projection $X \times \ell_2 \to X$ is shrinkable if X satisfies SDAP, Toruńczyk arrived at his characterization of ℓ_2-manifolds. He also employed the fact that $X \times \ell_2$ is an ℓ_2-manifold, a consequence of his factor theorem. A similar argument works for obtaining the characterization of manifolds modeled on nonseparable Hilbert spaces and of Q-manifolds. In the case of Q-manifolds, however, one makes use of Edwards' factor theorem, which is a consequence of the following two facts. For a CE map $f: M \to X$ between a Q-manifold M and a locally compact ANR space X, $f \times \mathrm{id}: M \times Q \to X \times Q$ is shrinkable. Every locally compact metrizable ANR is the CE image of a Q-manifold (the Miller–West resolution theorem).

CE maps commonly appear in ***decompositions*** $\mathcal{A}$ of a space X, and are realized as quotient maps $f: X \to Y$, where $Y = X/\mathcal{A}$. However, neither the ANR-property nor the finite-dimensionality of X is carried over to Y. Taylor constructed a CE map of Q onto a non AR. One may even require that the point inverses of such a map are homeomorphic to Q (van Mill's solution of a Borsuk problem). There exist CE maps from compact n-dimensional manifolds, $n \geqslant 6$, to infinite-dimensional compacta (combined efforts of Dranishnikov, Edwards, and Walsh). Dranishnikov's contribution [7] comes down to his famous example of an i-d compact metrizable space with (integral) ***cohomological dimension*** 3, a solution of a longstanding Alexandroff's problem. There are a few interesting results and questions on the AR images of CE maps of Q [vMR, pp. 527–531]. On the other hand, an ANR CE image of an i-d Hilbert manifold is a Hilbert manifold itself (Toruńczyk [13]).

One of the most interesting branches of i-d topology is related to group actions on Q-manifolds and ℓ_2-manifolds. Actions of compact groups on ℓ_2-manifolds are well-understood. For every separable metric group G and every closed $A \subset \ell_2$, there exists an action of G on ℓ_2 such that A is its set of fixed points (i.e., for every $g \neq 1$, $A = \{x \in \ell_2: gx = x\}$), a fact that supplements results mentioned earlier. Since Q and ℓ_2 can be represented as infinite products of their copies or of other factors, group actions on them can be defined as products of actions on the factors . Also the cone structure of Q and ℓ_2 provides interesting examples of group actions on these spaces (see [5, p. 123] and [9] for examples and the discussion on that subject). A group G acts on X **semi-freely** if there exists $p \in X$ such that G acts freely on $X \setminus \{p\}$. Anderson's question of whether a semi-free action of Z_2 on Q must be topologically conjugated with the standard antipodic action $x \mapsto -x$ on Q is a longstanding problem. It is known that this holds if the fixed point of the action has a basis of contractible invariant neighbourhoods (for other partial results see [9] and [vMR, p. 554]). The most fundamental group action problem, possibly involving i-d spaces, is the Hilbert–Smith conjecture, a generalization of Hilbert's Fifth Problem. It asks whether a locally compact topological group acting effectively on an n-dimensional manifold M must be a Lie group. If the answer is not affirmative, then the p-adic group A_p acts effectively on M (Montgomery–Zippin), in which case the orbit space M/A_p has (integer) cohomological dimension $n+2$, and therefore its ***covering dimension*** is either $n+2$ or ∞. Orbit spaces of group actions turn out to be interesting examples of i-d spaces. Here are two such intriguing spaces related to hyperspaces and the Banach space theory. For a compact Lie group G, G acts on the hyperspace 2^G, a topological copy of Q, by left translation. In general, the topological structure of the orbit space $2^G/G$ is unknown. But, in case of the circle group S^1, $(2^{S^1}/S^1) \setminus [S^1] = 2^{S^1} \setminus \{S^1\}/S^1$, where $[S^1]$ denotes the orbit of S^1, is a Q-manifold which is an ***Eilenberg–MacLane space*** of type $K(\mathbb{Q}, 2)$; here, $\mathbb{Q}$ stands for the rationals. Let $BM(n)$, $n \geqslant 1$, be the Banach–Mazur compactum, i.e., $BM(n)$ is the set of isometry classes of n-dimensional real Banach spaces topologized by the metric $d(E, F) = \ln \inf\{\|T\| \|T^{-1}\|: T: E \to F$ is an isomorphism$\}$. The space $BM(n)$ is (canonically) homeomorphic to the orbit space of the hyperspace $\mathcal{N}(n)$ of all symmetric compact convex bodies in $\mathbb{R}^n$ modulo the obvious action of the group $GL(n)$ of all isomorphisms of $\mathbb{R}^n$. The compactum $BM(2)$ is an i-d AR, but it is not homeomorphic to Q because the space $BM(2) \setminus [H] = \mathcal{N}(2) \setminus \{B\}/GL(n)$, where $[H]$ is the isometry class of the 2-dimensional Hilbert space H and B is the Euclidean 2-ball, is not contractible and has a complicated homotopy structure. In case $n > 2$ one only knows that $BM(n)$ is an AR.

The theory of i-d incomplete spaces is much more complex, even when it comes to the topological classification of ***normed spaces***. There are continuum many pairwise non-homeomorphic σ-compact such spaces. In general, there are 2^{2^ω} many pairwise non-homeomorphic separable normed spaces. The topological classification of incomplete normed linear spaces does not come down to considering all vector subspaces of a Hilbert space. There exists a dense vector subspace of $\ell_1 = \{x = (x_n) \in \mathbb{R}^\omega: \|x\| = \sum_{n\in\omega} |x_n| < \infty\}$ that is not homeomorphic with any convex subset of ℓ_2. Versions of topological characterizations of incomplete i-d spaces exist however. They rely on Bestvina–Mogilski's generalization of the notion of **absorbing sets** in Hilbert manifolds (or Q-manifolds) [3], an idea that goes back to the work of Anderson (on cap-sets and fd-cap-sets), Bessaga–Pełczyński (on skeletons), and Toruńczyk, West, and Chapman (on absorbing sets), see [1, 11]. For a **topological class** $\mathcal{C}$ of separable metrizable spaces (i.e., if $C \in \mathcal{C}$ then $h(C) \in \mathcal{C}$ for every homeomorphism h), $\mathcal{C}_\sigma$ stands for the class of all metrizable spaces of the form $Y = \bigcup_{n\in\omega} Y_n$, where each

$Y_n \in \mathcal{C}$ is closed in Y. A separable metrizable space $X \in \mathcal{C}_\sigma$ is a **$\mathcal{C}$-absorbing space** if X is an ANR with SDAP, and additionally X is **strongly $\mathcal{C}$-universal** and is a **Z_σ-space**. The latter means that X is a countable union of Z-sets in X, a property that is stronger than being of the ***first category*** in itself. To explain the strong $\mathcal{C}$-universality, we note that, designating $\mathcal{M}_1$ for the class of separable metrizable ***absolute G_δ-sets***, every ℓ_2-manifold M not only is a **$\mathcal{M}_1$-universal space** (that is, every element of $\mathcal{M}_1$ embeds onto a closed subset of M), but also M is strongly $\mathcal{M}_1$-universal, which means that, for any $A \in \mathcal{M}_1$ and a closed set $B \subset A$, any continuous map $f: A \to M$ such that $f|B$ is a **Z-embedding** (an embedding whose image is a Z-set in M) can be arbitrary closely approximated by Z-embeddings $g: A \to M$ with $g|B = f|B$. Similarly, Q-manifolds are strongly $\mathcal{A}_1$-universal, where $\mathcal{A}_1$ stands for the class of σ-compact metrizable spaces. The spaces $\Sigma = \{(x_n) \in \mathbb{R}^\omega\colon \sup|x_n| < \infty\}$ and $\sigma = \{(x_n) \in \mathbb{R}^\omega\colon x_n = 0 \text{ a.e.}\}$ are, respectively, $\mathcal{A}_1$- and $\mathcal{A}_1^{fd}$-absorbing spaces, where the class $\mathcal{A}_1^{fd}$ is a subclass of $\mathcal{A}_1$ that consists of countable unions of finite-dimensional compacta. An ANR space $X \in \mathcal{A}_1$ is a **Σ-manifold** (i.e., a manifold modeled on Σ) if and only if X is a strongly $\mathcal{A}_1$-universal Z_σ-space with SDAP, see [8]. Replacing the class $\mathcal{A}_1$ by $\mathcal{A}_1^{fd}$, we obtain a topological characterization of **σ-manifolds** among $\mathcal{A}_1^{fd}$-spaces which are ANRs. As an application, every σ-compact and universal for metrizable compacta, vector subspace E of a Fréchet space is homeomorphic to Σ. Also, every i-d metric linear space $E_0 \in \mathcal{A}_1^{fd}$ is homeomorphic to σ; here, one employs the fact that the local contractibility implies the ANR property for the ***countable-dimensional*** metrizable spaces (Haver). Since every $\mathcal{C}$-absorbing space X is also a $\mathcal{C}_\sigma$-absorbing space, in the above, the classes $\mathcal{A}_1$ and $\mathcal{A}_1^{fd}$ can be replaced by the class of metrizable compacta and the class of finite-dimensional metrizable compacta, respectively.

The fundamental tool in identifying i-d incomplete spaces is the **uniqueness theorem** for absorbing spaces, which states that, for a topological class $\mathcal{C}$, two $\mathcal{C}$-absorbing spaces X and X' are homeomorphic if and only if they are homotopy equivalent; in particular, if X and X' are AR's then X and X' are homeomorphic. Counterparts of the Z-set unknotting and triangulation theorems hold for $\mathcal{C}$-absorbing spaces (see [1]). In particular, Absolute Retract $\mathcal{C}$-absorbing spaces are homogeneous but they may not carry either a group or a convex structure. It turns out that every $\mathcal{C}$-absorbing space X can be nicely embedded as a dense subset of a certain ℓ_2-manifold M. A relative version of the strong universality leads to the notion of **absorbing pairs** (M, X), $X \subset M$, where M is either an ℓ_2-manifold or a Q-manifold. The uniqueness theorem remains true for the absorbing pairs. A variety of AR $\mathcal{C}$-absorbing spaces (respectively, absorbing pairs) are known [1]. With the use of the uniqueness theorem, many i-d incomplete spaces (respectively, pairs of spaces) have been topologically identified as such absorbing spaces (respectively, such absorbing pairs), see [1, 11]. For instance, for every countable ordinal $\alpha \geqslant 1$, there exists an $\mathcal{A}_\alpha$-absorbing space Λ_α, where $\mathcal{A}_\alpha$ denotes the class of separable ***absolute Borel sets*** of ***additive class*** α; similarly, for $\alpha \geqslant 2$, there exists an $\mathcal{M}_\alpha$-absorbing space Ω_α, where $\mathcal{M}_\alpha$ denotes the class of separable ***absolute Borel sets*** of ***multiplicative class*** α. There are also absorbing spaces Π_n for every ***projective class*** $\mathcal{P}_n$ (in contemporary notation, $\mathcal{P}_{2n-1} = \Sigma_n^1$ and $\mathcal{P}_{2n} = \Pi_n^1$). There exist copies of all those absorbing spaces X in ℓ_2 and Q so that (ℓ_2, X) or (Q, X) are absorbing pairs. We can set $\Lambda_1 = \Sigma$, $\Omega_2 = \Sigma^\omega$ and, in general, $\Omega_\alpha = \Lambda_\beta^\omega$ if $\alpha = \beta + 1$ and $\Omega_\alpha = \prod_{\xi<\alpha} \Lambda_\xi$ if α is a limit ordinal. Each such Ω_α can be canonically identified with a subset of $\mathbb{R}^\omega$; for $\alpha > 1$, letting $Z = \mathbb{R}^\omega \setminus \Omega_\alpha$ and fixing $p \in Z$, we set $\Lambda_\alpha = \{(x_n) \in Z^\omega\colon x_n = p \text{ a.e.}\}$. The product σ^ω is another $\mathcal{M}_2$-absorbing space; hence, Σ^ω is homeomorphic to σ^ω. The space $Q \setminus (-1,1)^\omega$ is also an $\mathcal{A}_1$-absorbing space. Furthermore, $(Q, Q \setminus (-1,1)^\omega)$, and $(\{x \in \ell_2\colon \|x\| \leqslant 1\}, \{x \in \ell_2\colon \|x\| < 1\} \cup \{p\})$, $p = (1,0,0,\dots) \in \ell_2$, with the ***weak topology***, are absorbing pairs. By the uniqueness theorem for the pairs, $(-1,1)^\omega$ is homeomorphic to $S_0 = \{x \in \ell_2\colon \|x\| = 1\} \setminus \{p\}$ with the weak topology. Since the weak and the norm topologies coincide on the unit sphere in ℓ_2, and since the stereographic projection yields a homeomorphism of S_0 onto an isometric copy of ℓ_2, we conclude that $\mathbb{R}^\omega$ is homeomorphic to ℓ_2. This was Bessaga–Pełczyński's proof of Anderson's theorem; in the same manner, they showed that every i-d separable Banach space was homeomorphic to $\mathbb{R}^\omega$.

The list of spaces that were topologically identified with the use of absorbing space theory is long; in most of the cases the Borel complexity of those spaces was established long ago. While we present the most spectacular applications, for further information, consult [1, 11], [HvM, Chapter 3], and references therein. The space $C_p(X)$ of all continuous real-valued functions on a countable nondiscrete ***regular space*** with the pointwise topology is homeomorphic to σ^ω if and only if $C_p(X) \in \mathcal{M}_2$ (Dobrowolski–Marciszewski–Mogilski), see [11]. Let us note that $C_p(X) \in \mathcal{M}_2$ for every countable metrizable X. The space $\{f \in C[0,1]\colon f$ is differentiable$\}$, as a subspace of the Banach space $C[0,1]$ of all continuous functions $f: [0,1] \to \mathbb{R}$ with the ***topology of uniform convergence***, is homeomorphic to Π_2. For an ***analytic set*** $A \subset [0,1]$, which is not σ-compact and is not dense in $[0,1]$, the space $\mathcal{L}_A = \{f \in C[0,1]\colon f$ vanishes on some neighbourhood of $A\}$ is homeomorphic to Π_2; the fact that $\mathcal{L}_A$ is ***co-analytic*** (i.e., $\mathcal{L}_A \in \mathcal{P}_2$) and not Borel is due to Banach–Kuratowski. The following hyperspaces are homeomorphic to σ^ω: $\{A \in 2^Q\colon \dim(A) = \infty\}$, $\{A \in 2^{\mathbb{R}^2}\colon A$ is an arc$\}$, $\{A \in 2^{\mathbb{R}^2}\colon A$ is an AR$\}$, $L(\mathbb{R}^n) = \{A \in 2^{\mathbb{R}^n}\colon A$ is a Peano continuum$\}$, $n \geqslant 3$ (the fact $L(\mathbb{R}^n) \in \mathcal{M}_2 \setminus \mathcal{A}_2$ was proved long ago by Kuratowski and Mazurkiewicz). The hyperspace $\{A \in 2^{\mathbb{R}^n}\colon A$ is an ANR$\}$, $n \geqslant 3$, is homeomorphic to Ω_3. The hyperspace $\{A \in 2^{[0,1]}\colon A$ is countable$\}$ is homeomorphic to Π_2, and the hyperspace $\{A \in 2^{\mathbb{R}^n}\colon A$ is an arcwise connected continuum$\}$, $n \geqslant 3$, is homeomorphic to Π_4.

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